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CONTENTS.

PAGE

Bosanquet, L. S., and Kestelman, H. The absolute convergence of series of integrals 88
Cooke, R. G., and Dienes, P. On the effective range of generalised limit processes 45
Cooper, J. L. B. The absolute Cesàro summability of Fourier integrals 425
Cossar, J. On conjugate functions 369
Davenport, H. On the product of three homogeneous linear forms (III) 98
Dienes, P., and Cooke, R. G. On the effective range of generalised limit processes 45
Hay, G. E. The method of images applied to the problem of torsion 382
Hirsch, K. A. On skew-groups 357
Hua, Loo-keng. On Waring's problem for fifth powers 144
Kestelman, H., and Bosanquet, L. S. The absolute convergence of series of integrals 88
Kober, H. Beziehungen zwischen Systemen selbstreziproker Funktionen 229
Kuttner, B. Some theorems on Riesz and Cesàro sums 398
Macaulay-Owen, P. Parseval's theorem for Hankel transforms ... 458
Macintyre, A. J. Laplace's transformation and integral functions... 1
Maitland, B. J. On analytic functions bounded at a double sequence of points 440
Rosser, B. The n-th prime is greater than $n \log n$ 21
Seetharaman, V. Differential invariants for higher path spaces ... 64
Stevenson, A. C. The torsion and flexure solutions for the elliptic limaçon cross-section 126
Todd, J. A. The geometrical invariants of algebraic loci (second paper) 410
Turing, A. M. Systems of logic based on ordinals 161
Whitehead, J. H. C. Simplicial spaces, nuclei, and m-groups ... 243
Wolf, F. On summable trigonometrical series: an extension of uniqueness theorems 328
Corrigenda 475

PAPERS

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LAPLACE'S TRANSFORMATION AND INTEGRAL FUNCTIONS

By A. J. Macintyre.

[Received 17 April, 1938.—Read 21 April, 1938.]

Part I.

The first part of the present paper is both comparatively simple and dependent only on the familiar methods available for integral functions of order 1. We are concerned with the development of a remark made by Pólya[†], that if $F(z)$ is an integral function satisfying

$$(1) \qquad |F(z)| < Me^{k|z|} \quad (k < \pi)$$

then the growth of the function along any radius $\arg z = \phi$ is determined by its growth at a special sequence of points, in the sense that

$$(2) \qquad \overline{\lim_{r\to\infty}} \frac{1}{r} \log |F(re^{i\phi})| = \overline{\lim_{n\to\infty}} \frac{1}{n} \log |F(ne^{i\phi})| \quad (n = 1, 2, \ldots).$$

This result is of a kind to be expected, since from Carlson's theorem[‡] a function satisfying (1) is completely determined by its values at the points $ne^{i\phi}$, and vague ideas of interpolation would suggest something like (2). Various generalizations of (2) have been made by other methods without any gain in "depth"[§]. A remarkable advance was recently made by Miss Cartwright† in proving that an integral function satisfying (1) and bounded at the positive and negative integers is bounded along the whole of the real axis. If further $F(n)$ has a limit as n tends to infinity by positive integer values, then $F(x)$ has the same limit as x tends to infinity by positive values. Miss Cartwright also established the natural extension of the first of these results to functions regular and of finite order in an angle. Her methods depended on the use of Lagrange's interpolation formula‡ and the so-called Phragmén-Lindelöf principle. We replace the latter by the Laplace transformation, when, as appears in part II, the extensions of both results to functions of finite order in an angle appear as simple corollaries of the special case. In this part we develop a number of interpolation formulae from the "Borel-Laplace" transformation, and take the opportunity of giving what may be considered a simpler proof and sundry generalisations of Miss Cartwright's results for integral functions of order one.

[†] G. Pólya, *Math. Zeitschrift*, 29 (1929), 549–640. See formula (70), p. 606.

[‡] *I.e.* An integral function satisfying (1) and vanishing at the positive integers is identically zero. F. Carlson, *Sur une classe de séries de Taylor* (Upsala, 1914). The theorem is more general than we have stated. An accessible account using methods very similar to those of the present paper is: G. H. Hardy, *Acta Math.*, 42 (1920), 327–339. The transformation used (330–332) is not that of Laplace and does not appear so convenient for present purposes.

[§] V. Bernstein, *Leçons sur les . . . séries de Dirichlet* (Paris, 1933), 244, and *Bull. Sci. Math.* 52 (1928), 420–436.

1. *Borel's case of the Laplace transformation and the resulting interpolation formulae.*

This transformation is sufficiently well known for us to quote the necessary results§. With the integral function

$$F(z) = \sum_0^\infty a_n z^n \tag{3}$$

satisfying (1) we associate the function

$$f(z) = \sum_0^\infty a_n\, n!\, z^{-n-1}. \tag{4}$$

The relation between these two functions may be expressed by means of the integrals

$$f(z) = \int_0^\infty e^{-t} F(t/z)\, dt/z, \tag{5}$$

$$F(z) = \oint e^{z\zeta} f(\zeta)\, d\zeta. \tag{6}$$

The function $f(z)$ is regular outside the circle $|z| = k$, so that the contour in (6) can be taken as a circle $|z| = k+\epsilon$. A second associated function also discussed by Pólya and others[†] is

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† M. L. Cartwright, *Quart. J. of Math.* (Oxford), 7 (1936), 46–55.

‡ See footnote †, page 5, for the terminology of these interpolation formulae.

§ A detailed account in Pólya, *loc. cit.*, 578–586.

$$\psi(z)=\tfrac{1}{2}F(0)+\sum_{1}^{\infty}F(n)\,e^{-nz}. \tag{7}$$

If we replace $F(n)$ in this expression by means of (6) we obtain

$$\psi(z)=\frac{1}{4\pi i}\oint\frac{1+e^{\zeta-z}}{1-e^{\zeta-z}}f(\zeta)\,d\zeta. \tag{8}$$

If now z, but no other value of $z+2n\pi i$ $(n=0,\ \pm1,\ \pm2,\ \ldots)$, is allowed to enter the contour, this formula must be modified to

$$\psi(z)=f(z)+\frac{1}{4\pi i}\oint\frac{1+e^{\zeta-z}}{1-e^{\zeta-z}}f(\zeta)\,d\zeta. \tag{9}$$

From this we deduce that, if $k<\pi$, then $\psi(z)-f(z)$ is regular for $|z|<2\pi-k$, and that $\psi(z)$ is regular for $k<|z|<2\pi-k$. Hence, if the contour of (6) is inside $|z|<2\pi-k$, we may replace $f(\zeta)$ by $\psi(\zeta)$ and obtain

$$F(z)=\frac{1}{2\pi i}\oint e^{\zeta z}\psi(\zeta)\,d\zeta. \tag{10}$$

We must observe that the expansion (7) follows from (8) when $R(z)<k$, while, when $R(z)<-k$, we obtain the expansion

$$\psi(z)=-\frac{1}{4\pi i}\oint\frac{1+e^{z-\zeta}}{1-e^{z-\zeta}}f(\zeta)\,d\zeta=-\tfrac{1}{2}F(0)-\sum_{1}^{\infty}F(-n)\,e^{nz}. \tag{11}$$

The formula (10) may be regarded as a species of interpolation formula since $\psi(z)$ depends only on $F(n)$, $n=1, 2, \ldots$; and with some other conditions interpolation formulae of more familiar type may be deduced. Suppose, for example, that the series (7) and (11) are convergent, the first for $R(z)>0$, and the second for $R(z)<0$. In this case the contour in (10) can be deformed into the rectangle $-\delta-\omega i$, $\delta-\omega i$, $\delta+\omega i$, $-\delta+\omega i$, where $k<\omega<2\pi-k$ and $\delta>0$. Hence

$$F(z)=\lim_{\delta\to0}\frac{1}{2\pi i}\int_{\delta-\omega i}^{\delta+\omega i}e^{\zeta z}\psi(\zeta)\,d\zeta-\lim_{\delta\to0}\frac{1}{2\pi i}\int_{-\delta-\omega i}^{-\delta+\omega i}e^{\zeta z}\psi(\zeta)\,d\zeta, \tag{12}$$

where the limit exists uniformly for z in any bounded region. Using the series (7) and (11) for $\psi(\zeta)$ and integrating term by term we deduce that

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[†] Pólya, *loc. cit.*, 598–610. E. Lindelöf, *Le calcul des résidues* (Paris, 1905). A. Ostrowski, *Math. Annalen*, 108 (1933), 718–756.

$$(13)\quad F(z)=\lim_{\delta\to 0}\left\{\frac{F(0)}{z}\,\frac{\sin\omega z}{\pi}\cosh\delta z+\frac{e^{\delta z}}{\pi}\sum_{1}^{\infty}\frac{F(n)\,e^{-n\delta}\sin\omega(z-n)}{z-n}\right.$$
$$\left.+\frac{e^{-\delta z}}{\pi}\sum_{1}^{\infty}F(-n)\,e^{-n\delta}\,\frac{\sin\omega(z+n)}{z+n}\right\}.$$

Hence

Theorem 1. *If $F(z)$ is an integral function satisfying* (1) *and if*

$$\overline{\lim_{n\to\infty}}\,\sqrt[n]{|F(\pm n)|}\leqslant 1\qquad (k<\omega<2\omega-k),$$

then†

$$(14)\qquad F(z)=\lim_{\delta\to 0}\frac{1}{\pi}\sum_{-\infty}^{\infty}\frac{\sin\omega(z-n)}{(z-n)}\,F(n)\,e^{-\delta|n|}.$$

In the special case $\omega=\pi$ this states that the cardinal series

$$(15)\qquad \frac{\sin\pi z}{\pi}\sum_{-\infty}^{\infty}\frac{(-)^n}{z-n}\,F(n)$$

is summable by Abel's method to $F(z)$.

2. *Some derived interpolation series.*

If we impose the additional hypothesis that $|F(n)|$ is bounded, then it follows from Littlewood's Tauberian theorem that the series (14) and (15) are not only Abel summable but convergent. We confine ourselves to more elementary methods. From (14) we evidently have

$$(16)\quad \omega^2F(z)+F''(z)=\frac{2}{\pi}\sum_{-\infty}^{\infty}\left\{\frac{\sin\omega(z-n)}{(z-n)^3}-\frac{\omega\cos\omega(z-n)}{(z-n)^2}\right\}F(n),$$

provided that the series on the right is convergent, so that Abel summation may be replaced by convergence. If we replace ω by $2\pi-\omega$ and subtract, there results

$$(17)\quad 4\pi(\pi-\omega)\,F(z)=\frac{2}{\pi}\sum_{-\infty}^{\infty}\left\{\frac{\sin(2\pi-\omega)(z-n)-\sin\omega(z-n)}{(z-n)^3}\right.$$
$$\left.-\frac{(2\pi-\omega)\cos(2\pi-\omega)(z-n)-\omega\cos\omega(z-n)}{(z-n)^2}\right\}F(n).$$

In the more familiar case where ω is taken to be π we may write

$$(18)\qquad F(z)=\lim_{\delta\to 0}\frac{\sin\pi z}{\pi}\sum_{-\infty}^{\infty}\frac{(-)^n F(n)}{z-n}\,e^{-|n|\delta};$$

---

† It is obvious that $\overline{\lim}\,\sqrt[n]{|F(\pm n)|}<1$ can only arise if $F(z)\equiv 0$.

and, so long as the right-hand series is convergent, we have the set of interpolation formulae

$$F(z)-\frac{\sin \pi z}{\pi} F'(0)=\frac{\sin \pi z}{\pi}\left\{\frac{F(0)}{z}+\sum_{-\infty}^{\infty}{}' \frac{(-)^n z F(n)}{n(z-n)}\right\}, \tag{19}$$

$$F(z)-\frac{\sin \pi z}{\pi} F'(0)-\frac{z \sin \pi z}{2\pi} F''(0) \tag{20}$$

$$=\frac{\sin \pi z}{\pi}\left\{\left(\frac{1}{z}+\frac{\pi^2 z}{3\,!}\right) F(0)+\sum_{-\infty}^{\infty}{}' \frac{(-)^n z^2 F(n)}{n^2(z-n)}\right\}.$$

The first of these is commonly known as Tschakaloff's series but the whole family had been discussed before by Valiron under the name of Lagrange's series†.

3. *The theorems of Miss Cartwright and some generalizations.*

The theorems of Miss Cartwright quoted in the introduction are almost immediate consequences of the formula (17). On the one hand the factor multiplying $F(n)$ attains its maximum absolute value

$$2\pi^2(\pi-\omega)+\frac{2(\pi-\omega)^3}{3}$$

when $z=n$, and in any case is less than

$$\frac{2}{|z-n|^3}+\frac{2\pi}{|z-n|^2}$$

for real z. Since we may take ω arbitrarily near k we have

THEOREM 2. *If $F(z)$ is an integral function satisfying* (1) *and if $|F(n)|<A$, $n=0, \pm 1, \pm 2, \ldots$, then, for all real z,*

$$|F(z)| \leqslant \frac{1}{\pi^2(\pi-k)}\left\{\frac{2(\pi-k)^3}{3}+2\pi^2(\pi-k)+\sum_1^{\infty} \frac{2}{n^3}+\sum_1^{\infty} \frac{2\pi}{n^2}\right\} A \tag{21}$$

$$=K(k) A.$$

This is the first of the theorems, with an explicit value for $K(k)$. After our remarks concerning the formula (17) it is scarcely necessary to say anything concerning the proof of

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† G. Valiron, *Bull. Sc. Math.* (2), 49 (1925), 181–192, and 203–224. L. Tschakaloff, *Jahresb. der D.M.V.*, 43 (1934), 12.

THEOREM 3. *If in addition to satisfying the hypotheses of Theorem 2 $F(z)$ tends to zero as z tends to infinity by positive integer values, then $F(z)$ also tends to zero as z tends to infinity by continuous positive values.*

This is the second theorem quoted.

The example of $\sin \pi z$ shows that in Theorem 2, condition (1) cannot be relaxed to include $k = \pi$, but it is conceivable that a similar result may hold with $|F(x)| < K(M, A)$, and in any case some curious if rather trivial results can be obtained. If $F(z)$ satisfies (1) with $k = \pi$, and $|F(n)| < A$, $n = 0, \pm 1, \pm 2, \ldots$, then (19) is still valid† and

$$F'(z) = -\frac{\sin \pi z}{\pi} \sum_{-\infty}^{\infty} \frac{(-)^n F(n)}{(z-n)^2} + \pi \cot \pi z \, F(z). \tag{22}$$

From this it follows that, if p is an integer,

$$\begin{aligned} |F'(p+\tfrac{1}{2})| &\leqslant \frac{2}{\pi} \left\{ \frac{1}{(\frac{1}{2})^2} + \frac{1}{(\frac{3}{2})^2} + \ldots \right\} A \\ &= \pi A. \end{aligned} \tag{23}$$

Also

$$F''(z) + \pi^2 F(z) = 2 \sum_{-\infty}^{\infty} (-)^n F(n) \left(\frac{\sin \pi z}{\pi (z-n)^3} - \frac{\cos \pi z}{(z-n)^2} \right) \tag{24}$$

so that

$$|F''(z) + \pi^2 F(z)| \leqslant 2A \left\{ 2\pi^2 + \frac{2}{\pi} \left(\frac{1}{1^3} + \frac{1}{2^3} + \ldots \right) + 2 \left(\frac{1}{1^2} + \frac{1}{2^2} + \ldots \right) \right\}, \tag{25}$$

for all real z.

It follows also that

$$F^{(n+2)}(z) + \pi^2 F^{(n)}(z) \tag{26}$$

is bounded for all real z, so that any even derivative of $F(z)$ is bounded at the integers, while any odd derivative is bounded at the halves of the odd integers. This result just fails to bring the problem into J. M. Whittaker's theory of two point expansions‡.

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† G. Valiron, *loc. cit.*, 206.

‡ J. M. Whittaker, *Proc. London Math. Soc.* (2), 36 (1933), 451–469.

It is also immediately evident from (19) that, for real z,

$$|F(z)| \leqslant |F'(0)| + A\left\{1 + 2\left(1 + \tfrac{1}{2} + \ldots + \frac{1}{p}\right)|z| + 4|z| \sum_{p+1}^{\infty} \frac{1}{p^2}\right\} \tag{27}$$

$$(p = [2|z|] + 1),$$

and, since $F(z+m)$ satisfies the same conditions as $F(z)$ when m is an integer, we have, for example,

THEOREM 4. *If $F(z)$ satisfies (1) with $k = \pi$, and if $|F(n)| \leqslant A$ for all integers n and $|F'(q)| \leqslant B$ for some integer q in every interval $(x, x+C)$, then*

$$|F(z)| \leqslant K(A, B, C) \tag{28}$$

for all real z.

PART II.

4. The method of the Borel-Laplace transform used in Part I is nearly equivalent in the information it gives about integral functions to a special case of the direct argument used by Phragmén and Lindelöf[†]. It has the advantage of giving a clear geometrical interpretation of the issues involved, and the question of generalizing it to deal with all integral functions has been considered for some time. Neglecting the matter of proximate orders, we should at least be able to find the appropriate modification of (3) and (4) to investigate the properties of

$$h(\theta) = \varlimsup_{r \to \infty} \frac{1}{r^{\rho}} \log |F(re^{i\theta})| \tag{29}$$

for an integral function satisfying

$$|F(z)| < M \exp(k|z|^{\rho}). \tag{30}$$

Alternatively it would be sufficient to generalize the Laplace transform to the case of a function regular in an angle $|\arg z| < \alpha$ and satisfying

$$|F(z)| < Me^{k|z|}. \tag{31}$$

Of course all the necessary transform theory is already available from the general theory of transforms, but there remains the question of selecting a form with certain function theory properties. The transform to replace

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[†] E. Phragmén and E. Lindelöf, *Acta Math.*, 31 (1908), 381–406.

(3) and (4) which is usually studied[†] is given by

$$F(z) = \sum_0^\infty a_n z^n, \tag{32}$$

$$f(z) = \sum_0^\infty a_n \Gamma(1+n\sigma) z^{-n-1} \quad (\sigma = 1/\rho). \tag{33}$$

Subbotin[‡] made use of the associated function

$$f(z) = \sum_0^\infty a_n \Gamma(\tau+n\sigma) z^{-n-1}. \tag{34}$$

These transformations do in fact deal with $h(\theta)$ as defined by (29) just as in Pólya's special case $\rho = 1$, but only so long as $h(\theta)$ is positive. In order to deal with negative values of $h(\theta)$ also it is only necessary to modify (34) into

$$f(z) = \sum_0^\infty a_n \Gamma(\sigma+n\sigma) z^{-n-1}. \tag{35}$$

This fact was recently discovered by Bernstein[§] in that he used it to construct integral functions with a prescribed $h(\theta)$. We propose here to exploit the properties of the associated function (35) in a manner parallel to that of Part I. It continues to be of interest when σ is complex[||] and yields us some rather involved properties of integral functions related to logarithmic spirals. It is not difficult to extend the theory further so as to refer to functions satisfying (31) in an angle[¶] and it is, in fact, this form with which we are mainly concerned. We have have only to divide the contour of the integral defining $F(z)$ in terms of the associated function to see that $F(z)$ can be approximated to by an integral function in such a way that the theorems of Part I apply immediately. Thus

THEOREM 6. *If* $F(z)$ *is regular for* $|\arg z| < \alpha < \frac{1}{2}\pi$, *satisfies* (31) *with* $k < \pi \sin \alpha$, *and is bounded for* $z = 1, 2, \ldots$, *then* $F(z)$ *is bounded for all positive* z.

And

THEOREM 7. *If, further,* $F(z)$ *tends to a limit as* z *tends to infinity by*

[†] G. Valiron, *Comptes rendus*, 194 (1932), 1305–1308 and 1552–1555. V. Bernstein, *ibid.*, 1887–1890.

[‡] Th. Subbotin, *Math. Annalen*, 104 (1931), 377–386.

[§] V. Bernstein, *Mem. Accad. Ital.*, 7 (1936), 131–189.

[||] Following a suggestion of Prof. Wright that a "complex order" might be of interest.

[¶] The substitution $z^{1/\rho}$ for z is to be borne in mind.

positive integer values then it tends to a limit as z tends to infinity by continuous positive values.

The first of these results was obtained by Miss Cartwright[†].

The choice of the proper form of these transformations is of interest also in the study of the analytic continuation of functions of the form $\Sigma\, F(n)\, z^n$, and our discussion gives for example an immediate extension of some theorems of Ostrowski[‡] as well as effecting a considerable simplification in their proofs. We are also able to add a few remarks concerning gap and similar theorems.

5. *A form of Laplace transformation for integral functions.*

If $F(z)$ is an integral function defined by the power series (32) and satisfying condition (30), then, formally, using Euler's formula

$$\Gamma(n\sigma+\sigma)=\int_0^\infty e^{-t}\, t^{n\sigma+\sigma-1}\, dt \tag{36}$$

in (35) and interchanging the order of integration and summation, we have

$$f(z)=\int_0^\infty e^{-t}\, t^{\sigma-1}\, F(t^\sigma/z)\, dt/z. \tag{37}$$

$f(z)$, which we define by this formula, is evidently regular for $|z|>k$ and has a zero at infinity. The relation between the power series of $F(z)$ and $f(z)$ is then seen to be given by (32) and (35) by considering that

$$\begin{aligned} f(z)={}&\sum_0^p a_n\, \Gamma(n\sigma+\sigma)\, z^{-n-1} \\ &+\int_0^\infty e^{-t}\, t^{p\sigma+\sigma-1}\left(\frac{z}{t^\sigma}\right)^p \left\{F\left(\frac{t^\sigma}{z}\right)-\sum_0^p a_n\, t^{n\sigma}\, z^{-n}\right\} \frac{dz}{z^{p+1}}. \end{aligned} \tag{38}$$

(37) takes a more useful form after the change of variable

$$z=re^{i\phi}, \quad t=(ru)^\rho, \quad t^\sigma=zue^{-i\phi},$$

which gives

$$f(re^{i\phi})=\int_0^\infty \exp\{-(ru)^\rho\}\, F(ue^{-i\phi})\, \rho e^{-i\phi}\, du. \tag{39}$$

[†] M. L. Cartwright, *Quart. J. of Math.* (Oxford), 7 (1936), 46–55, Theorem 10.

[‡] A. Ostrowski, *Math. Annalen*, 108 (1933), 718–756.

Now put $r = z^\sigma$, considering z to be positive at first, and we have

(40) $$f(z^\sigma e^{i\phi}) = \rho e^{-i\phi} \int_0^\infty \exp(-zt^\rho)\, F(te^{-i\phi})\, dt,$$

which is seen to define a function regular for

(41) $$R(z) > h(-\phi).$$

It is to be noted that the analytic continuation over this half plane starts from a radius and that $f(z^\sigma e^{i\phi})$ will be continuable past the origin if $h(-\phi)$ is negative. To obtain an integral defining $F(z)$ in terms of $f(z)$ we start from

(42) $$\frac{1}{\Gamma(n\sigma+\sigma)} = \frac{1}{2\pi i} \int_{-\infty}^{(0)} e^\zeta \zeta^{-n\sigma-\sigma}\, d\zeta$$

instead of (36) and we evidently have

(43) $$F(z) = \frac{1}{2\pi i} \int_{-\infty}^{(0)} e^\zeta f(\zeta^\sigma/z)\, d\zeta/z,$$

valid so long as the contour encloses the circle $|\zeta| < k|z|$. Now put $z = re^{i\phi}$ and $r^\rho \zeta$ in place of ζ; this gives

(44) $$F(re^{i\phi}) = \frac{r^{\rho-1} e^{-i\phi}}{2\pi i} \int_{-\infty}^{(0)} \exp(r^\rho \zeta) f(\zeta^\sigma e^{-i\phi})\, d\zeta.$$

Now, if $f(\zeta^\sigma e^{-i\phi})$ is regular in the half plane

(45) $$R(\zeta) > \lambda,$$

the path of integration may be deformed so that on it $R(\zeta) < \lambda + \epsilon$, where ϵ is any positive number, and $F(z)$ must satisfy

(46) $$\overline{\lim_{r\to\infty}} \frac{1}{r^\rho} \log|F(re^{i\phi})| \leqslant \lambda.$$

Comparing this result with the half plane defined by (41), we see that the largest half plane $R(z) > \lambda$ in which the function $f(z^\sigma e^{i\phi})$ is regular is the half plane $R(z) > h(-\phi)$. By the use of this transformation Pólya's theory of $h(\phi)$ is immediately extended from the particular case $\rho = 1$ with only trivial modifications†. We leave the developments made in Part I until we have obtained the theory of the more general transform appropriate to functions regular in an angle.

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† G. Pólya, *Math. Zeitschrift*, 29 (1929), 549–640 (571–597).

Subbotin's work was based on the function[†]

$$\psi(z, \phi) = \sum_{1}^{\infty} F(n^{\sigma} e^{i\phi})\, n^{\tau-1} z^{n-1}. \tag{47}$$

He did not discuss completely the relation between $\psi(z, \phi)$, $f(z)$ and $h(\phi)$, but it should be sufficient for us to remark that by taking $\tau = \sigma$ the matter is simplified considerably. We have

$$\psi(z, \phi) = \frac{1}{2\pi i} \int_{-\infty}^{(0)} f(\zeta^{\sigma} e^{-i\phi}) \frac{e^{\zeta}\, d\zeta}{1 - ze^{\zeta}}, \tag{48}$$

and evidently, just as in Part I for the case $\sigma = 1$, when $k < \pi$,

$$\overline{\lim_{r\to\infty}} \frac{1}{r} \log |F(r^{\sigma} e^{i\phi})| = \overline{\lim_{n\to\infty}} \frac{1}{n} \log |F(n^{\sigma} e^{i\phi})| \quad (n = 1, 2, \ldots) \tag{49}$$

together with generalizations like those given by Pólya[‡] in the special case.

6. *The transformations for functions defined in an angle, and their applications.*

Here we suppose that $F(z)$ is regular in an angle $|\arg z| \leqslant \alpha \leqslant \frac{1}{2}\pi$ and that

$$|F(z)| \leqslant Me^{k|z|}. \tag{50}$$

Define the Laplace transform

$$f(z) = \int_{0}^{\infty} e^{-zt} F(t)\, dt, \tag{51}$$

which is evidently regular provided that $R(z) > k$. $f(z)$ may be continued analytically by rotating the line of integration, or by the changes of variable used before, and we have the representation

$$f(z) = e^{-i\phi} \int_{0}^{\infty} \exp(-ze^{-i\phi} t)\, F(te^{-i\phi})\, dt, \tag{52}$$

which defines $f(z)$ in the half plane

$$R(ze^{-i\phi}) > h(-\phi), \tag{53}$$

where

$$h(\phi) = \overline{\lim_{r\to\infty}} \frac{1}{r} \log |F(re^{i\phi})|. \tag{54}$$

[†] Th. Subbotin, *loc. cit.*

[‡] G. Pólya, *loc. cit.*, 606.

$f(z)$ is thus defined on the right of the contour consisting of the arc of the circle $|z| = k$ for which $|\arg z| < \alpha$ together with the tangents at the ends of this arc. If $h(\phi)$ is ever less than k the area in which $f(z)$ is defined is correspondingly increased. Evidently $f(z)$ is bounded provided that z is more than a fixed positive distance from this boundary, and is $O(1/|z|)$ uniformly as z tends to infinity with $|\arg z| < \alpha + \frac{1}{2}\pi - \delta$.

Now define

$$F^*(z) = \frac{1}{2\pi i} \int_\Gamma e^{z\zeta} f(\zeta)\, d\zeta, \tag{55}$$

where the contour is parallel to the boundary of the region in which $f(z)$ is defined, and a small distance inside. It is evident that $F^*(z)$ is regular in the same angle as $F(z)$ and satisfies a similar inequality. $F^*(z)$ can be determined in terms of $F(z)$ if we replace $f(z)$ by one of its definitions. It is clear that different forms will have to be taken on different parts of the contour; to reduce the number of them as much as possible, we first deform the contour into a pair of lines meeting in a point B of the real axis with B sufficiently large, and making angles $\pm(\frac{1}{2}\pi + \beta)$ with the real axis. If we take $\beta < \alpha$, and $B > k \sec \alpha$, the contour defines $F^*(z)$ in the angle $|\arg z| < \beta$, but, restricting z to be real to simplify the discussion, we write

$$F^*(x) = \frac{1}{2\pi i} \int_{\Gamma_1} e^{x\zeta} f(\zeta)\, d\zeta + \frac{1}{2\pi i} \int_{\Gamma_2} e^{x\zeta} f(\zeta)\, d\zeta, \tag{56}$$

where Γ_1 is the lower arm and Γ_2 is the upper arm of the contour. On Γ_1, since $R(\zeta e^{i\beta}) > k$, we may put

$$f(\zeta) = \int_0^\infty e^{-z\zeta} F(z)\, dz \tag{57}$$

with the line of integration $\arg z = \beta$, while on Γ_2 we may use the same integral with the line of integration $\arg z = -\beta$. This gives

$$F^*(x) = \frac{1}{2\pi i} \int_{\Gamma_1} \int_0^\infty e^{(x-z)\zeta} F(z)\, dz\, d\zeta + \frac{1}{2\pi i} \int_{\Gamma_2} \int_0^\infty e^{(x-z)\zeta} F(z)\, dz\, d\zeta, \tag{58}$$

and we must remember that the integration from 0 to infinity in the two integrals is along different radii. It is easily verified that the order of integration can be interchanged, and that the integration with respect to ζ can be effected, which gives

$$F^*(x) = -\frac{1}{2\pi i} \int_0^\infty \frac{e^{B(x-z)}}{x-z} F(z)\, dz + \frac{1}{2\pi i} \int_0^\infty \frac{e^{B(x-z)}}{x-z} F(z)\, dz. \tag{59}$$

It is to be remarked that, in the first integral the path of integration is along the arg $z=\beta$, while, in the second, it is along arg $z=-\beta$, so that the two integrals combine to give $F(x)$. The integral (55) is thus the solution of the integral equation (51), under the stated conditions. The converse is even simpler to prove, since we do not need the device of dividing the contour into two parts. Assuming $F(z)$ to be defined by the integral (55), where $f(z)$ satisfies the conditions stated, in the integral (51) we can substitute the definition (55) for $F(z)$, interchange the order of integration, and perform one of the integrations. There is, of course, no complete information about $F(z)$ on the boundary of the angle. If we take x to be real in the first place, the formal process (easily justifiable) is

$$(60)\qquad \int_\delta^\infty e^{-xt}F(t)\,dt=\frac{1}{2\pi i}\int_\delta^\infty\int_\Gamma e^{(\zeta-x)t}f(\zeta)\,d\zeta\,dt$$
$$=\frac{1}{2\pi i}\int_\Gamma\frac{e^{(\zeta-x)\delta}}{\zeta-x}f(\zeta)\,d\zeta.$$

From this it appears that a sufficient condition to ensure that

$$(61)\qquad \lim_{\delta\to 0}\int_\delta^\infty e^{-xt}F(t)\,dt$$

exists and is equal to $f(x)$ is that, uniformly on and on the right of Γ,

$$(62)\qquad |f(z)|=O(1/|z|^\delta)\quad(\delta>0).$$

We thus have

THEOREM 5. *If $F(z)$ is regular in the angle $|\arg z|\leqslant\alpha\leqslant\frac{1}{2}\pi$ and satisfies the condition*

$$(63)\qquad |F(re^{i\phi})|\leqslant Me^{kr},$$

then its Laplace transform

$$(64)\qquad f(z)=\int_0^\infty e^{-zt}F(t)\,dt$$

is regular in the region to the right of the arc of the circle $|z|=k$ with $|\arg z|\leqslant\alpha$ and the tangents to it at the extremities. $F(z)$ is defined by the inverse formula

$$(65)\qquad F(z)=\frac{1}{2\pi i}\int_\Gamma e^{z\zeta}f(\zeta)\,d\zeta\quad(|\arg z|<\alpha),$$

where Γ is a contour inside the region and parallel to the boundary. Conversely, if $f(z)$ is regular in the region described and satisfies

$$(66)\qquad |f(z)|=O(1/|z|^\delta)\quad(\delta>0),$$

then the function defined by (65) *is regular in the angle* $|\arg z| < a$ *and* $f(z)$ *is defined by* (64).

Since the preparation of this paper several convenient accounts of this transformation depending on real variable methods have been published, but in view of its importance in complex analysis a proof depending on Cauchy's theorem seems desirable†.

Now define

$$\psi(z) = \sum_{1}^{\infty} F(n)\, e^{-nz}, \tag{67}$$

the series being evidently convergent for $R(z) > h(0)$. If the real part of z is greater than that of any point of the contour Γ, we may express $F(n)$ by (55), sum under the integral sign, and obtain

$$\psi(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)\, e^{\zeta-z}}{1-e^{\zeta-z}}\, d\zeta. \tag{68}$$

If now z enters the contour without encountering any singularity of $f(z)$ and without any other value of $z+2n\pi i$ crossing the contour, this must be modified to

$$\psi(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)\, e^{\zeta-z}}{1-e^{\zeta-z}}\, d\zeta + f(z). \tag{69}$$

Thus, so long as no further value of $z+2n\pi i$ crosses the contour, $f(z)-\psi(z)$ is a regular function which we shall denote by $\phi(z)$. In order to obtain any precise results about the region of regularity of $\phi(z)$ we must have some definite information about the possible form of the contour Γ. We make the additional assumption

$$h(a) < \pi \sin a, \quad h(-a) < \pi \sin a. \tag{70}$$

The contour Γ can then be distorted into an angle just enclosing that excluded from the two lines

$$R(ze^{ia}) > h(a), \quad R(ze^{-ia}) > h(-a), \tag{71}$$

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† E. C. Titchmarsh, *Introduction to the theory of Fourier integrals* (Oxford, 1937) 6, 48. V. Bernstein, *Leçons sur . . . les séries de Dirichlet* (Paris, 1933), 294–311. The work of the present section extends immediately to the case of proximate orders in view of their connection with the conformal representation of an angle on a similar region (Valiron, *Integral functions*, p. 127). The second part of Theorem 5 is of interest in connection with some work of Ostrowski (*loc. cit.*) where it can replace the complicated considerations of 725–735 relating to the inversion of the order of repeated Stieltjes integrals, with a gain in generality.

and having an intercept less than 2π on the imaginary axis. If $\alpha = \frac{1}{2}\pi$ this statement needs a slight modification. The contour can consist of two half lines parallel to the real axis and distant less than 2π apart together with a portion of the line $R(z) = h(0)+\epsilon$. In either case $\phi(z)$ is regular inside the contour to the right of the imaginary axis and on the segment of the axis as well. If we now introduce the condition that $\psi(z)$ is regular for $R(z) > 0$, then a little consideration shows that $f(z)$ is also regular to the right of the imaginary axis and that $\phi(z)$ is also regular on the imaginary axis provided that

$$2\pi - h(\alpha)\operatorname{cosec}\alpha > I(z) > -2\pi + h(-\alpha)\operatorname{cosec}\alpha. \tag{72}$$

We are now in a position to obtain interpolation formulae similar to those of Part I, with the difference that in the present case they are only approximate. In the definition of $F(z)$ by (55) we choose the contour Γ to pass through the points $-\pi i$ and πi. On the part Γ_1 on the right of the imaginary axis we replace $f(z)$ by $\phi(z)+\psi(z)$ and then distort the contour of the integral containing $\phi(z)$ until it lies on the left of the imaginary axis except for its ends. We denote this curve by Γ_3 and use Γ_2 for the part of Γ on the left of the imaginary axis. $F(z)$ is thus represented as the sum of the three functions

$$\left\{\begin{aligned} &F_1(z) = \frac{1}{2\pi i}\int_{\Gamma_1} e^{z\zeta}\psi(\zeta)\,d\zeta, \quad F_2(z) = \frac{1}{2\pi i}\int_{\Gamma_2} e^{z\zeta} f(\zeta)\,d\zeta, \\ &\qquad\qquad F_3(z) = \frac{1}{2\pi i}\int_{\Gamma_3} e^{z\zeta}\phi(\zeta)\,d\zeta. \end{aligned}\right. \tag{73}$$

$F_1(z)$ and $F_3(z)$ are integral functions and $F_2(z)$ and $F_3(z)$ each satisfy

$$F_p(z) = O\left(\frac{1}{r}\, e^{\pi r|\sin\phi|}\right) \quad (z = re^{i\phi}, \quad p = 2,\, 3) \tag{74}$$

uniformly for $|\arg z| \leqslant \beta < \alpha$. In particular, the difference between $F(z)$ and $F_1(z)$ tends to zero as z tends to infinity by positive values†. The dominant element $F_1(z)$ can be treated as in Part I, because

$$F_1(z) = \lim_{\delta\to 0} \frac{1}{2\pi i}\int_{-\pi i+\delta}^{\pi i+\delta} \psi(\zeta)\, e^{z\zeta}\, d\zeta, \tag{75}$$

and the series for $\psi(z)$ can be integrated term by term, giving

$$F_1(z) = \lim_{\delta\to 0} \frac{\sin\pi z}{\pi} \sum_1^\infty \frac{(-)^n F(n)\, e^{-n\delta}}{z-n}. \tag{76}$$

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† M. L. Cartwright, *loc. cit.*, Theorem 6.

This series shows that $F_1(z) = F(z)$ for $z = 1, 2, \dots$ and defines an integral function of order 1 satisfying (30) with $k = \pi$ and $\rho = 1$. We cannot immediately apply Theorems 2 and 3 since for them $k < \pi$ is required, but a simple artifice suffices. The contour for the integral defining $F_1(z)$ can be deformed to include at each end a short segment of the imaginary axis. The contribution of these two parts is an integral function which tends to zero along the real axis, while the remainder is an integral function satisfying (1) to which Theorems 2 and 3 apply. We thus obtain

THEOREM 6. *If* $F(z)$ *is regular in an angle* $|\arg z| \leqslant \alpha \leqslant \frac{1}{2}\pi$, *satisfies* (50) *for all* z *in the angle and the same inequality with* $k < \pi \sin \alpha$ *for* $\arg z = \pm\alpha$, *then if* $F(n)$ $(n = 1, 2, \dots)$ *is bounded* $F(z)$ *is bounded for all positive* z†, and

THEOREM 7. *If in addition to satisfying the hypotheses of Theorem* 6 $F(z)$ *tends to zero as* z *tends to infinity by positive integer values, then* $F(z)$ *also tends to zero as* z *tends to infinity by continuous positive values.*

We may mention as generalizations obtainable by trivial refinements that these results still hold when $F(z)$ is defined in the angle

$$-\beta < \arg z < \alpha \quad (\alpha < \tfrac{1}{2}\pi, \;\; \beta < \tfrac{1}{2}\pi) \tag{77}$$

and when (70) is replaced by

$$h(\alpha) \operatorname{cosec} \alpha + h(-\beta) \operatorname{cosec} \beta < 2\pi. \tag{78}$$

7. *Transformations for integral functions using complex parameters.*

$F(z)$ in this section is again an integral function satisfying (30). We still define $f(z)$ by (35) but now take $\sigma = \rho^{-1} + i\tau$ to be complex, with $\tau \geqslant 0$ to facilitate certain geometrical statements. The formulae (36) and (37), which we restate as

$$f(z) = \int_0^\infty e^{-t} t^{\sigma-1} F(t^\sigma/z)\, dt/z \tag{79}$$

remain true. With our change in σ we must now state the condition of convergence of this integral as $|z| > k^{1/\rho}$. Put $ze^{-i\phi}$ for z, which gives

$$f(ze^{-i\phi}) = e^{i\phi} \int_0^\infty e^{-t} t^{\sigma-1} F(t^\sigma e^{i\phi}/z)\, dt/z. \tag{80}$$

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† This is Theorem 10 of M. L. Cartwright, *loc. cit.* It is clear that by considering the various integrals in more detail an explicit value could be obtained for a bound for $F(z)$ $(z > 0)$ depending on the bound for $F(n)$, α, and the constants in (50).

Now supposing $z^{1/\sigma}$ to be real, positive, and greater than k, so that z is uniquely defined as $\exp(\sigma \log z^{1/\sigma})$ and $|z| > k^{1/\rho}$, make the change of variable $t = z^{1/\sigma} v^\rho$, and the integral becomes

$$f(ze^{-i\phi}) = \rho e^{i\phi} \int_0^\infty \exp(-z^{1/\sigma} v^\rho)\, v^{\rho\sigma-1} F(v^{\rho\sigma} e^{i\phi})\, dv.$$

Now put $w = z^{1/\sigma}$ and obtain

$$f(w^\sigma e^{-i\phi}) = \rho e^{i\phi} \int_0^\infty \exp(-wv^\rho)\, v^{\rho\sigma-1} F(v^{\rho\sigma} e^{i\phi})\, dv. \tag{81}$$

This integral is, however, convergent for $R(w) > h(\phi)$, where

$$h(\phi) = \overline{\lim_{r\to\infty}} \frac{1}{r^\rho} \log |F(r^{\rho\sigma} e^{i\phi})|. \tag{82}$$

This new definition of $h(\phi)$ makes it dependent on the behaviour of $F(z)$ along a spiral.

The radius of convergence of the series for $f(z)$ is actually less than $k^{1/\rho}$, since†

$$\log \Gamma(n\sigma+\sigma) \cong (n\sigma+\sigma-\tfrac{1}{2}) \log(n\sigma+\sigma) - (n\sigma+\sigma) + \tfrac{1}{2}\log(2\pi),$$

$$\frac{1}{n} \log |\Gamma(n\sigma+\sigma)| \cong \frac{1}{\rho} \log |n\sigma| - \frac{1}{\rho} - r \arg \sigma,$$

$$\overline{\lim}\, \sqrt[n]{|a_n \Gamma(n\sigma+\sigma)|} = \left(\frac{ke\rho}{n}\right)^{1/\rho} (n|\sigma|)^{1/\rho} \exp\left(-\frac{1}{\rho} - \tau \arg \sigma\right)$$

$$= k^{1/\rho} |\rho\sigma|^{1/\rho} \exp(-\tau \tan^{-1} \tau\rho).$$

The radius of convergence is thus

$$a = k^{1/\rho} \exp\{\tfrac{1}{2}\rho^{-1} \log(1+\rho^2\tau^2) - \tau \tan^{-1} \rho\tau\} \tag{83}$$

and it is easily seen that the exponential factor is less than unity. We compare this result with the half plane connected with (81). With $z = w^\sigma = (re^{i\theta})^\sigma$, the circle $|z| = a$ corresponds to

$$\rho^{-1} \log r - \tau\theta = \log a, \quad i.e. \quad r = a^\rho e^{\tau\rho\theta} \tag{84}$$

---

† We have used the fact that, for an integral function satisfying (30),

$$\overline{\lim_{n\to\infty}}\, n^{1/\rho} |a_n|^{1/n} = (ke\rho)^{1/\rho},$$

a result due to E. Lindelöf, *Bull. Sc. Math.* (2), 27 (1903), 213–226. This result can easily be deduced from our paragraph 5.

(we have put $\phi = 0$), and with $|\theta| < \pi$ we see that $f(w^{\sigma})$ is regular if

$$r > a^{\rho} e^{\tau\rho\theta}. \tag{85}$$

The tangent at the point θ to this spiral is parallel to the imaginary axis if $\theta = \tan^{-1} \tau\rho$; and for this point

$$r \cos\theta = \frac{a^{\rho} e^{\tau\rho\theta}}{\sqrt{(1+\tau^2\rho^2)}} = k, \tag{86}$$

from the value (83) of a. Thus, independently of the integral (81), we see that $f(w^{\sigma})$ is regular for the half plane $R(w) > k$. Of course, some properties of the spiral defining $h(\phi)$ could be obtained from known results by considering $F(z^{\sigma})$, and, in particular, Miss Cartwright's theorems are of interest in this case, but the fact that $F(z)$ is an *integral* function imposes restrictions which cannot easily be obtained in this way. Since, for example, $f(z)$ has a singularity somewhere on its circle of convergence $|z| = a$, if we continue $f(w^{\sigma})$ analytically from the half plane $R(w) > k$, along the outside of the spiral

$$r = a^{\rho} e^{\tau\rho\theta} \quad (w = re^{i\theta}) \tag{87}$$

starting from the point $\theta = \tan^{-1} \rho\tau$, then we shall eventually reach a singularity of $f(w^{\sigma})$. If we choose the direction of decreasing θ, so that the spiral approaches the origin, the whole of this part of the curve lies on the right of a line easy to determine. For the tangent to the spiral is again parallel to the imaginary axis when $\theta = \tan^{-1} \rho\tau - \pi$, and thus by analytic continuation along the spiral we must encounter a singularity of $f(w^{\sigma})$ somewhere in the half plane

$$\begin{aligned} R(w) &> -a^{\rho} \exp(-\tau\rho\pi + \tau\rho \tan^{-1} \tau\rho)(1+\rho^2\tau^2)^{-\frac{1}{2}} \\ &= -k \exp(-\tau\rho\pi). \end{aligned} \tag{88}$$

Thus $f(w^{\sigma})$ cannot be regular in a larger half plane; and $h(0)$, and hence $h(\phi)$, for all ϕ, satisfies the inequality

$$h(\phi) \geqslant -k \exp(-\tau\rho\pi). \tag{89}$$

This generalizes the well-known result which arises when $\tau = 0$. If the order ρ is less than unity, the case $\tau = 0$ can be improved to $h(\phi) \geqslant \cos \pi\rho$. This result may also be improved for the spirally defined $h(\phi)$ although the exact statement is somewhat awkward. The improved result is obtained if we remark that the circumference $|z| = a$, taken once, corresponds to an arc of the spiral which subtends an angle $2\pi\rho/(1+\rho^2\tau^2)$ at the origin. If this angle is less than that subtended at the origin between the point $\theta=-\pi+\tan^{-1}\rho\tau$ and the first point on the spiral above the real axis with the same real part as this one, then by the process of analytic continuation along the spiral we shall reach a singularity of $f(w^\sigma)$ before the real part of w has reached the value $-k\exp(-\tau\rho\pi)$. If an arc of the spiral subtending the angle $2\pi\rho/(1+\rho^2\tau^2)$ at the origin then has its ends at points having the same real part x, then our result is $h(\phi)\geqslant x$. The analytical form of this geometrical statement is as follows.

If the root γ of the equation in θ

$$\sqrt{(1+\rho^2\tau^2)}\cos\theta=-\exp\rho\tau(\tan^{-1}\rho\tau-\theta-\pi) \tag{90}$$

which lies between $\frac{1}{2}\pi$ and π satisfies the condition

$$\omega=2\pi\rho/(1+\rho^2\tau^2)<\pi+\gamma-\tan^{-1}\rho\tau, \tag{91}$$

then the inequality (89) can be improved to

$$h(\phi)\geqslant k\cos\theta\sqrt{(1+\rho^2\tau^2)}\exp\rho\tau(\theta-\tan^{-1}\rho\tau), \tag{92}$$

where θ is the smallest positive root of

$$e^{\rho\tau\theta}\cos\theta=e^{\rho\tau(\theta-\omega)}\cos(\theta-\omega). \tag{93}$$

A particular case of some interest occurs when $1+\rho^2\tau^2>2\rho$, so that the relevant arc of the spiral subtends an angle less than π at the origin and our result is $h(\phi)>0$. This is very similar to Ahlfors' theorem on asymptotic paths†.

The use of a complex parameter σ is also of interest in connection with the developments of the next article.

8. *The connection with theorems on singularities of power series.*

The large number of theorems concerning the coefficients and singularities of power series of finite radius of convergence can all be interpreted, if sometimes awkwardly, as properties of integral functions. The subject has been discussed extensively by Pólya‡, many of whose theorems are automatically extended by the transformations of §§ 5, 6, and 7. We select a few examples.

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† L. Ahlfors, *Acta. Soc. Sci. Fenn.* (Nova Series), A.1. (1930), No. 9, §4.

‡ G. Polyá, *loc. cit.*, and *Annals of Math.* (2), 34 (1933), 731–777.

Suppose that the coefficients of the power series (9) of the integral function $F(z)$ almost all vanish, in the sense that the number $\lambda(n)$ out of the first n which do not vanish satisfies the condition $\lambda(n) = o(n)$. The same condition is evidently satisfied by the series (12) for $f(z)$, which accordingly, by Fabry's theorem[†], has singularities at every point of its circle of convergence. We deduce that $h(\phi)$ as defined by (29) or by (82) is a constant. In other words the integral function increases at the same rate along all radii and along all equiangular spirals.

Suppose that $F(n)/F(n+1)$ has a definite limit a as n tends to infinity, where $F(z)$ satisfies the conditions of § 6. From Fabry's theorem[‡] the function $\psi(z)$ defined by (70) has all the values of $\log a$ as singularities. The series is, of course, convergent for $R(z) > \log|a|$, and we see that $f(z)$ has a singularity at at least one of the values of $\log a$. The type k of the function $F(z)$ must be at least $|\log a|$. We deduce that, if $F(z)$ is an integral function, and $F(n^\sigma)/F\{(n+1)^\sigma\}$ tends to a limit different from unity, then $F(z)$ is of at least mean type and order $1/\sigma$.

If we had interchanged the application of these two theorems we should have had to all intents the starting points of the proofs given by Pólya[§].

A theorem of Szego[||] states that, if the coefficients of a power series include only a finite number of different values, then the power series is either not continuable, or is a meromorphic function of a special type. Restricting the application of this theorem to functions of order one, in order to obtain a simple result, we deduce the result that, if the values $F(n)$ for $n = 1, 2, \ldots$ of the integral function $F(z)$ only repeat a finite set, then $F(z)$ is a sum of exponential functions of the kind $\exp(2\pi\, kiz/m)$ or else it is of at least type π, order 1.

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[†] *E.g.* P. Dienes, *The Taylor series* (Oxford, 1931), 376.

[‡] P. Dienes, *loc. cit.*, 377.

[§] G. Pólya, *loc. cit.*, 615–627.

[||] P. Dienes, *loc. cit.*, 324.

THE n-TH PRIME IS GREATER THAN $n \log n$

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1. *Summary of results and outline of proof.*

Counting 2 as the first prime, denote the n-th prime by $p(n)$. For large n, $p(n) = n \log n + n \log\log n - n + O(n \log\log n / \log n)$, so that, for large n, $p(n) > n \log n$. In this paper it is shown that, for all positive n, $p(n) > n \log n$. As a consequence of this, one can prove that, if B_n, C_n, and D_n are defined for $n \geqslant 2$ by

$$\sum_{m=1}^{n} \frac{1}{p(m)} = \log\log n + B_n, \qquad \prod_{m=1}^{n} \left(1 - \frac{1}{p(m)}\right) = \frac{C_n}{\log n},$$

and

$$\prod_{m=2}^{n} \left(1 - \frac{2}{p(m)}\right) = \frac{D_n}{\log^2 n},$$

then $B_n > B_{n+1}$, $C_{n+1} > C_n$, and $D_{n+1} > D_n$. It is readily shown that all three sequences approach limits, and the numerical values of these limits are given to ten places of decimals.

These results will be used in the author's paper "An improvement of Brun's method in number theory".

For completeness, the following theorems are also proved: *If* $n > 3$, $p(n) < n \log n + 2n \log\log n$. *If* $n > 1$,

$$n \log n + n \log\log n - n - 9n < p(n) < n \log n + n \log\log n - n + 9n.$$

The general procedure of the proof that $p(n) > n \log n$ is as follows. By use of Lehmer's table of primes, it is shown that, for $n \leqslant 1071$, $p(n) > n \log n$. If $p(n) > n \log n$ for all $n \leqslant m$, then

$$\psi\big(p(m)\big) \geqslant \theta\big(p(m)\big) = \sum_{r=1}^{m} \log\big(p(r)\big) > \sum_{r=1}^{m} \log(r \log r).$$

We can get a lower bound for the latter expression by straightforward methods; in particular, we can show that, if $m > 1000$,

$$\psi\big(p(m)\big) > (m+1)\log(m+1) \times\left[1+\frac{\log\log m}{\log m}-\frac{1}{\log m}-\frac{1}{\log^2 m}-\frac{2}{\log^3 m}-\frac{2}{m}\right].$$

If $H(x)$ is never zero, and is so small that $x\big(1+H(x)^{-1}\big) \geqslant \psi(x)$, then

$$p(m)\left(1+\frac{1}{H\big(p(m)\big)}\right) \geqslant \psi\big(p(m)\big) > (m+1)\log(m+1) \times\left[1+\frac{\log\log m}{\log m}-\frac{1}{\log m}-\frac{1}{\log^2 m}-\frac{2}{\log^3 m}-\frac{2}{m}\right],$$

and so, if

$$\frac{1}{H\big(p(m)\big)} \leqslant \frac{\log\log m}{\log m}-\frac{1}{\log m}-\frac{1}{\log^2 m}-\frac{2}{\log^3 m}-\frac{2}{m},$$

then $p(m+1) > p(m) > (m+1)\log(m+1)$. Hence $p(n) > n\log n$ for all $n \leqslant m+1$. This gives the means of proving $p(n) > n\log n$ by induction, provided that $x\big(1+H(x)^{-1}\big) \geqslant \psi(x)$ can be proved for a sufficiently large $H(x)$. For most x, the standard theory of the approximation of $\psi(x)$ can be used by evaluating the various constants which appear. For the smaller values of x ($< e^{3000}$), a special method was devised, which gave a proof of $x\big(1+H(x)^{-1}\big) \geqslant \psi(x)$ for a somewhat larger $H(x)$ than could be used under the standard theory.

2. *Preliminary lemmas.*

The word "Ingham" refers to Ingham's "The distribution of prime numbers", *Cambridge Tract*, No. 30, the word "Landau" to the first volume of Landau's *Handbuch der Lehre von der Verteilung der Primzahlen*, and the word "Davis" to the first volume of H. T. Davis's *Tables of the higher mathematical functions*.

Throughout the paper s denotes a complex number, and σ and t denote its real and imaginary parts.

Lemma 1. *If* $t > 12$, *then* $\log\Gamma(s) = (s-\frac{1}{2})\log s - s + \frac{1}{2}\log(2\pi) + \theta/(3t)$, *where* $|\theta| < 1$.

Proof. For all s except on the negative real axis,

$$\log \Gamma(s) = (s-\tfrac{1}{2}) \log s - s + \tfrac{1}{2} \log (2\pi) + \int_0^\infty \frac{P_1(x)\,dx}{s+x},$$

where* $P_1(x) = [x] - x + \frac{1}{2}$.

$$\int_0^\infty \frac{P_1(x)\,dx}{s+x} = \sum_{n=0}^{\infty} \int_0^1 \frac{P_1(n+\theta)\,d\theta}{s+n+\theta}$$

$$= \sum_{n=0}^{\infty} \left[(\tfrac{1}{2}+s+n) \log \left(\frac{s+n+1}{s+n}\right) - 1\right].$$

If $t > 12$, $|s+n| > 12$, and $\log\big((s+n+1)/(s+n)\big)$ can be expanded into a power series in $(s+n)^{-1}$. Multiplying this by $\frac{1}{2}+s+n$ and arranging according to powers of $s+n$, we have

$$\int_0^\infty \frac{P_1(x)\,dx}{s+x}$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{12(s+n)^2} - \frac{1}{12(s+n)^3} + \frac{3}{40(s+n)^4} + \dots + \frac{(r-1)(-1)^r}{2r(r+1)(s+n)^r} + \dots\right].$$

Since $|s+n| > 12$,

$$\left|\int_0^\infty \frac{P_1(x)\,dx}{s+x}\right| < \sum_{n=0}^{\infty} \left|\frac{1}{11(s+n)^2}\right| < \sum_{-\infty}^{\infty} \left|\frac{1}{11(s+n)^2}\right|$$

$$< \frac{1}{11t^2} + \frac{1}{11} \int_{-\infty}^{\infty} \frac{dx}{t^2+x^2} < \frac{1}{3t}.$$

LEMMA 2. *If* $0 < \sigma \leqslant 2$, *and* $1400 \leqslant t$, *then*

$$|\Gamma(s)| < (1{\cdot}0003)\, t^{\sigma-\frac{1}{2}} (2\pi)^{\frac{1}{2}} e^{-\frac{1}{2}\pi t}.$$

Proof. Take $0 < \sigma \leqslant 2$ and $1400 \leqslant t$.

$$\log |\Gamma(s)| = \mathfrak{R}\big(\log \Gamma(s)\big)$$

$$< (\sigma - \tfrac{1}{2}) \log (\sigma^2+t^2)^{\frac{1}{2}} - t \operatorname{arc\,tan} \frac{t}{\sigma} - \sigma + \tfrac{1}{2} \log (2\pi) + 1/(3t).$$

However,

$$\operatorname{arc\,tan} x = \tfrac{1}{2}\pi - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \dots$$

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* Bieberbach, *Lehrbuch der Funktionentheorie*, 1 (1921), 308 and 298.

for $x > 1$, and $\log(\sigma^2+t^2)^{\frac{1}{2}} < \log t + \sigma^2/(2t^2)$. So

$$\log|\Gamma(s)| < (\sigma-\tfrac{1}{2})\log t - \tfrac{1}{2}\pi t + \tfrac{1}{2}\log(2\pi) + 1/(3t) + 4/t^2.$$

The lemma follows.

LEMMA 3. *If* $0 \leqslant \sigma \leqslant 2$, *and* $700 \leqslant t$, *then*

$$\mathfrak{R}\left(\frac{\Gamma'(s)}{\Gamma(s)}\right) < \log t + \frac{4}{t^2}.$$

Proof. Take $0 \leqslant \sigma \leqslant 2$ and $700 \leqslant t$. Put

$$f(s) = \log\Gamma(s) - (s-\tfrac{1}{2})\log s + s - \tfrac{1}{2}\log(2\pi).$$

Then
$$f'(s) = \frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{(w-s)^2}.$$

Take C to be a circle with centre at s and radius $\frac{1}{3}t$. Then, by Lemma 1, $|f(w)| < 1/(2t)$ on C. Also $|w-s| = \frac{1}{3}t$ on C. Hence $|f'(s)| < 3/(2t^2)$. Hence

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + \frac{3\theta}{2t^2},$$

where $|\theta| < 1$. Hence

$$\mathfrak{R}\left(\frac{\Gamma'(s)}{\Gamma(s)}\right) < \log(\sigma^2+t^2)^{\frac{1}{2}} - \frac{\sigma}{2\sigma^2+2t^2} + \frac{3}{2t^2}.$$

The lemma follows because $\log(\sigma^2+t^2)^{\frac{1}{2}} < \log t + \sigma^2/(2t^2)$.

LEMMA 4. *If* $\frac{1}{13} \leqslant \sigma$, *and* $1400 \leqslant t$, *then* $|\zeta(s)| \leqslant 7t^{\frac{1}{2}}$.

Proof. Take $\frac{1}{13} \leqslant \sigma$ and $1400 \leqslant t$. By Ingham, formula (7), p. 27,

$$|\zeta(s)| < \sum_{n \leqslant X} \frac{1}{n^\sigma} + \frac{1}{tX^{\sigma-1}} + \frac{1}{X^\sigma} + |s|\int_X^\infty \frac{dx}{x^{\sigma+1}}.$$

Put $X = t$. Then

$$|\zeta(s)| < \sum_{n \leqslant t} \frac{1}{n^\sigma} + \frac{2}{t^\sigma} + \frac{t+\sigma}{\sigma t^\sigma}.$$

For $0 < \sigma$, the right-hand side of this inequality is a decreasing function of σ, and so

$$|\zeta(s)| < \sum_{n \leqslant t} \frac{1}{n^{\frac{1}{2}}} + \frac{2}{t^{\frac{1}{2}}} + \frac{t+\frac{1}{2}}{\frac{1}{2}t^{\frac{1}{2}}}$$

if $\frac{1}{2} \leqslant \sigma$. Now, for $0 < \sigma < 1$,

$$\sum_{n \leqslant t} \frac{1}{n^\sigma} + \frac{2}{t^\sigma} + \frac{t+\sigma}{\sigma t^\sigma} < \int_0^t \frac{dx}{x^\sigma} + \frac{3}{t^\sigma} + \frac{t^{1-\sigma}}{\sigma} \leqslant t^{1-\sigma}\left(\frac{1}{1-\sigma} + \frac{1}{\sigma} + \frac{1}{400}\right).$$

Hence, for $\frac{1}{2} \leqslant \sigma$,

$$|\zeta(s)| < (4\tfrac{1}{400})\, t^{\frac{1}{2}} < 7t^{\frac{1}{2}}.$$

Now, by Landau, p. 285,

$$\zeta(1-s) = \frac{2}{(2\pi)^s}\, \Gamma(s) \cos \tfrac{1}{2}\pi s\, \zeta(s).$$

Now $|1/(2\pi)^s| = 1/(2\pi)^\sigma$. Also, since $1400 \leqslant t$,

$$|\cos \tfrac{1}{2}\pi s| < \tfrac{1}{2}(1{\cdot}00001)\, e^{\frac{1}{2}\pi t}.$$

So, for $\frac{1}{2} \leqslant \sigma < 1$, we have, by Lemma 2,

$$|\zeta(1-s)| < (1{\cdot}0004) \left(\frac{t}{2\pi}\right)^{\sigma-\frac{1}{2}} |\zeta(s)|.$$

However, it has just been shown that, for $\frac{1}{2} \leqslant \sigma < 1$,

$$|\zeta(s)| < t^{1-\sigma} \left(\frac{1}{1-\sigma} + \frac{1}{\sigma} + \frac{1}{400}\right).$$

Hence $$|\zeta(1-s)| < (1{\cdot}0004)\, \frac{(2\pi t)^{\frac{1}{2}}}{(2\pi)^\sigma} \left(\frac{1}{1-\sigma} + \frac{1}{\sigma} + \frac{1}{400}\right).$$

So, for $\frac{1}{2} \leqslant \sigma \leqslant \frac{12}{13}$, $|\zeta(1-s)| \leqslant 7t^{\frac{1}{2}}$. This, together with the inequality for $|\zeta(s)|$ when $\frac{1}{2} \leqslant \sigma$, proves the lemma.

Whenever ρ, β, and γ are used henceforth, it is to be understood that ρ is a complex zero of $\zeta(s)$ and β and γ are the real and imaginary parts of ρ. $N(T)$ denotes the number of ρ's for which $0 < \gamma \leqslant T$.

LEMMA 5. *If* $1450 \leqslant T$, *then*

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \tfrac{7}{8} + \theta[1{\cdot}12 \log T + 9{\cdot}5],$$

where $|\theta| \leqslant 1$.

Proof. If $1 < \sigma$, then

$$|\zeta(s)| = \prod_p |1 - 1/p^s|^{-1}$$

[Ingham, formula (12), p. 17]. However,

$$|1 - 1/p^s|^{-1} \geqslant (1 + 1/p^\sigma)^{-1} = \frac{(1-1/p^{2\sigma})^{-1}}{(1-1/p^\sigma)^{-1}},$$

Hence $|\zeta(s)| \geqslant \zeta(2\sigma)/\zeta(\sigma)$. However, $\zeta(1{\cdot}25) = 4{\cdot}595$ and $\zeta(2{\cdot}5) = 1{\cdot}341$*.

Hence
$$\frac{1}{|\zeta(\frac{5}{4}+it)|} \leqslant \frac{\zeta(1{\cdot}25)}{\zeta(2{\cdot}5)} = 3{\cdot}425.$$

Hence either
$$\frac{1}{\left|\mathfrak{R}\left(\zeta(\frac{5}{4}+it)\right)\right|} \leqslant 4{\cdot}844 \quad \text{or} \quad \frac{1}{\left|\mathfrak{I}\left(\zeta(\frac{5}{4}+it)\right)\right|} \leqslant 4{\cdot}844.$$

Let $g(s,\, T)$ be $\frac{1}{2}\left(\zeta(s+iT)+\zeta(s-iT)\right)$ if
$$\frac{1}{\left|\mathfrak{R}\left(\zeta(\frac{5}{4}+iT)\right)\right|} \leqslant 4{\cdot}844,$$
and let it be $\frac{1}{2}\left(\zeta(s+iT)-\zeta(s-iT)\right)$ otherwise. Then, for all T,
$$\frac{1}{|g(\frac{5}{4},\, T)|} \leqslant 4{\cdot}844.$$

On pp. 68–69 of Ingham, it is shown that
$$N(T) = \frac{1}{\pi}\left[\pi - \tfrac{1}{2}T\log\pi + \mathfrak{I}\left(\log\Gamma(\tfrac{1}{4}+\tfrac{1}{2}Ti)\right) + [\arg\zeta(s)]_L\right],$$

where by $[\arg\zeta(s)]_L$ is meant the increase in $\arg\zeta(s)$ as s traverses the pair of straight lines going from 2 to $2+Ti$ and from $2+Ti$ to $\frac{1}{2}+Ti$. By Lemma 1, if $1450 \leqslant T$, then
$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \tfrac{7}{8} + \frac{\theta}{3T} + \frac{1}{\pi}[\arg\zeta(s)]_L,$$

where $|\theta| \leqslant 1$. Now the change in $\arg\zeta(s)$ along the straight line from 2 to $2+Ti$ is less than $\frac{1}{2}\pi$, since $\arg\zeta(2) = 0$ and $\mathfrak{R}\left(\zeta(2+ti)\right) > \frac{1}{4}$. Hence, if m is the number of zeros of $g(s,\, T)$ on the line $\frac{1}{2} \leqslant s \leqslant 2$,
$$|[\arg\zeta(s)]_L| \leqslant (m+\tfrac{3}{2})\pi$$

(Ingham, p. 69). Now apply Theorem D (Ingham, p. 49) to $g(s,\, T)$ and the circles $|s-\frac{5}{4}| \leqslant \frac{6}{5}\frac{1}{2}$ and $|s-\frac{5}{4}| \leqslant \frac{3}{4}$. If $1450 \leqslant T$, then, by Lemma 4, $|g(s,\, T)| < 7(T+2)^{\frac{1}{2}}$ on the larger circle. Also
$$\frac{1}{|g(\frac{5}{4},\, T)|} \leqslant 4{\cdot}844.$$

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* J. P. Gram, "Tafeln für die Riemannsche Zetafunktion, herausgegeben von N. E. Nörlund", *D. Kgl. Danske Vidensk. Selsk. Skrifter, Kopenhagen* (8), 10 (1925–26), 313–325.

So, for $1450 \leqslant T$, $(\frac{61}{39})^r \leqslant 34T^{\frac{1}{2}}$, where r is the number of zeros of $g(s, T)$ in the circle $|s-\frac{5}{4}| \leqslant \frac{3}{4}$. Hence $|[\arg \zeta(s)]_L| \leqslant (1{\cdot}12 \log T + 9{\cdot}4)\pi$. The lemma follows.

Denote $$\sum_{1468 \leqslant \gamma \leqslant e^8} \frac{1}{\gamma^2} \text{ by } S_8,$$

and, if $r > 8$, $$\sum_{e^{r-1} \leqslant \gamma \leqslant e^r} \frac{1}{\gamma^2} \text{ by } S_r.$$

LEMMA 6. $$S_8 < 0{\cdot}000{,}332.$$

Proof. By Lemma 5, if $1450 \leqslant T \leqslant e^8$,

$$\begin{aligned} N(T) &\leqslant \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \tfrac{7}{8} + 1{\cdot}12 \log e^8 + 9{\cdot}5 \\ &< \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + 20. \end{aligned}$$

Put $$F(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}.$$

Then $F(1468) = 1040{\cdot}58$ and $F(e^8) = 2449{\cdot}09$. Since $F(T)$ is a concave function, $F(1468 + 1513n/1409) < 1041 + n$ for $0 \leqslant n \leqslant 1409$. Hence

$$N(1468 + 1513n/1409) < 1061 + n \quad \text{for} \quad 0 \leqslant n \leqslant 1409.$$

If we let γ_m stand for the m-th positive γ in order of magnitude and count duplications, it follows that $1468 + 1513n/1409 < \gamma_{n+1061}$ for $0 \leqslant n \leqslant 1409$. However, the first 1041 ρ's all have $\beta = \frac{1}{2}$ and $0 < \gamma \leqslant 1468$, and the 1042-nd ρ has $\gamma \geqslant 1468$, and $\gamma_{1040} < 1468$*. Hence

$$\begin{aligned} S_8 &= \sum_{1468 \leqslant \gamma \leqslant e^8} \frac{1}{\gamma^2} \\ &< \frac{21}{(1468)^2} + \sum_{n=1}^{1409} \frac{1}{(1468 + 1513n/1409)^2} \\ &< \frac{21}{(1468)^2} + \int_0^{1409} \frac{dx}{(1468 + 1513x/1409)^2} \\ &= \frac{21}{(1468)^2} + \frac{1409}{1513} \left[\frac{1}{1468} - \frac{1}{2981} \right] \\ &= 0{\cdot}000{,}331{,}72. \end{aligned}$$

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* Titchmarsh, "The zeros of the Riemann zeta-function", *Proc. Royal Soc. London* A) 157 (1936), 261–263.

LEMMA 7. *If* $r > 8$, $S_r < 16\cdot 6(r+9)e^{-2r} + 0\cdot 274(r-2\cdot 25)e^{-r}$.

Proof. $F(e^{r-1}) = \dfrac{e^{r-1}}{2\pi}[r-2-\log 2\pi]$ and $F(e^r) = \dfrac{e^r}{2\pi}[r-1-\log 2\pi]$.

So
$$F\left(e^{r-1} + \frac{2\pi n}{r-1-\log 2\pi + 1/(e-1)}\right) \leqslant F(e^{r-1}) + n,$$
for $0 \leqslant n \leqslant F(e^r) - F(e^{r-1})$. Hence
$$e^{r-1} + \frac{2\pi n}{r-1-\log 2\pi + 1/(e-1)} < \gamma_M,$$
with $M = n + F(e^{r-1}) + 11 + 1\cdot 12r$, for $0 \leqslant n \leqslant F(e^r) - F(e^{r-1})$. However,
$$\gamma_m < e^{r-1}$$
for $m \leqslant F(e^{r-1}) - 9 - 1\cdot 12(r-1)$. Hence
$$\sum_{e^{r-1} \leqslant \gamma \leqslant e^r} \frac{1}{\gamma^2} < \frac{19 + 2\cdot 24r}{(e^{r-1})^2} + \sum_{n=1}^{F(e^r)-F(e^{r-1})} \left(e^{r-1} + \frac{2\pi n}{r-1-\log 2\pi + 1/(e-1)}\right)^{-2}.$$
So
$$S_r < \frac{2\cdot 24(r+9)}{e^{2r-2}} + \frac{r-1-\log 2\pi + 1/(e-1)}{2\pi}\left[\frac{1}{e^{r-1}} - \frac{1}{e^r}\right]$$
$$< 16\cdot 6(r+9)e^{-2r} + 0\cdot 274(r-2\cdot 25)e^{-r}.$$

COROLLARY. $S_9 < 0\cdot 000{,}233$.

COROLLARY. $S_{10} < 0\cdot 000{,}097{,}1$.

COROLLARY. $S_{11} < 0\cdot 000{,}040{,}2$.

COROLLARY. $S_{12} < 0\cdot 000{,}016{,}5$.

Clearly the upper bound for S_{r+1} is less than half of the upper bound for S_r for $r > 8$, and so:

COROLLARY. $S_{13} + S_{14} + \ldots < 0\cdot 000{,}016{,}5$.

LEMMA 8. $$\sum_\rho \frac{1}{\gamma^2} < 0\cdot 0463.$$

Proof. $$-\frac{\zeta'(s)}{\zeta(s)} = -b + \frac{1}{s-1} + \frac{1}{2}\frac{\Gamma'(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}s+1)} - \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

Ingham, p. 58). However,
$$\lim_{s\to 1}\left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right) = C$$

(Landau, p. 165). Hence

$$\sum_\rho \left(\frac{1}{1-\rho}+\frac{1}{\rho}\right) = -b+C+\tfrac{1}{2}\frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})}.$$

Now $2^{2x}\Gamma(x)\Gamma(x+\frac{1}{2}) = 2\pi^{\frac{1}{2}}\Gamma(2x)$ (Davis, p. 180). Differentiating logarithmically, we get

$$2\log 2+\frac{\Gamma'(x)}{\Gamma(x)}+\frac{\Gamma'(x+\frac{1}{2})}{\Gamma(x+\frac{1}{2})} = 2\frac{\Gamma'(2x)}{\Gamma(2x)}.$$

Putting $x=1$ and using $\Gamma'(1)/\Gamma(1) = -C$ and $\Gamma'(2)/\Gamma(2) = 1-C$ (Davis, pp. 277–278), we get

$$\frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})} = 2-C-2\log 2.$$

Hence

$$\begin{aligned}\sum_\rho \left(\frac{1-\beta}{(1-\beta)^2+\gamma^2}+\frac{\beta}{\beta^2+\gamma^2}\right) &= -b+C+1-\tfrac{1}{2}C-\log 2\\ &= 2+C-\log(4\pi),\end{aligned}$$

since $b = \log(2\pi)-1-\frac{1}{2}C$ (Ingham, p. 58). However (Ingham, p. 48) the complex zeros of $\zeta(s)$ lie symmetrically about the lines $t=0$ and $\sigma=\frac{1}{2}$, and so for every zero $\rho=\beta+i\gamma$, there is a zero $\rho=(1-\beta)+i\gamma$. Hence

$$\begin{aligned}\sum_\rho \frac{\beta}{\beta^2+\gamma^2} &= \tfrac{1}{2}\sum_\rho \left(\frac{1-\beta}{(1-\beta)^2+\gamma^2}+\frac{\beta}{\beta^2+\gamma^2}\right)\\ &= 1+\tfrac{1}{2}C-\tfrac{1}{2}\log(4\pi)\\ &= 0{\cdot}023{,}095{,}709.\end{aligned}$$

Now, for $\gamma < 1468$, we have $\beta=\frac{1}{2}$, and the smallest value of γ is $14{\cdot}1$*. Hence

$$\frac{1}{\gamma^2} < 1{\cdot}0013\frac{1}{\beta^2+\gamma^2}$$

for all ρ. Hence

$$\sum_\rho \frac{\beta}{\gamma^2} < 1{\cdot}0013\sum_\rho \frac{\beta}{\beta^2+\gamma^2}.$$

When $\beta=\frac{1}{2}$,
$$\frac{1}{\gamma^2} = 2\frac{\beta}{\gamma^2}.$$

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* J. I. Hutchinson, "On the roots of the Riemann zeta-function", *Trans. American Math. Soc.*, 27 (1925), 49–60.

When $\beta \neq \frac{1}{2}$, $\beta+i\gamma$ and $(1-\beta)+i\gamma$ are both zeros and

$$\frac{1}{\gamma^2}+\frac{1}{\gamma^2}=2\left[\frac{\beta}{\gamma^2}+\frac{1-\beta}{\gamma^2}\right].$$

Hence we can conclude that

$$\sum_{\rho}\frac{1}{\gamma^2}=2\sum_{\rho}\frac{\beta}{\gamma^2}<2\cdot 0026\sum_{\rho}\frac{\beta}{\beta^2+\gamma^2}<0\cdot 0463.$$

LEMMA 9. *If* $1400 \leqslant \gamma$, *then*

$$\beta<1-\frac{1}{19\log\gamma}.$$

The proof is just like Landau's proof that, for large γ,

$$\beta<1-\frac{1}{18\cdot 52\ldots\log\gamma}$$

(Landau, pp. 318-324), except that certain constants are evaluated numerically.

Proof.

$$-\frac{\zeta'(s)}{\zeta(s)}=-0\cdot 549+\frac{1}{s-1}+\tfrac{1}{2}\frac{\Gamma'(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}s+1)}-\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)$$

(Ingham, p. 58). By Lemma 3, if $1\leqslant\sigma\leqslant 2$ and $1400\leqslant t$, then

$$-\mathfrak{R}\left(\frac{\zeta'(s)}{\zeta(s)}\right)<-0\cdot 549+\frac{1}{t^2}+\tfrac{1}{2}\log\tfrac{1}{2}t+\frac{8}{t^2}-\sum_{\rho}\frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2}-\sum_{\rho}\frac{\beta}{\beta^2+\gamma^2}.$$

With $1\leqslant\sigma$, $\sigma-\beta$ and β are always positive, so that we can deduce that

$$-\mathfrak{R}\left(\frac{\zeta'(s)}{\zeta(s)}\right)<-0\cdot 89+\tfrac{1}{2}\log t.$$

Also, if we choose a $\gamma\geqslant 1400$ and put $t=\gamma$, we get

$$\frac{1}{\sigma-\beta}<\mathfrak{R}\left(\frac{\zeta'(\sigma+i\gamma)}{\zeta(\sigma+i\gamma)}\right)-0\cdot 89+\tfrac{1}{2}\log\gamma.$$

For $\frac{3}{2}\leqslant\sigma\leqslant\frac{61}{40}$,

$$\mathfrak{R}\left(\frac{\Gamma'(\sigma)}{\Gamma(\sigma)}\right)=\frac{\Gamma'(\sigma)}{\Gamma(\sigma)}\leqslant\frac{\Gamma'(\frac{61}{40})}{\Gamma(\frac{61}{40})}=0\cdot 0596$$

(Davis, p. 324), so, for $1 < \sigma \leqslant 1{\cdot}05$,

$$-\mathfrak{R}\left(\frac{\zeta'(\sigma)}{\zeta(\sigma)}\right) < -0{\cdot}549 + \frac{1}{\sigma-1} + \tfrac{1}{2}\frac{\Gamma'(\frac{1}{2}\sigma+1)}{\Gamma(\frac{1}{2}\sigma+1)} \leqslant -0{\cdot}51 + \frac{1}{\sigma-1}.$$

So for $1400 \leqslant \gamma$ and $1 < \sigma \leqslant 1{\cdot}05$,

$$\begin{aligned}\frac{1}{\sigma-\beta} &< -0.89 + \tfrac{1}{2}\log\gamma + \tfrac{5}{8}\left(-0{\cdot}51 + \frac{1}{\sigma-1}\right) \\ &\qquad + \tfrac{1}{2}\left(-0{\cdot}89 + \tfrac{1}{2}\log(2\gamma)\right) + \tfrac{1}{8}\left(-0{\cdot}89 + \tfrac{1}{2}\log(3\gamma)\right) \\ &< \tfrac{13}{16}\log\gamma + \tfrac{5}{8}\frac{1}{\sigma-1}.\end{aligned}$$

Put $\sigma = 1 + 1/(5\log\gamma)$. Then $1 < \sigma < 1{\cdot}05$. Thus

$$\beta < 1 - \frac{1}{19\log\gamma}.$$

Take
$$\theta(x) = \sum_{p \leqslant x} \log p, \qquad \psi(x) = \sum_{p^m \leqslant x} \log p,$$

and
$$\psi_1(x) = \int_1^x \psi(u)\,du$$

(Ingham, p. 30).

LEMMA 10. *If*

$$K_1(x) = \sum_{r=8}^{\infty} S_r x^{-1/(19r)} + 0{\cdot}0463x^{-\frac{1}{2}} + 1{\cdot}9x^{-1} + 3x^{-2},$$

then, for $x > 1$,

$$\psi(x) < x[1 + 2K_1(x)^{\frac{1}{2}} + 2K_1(x) + 2K_1(x)\,K_1(x)^{\frac{1}{2}}].$$

Proof.

$$\psi_1(x) = \tfrac{1}{2}x^2 - \sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)} - x\frac{\zeta'(0)}{\zeta(0)} + \frac{\zeta'(-1)}{\zeta(-1)} - \sum_{r=1}^{\infty}\frac{x^{1-2r}}{2r(2r-1)}$$

(Ingham, p. 73). Now $\zeta'(0)/\zeta(0) = 1{\cdot}837{,}877$ and*

$$\zeta'(-1)/\zeta(-1) = 1{\cdot}985{,}054.$$

Also
$$\left|\sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)}\right| < \sum_{\rho}\frac{x^{\beta+1}}{\gamma^2}.$$

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[*] A. Walther, "Anschauliches zur Riemannschen Zetafunktion", *Acta Math.*, 48 (1926), 393–400.

Now $$\sum_{\beta\leqslant\frac{1}{2}} \frac{x^{\beta+1}}{\gamma^2} \leqslant x^{\frac{3}{2}} \sum_{\beta\leqslant\frac{1}{2}} \frac{1}{\gamma^2} \leqslant x^{\frac{3}{2}} \sum_{\rho} \frac{1}{\gamma^2} < 0{\cdot}0463 x^{\frac{3}{2}}$$

by Lemma 8. Also

$$\sum_{\beta>\frac{1}{2}} \frac{x^{\beta+1}}{\gamma^2} = 2x^2 \sum_{\substack{\beta>\frac{1}{2}\\ \gamma>0}} \frac{x^{\beta-1}}{\gamma^2} = 2x^2 \sum_{\substack{\beta>\frac{1}{2}\\ \gamma\geqslant 1468}} \frac{x^{\beta-1}}{\gamma^2},$$

since $\beta=\frac{1}{2}$ for $\gamma<1468$. Hence, by Lemma 9,

$$\sum_{\beta>\frac{1}{2}} \frac{x^{\beta+1}}{\gamma^2} < 2x^2 \sum_{\substack{\beta>\frac{1}{2}\\ \gamma\geqslant 1468}} \frac{x^{-1/(19\log\gamma)}}{\gamma^2}.$$

However, for every $\beta>\frac{1}{2}$, $(1-\beta)+i\gamma$ is also a zero. Thus

$$\sum_{\beta>\frac{1}{2}} \frac{x^{\beta+1}}{\gamma^2} < x^2 \sum_{\gamma\geqslant 1468} \frac{x^{-1/(19\log\gamma)}}{\gamma^2} < x^2 \sum_{r=8}^{\infty} S_r x^{-1/(19r)}.$$

So, for $x>1$, $\psi_1(x)=\frac{1}{2}x^2+\theta x^2 K_1(x)$, where $|\theta|<1$. Denote $2xK_1(x)^{\frac{1}{2}}$ by h. Since $\psi(x)$ is a non-decreasing function of x,

$$\psi(x) \leqslant \frac{1}{h}\int_x^{x+h} \psi(y)\,dy = \frac{\psi_1(x+h)}{h} - \frac{\psi_1(x)}{h}$$

$$< x+\tfrac{1}{2}h+\frac{(x+h)^2 K_1(x+h)}{h}+\frac{x^2 K_1(x)}{h}.$$

However, $K_1(y)$ is a decreasing function of y, and so the lemma results after replacing $K_1(x+h)$ by $K_1(x)$ and h by $2xK_1(x)^{\frac{1}{2}}$ in the above inequality.

For particular choices of $x>e^{50}$, it is possible to get smaller upper bounds for $\psi(x)$ than those given in Lemma 10, by giving the proper values to a in

LEMMA 11. *If* $x>e^{50}$, *and*

$$K_2(x) = 0{\cdot}000{,}000{,}2\, x^{-\frac{1}{152}} + \sum_{r=9}^{\infty} \frac{S_r}{e^{r-1}}\, x^{-1/(19r)},$$

and a *is a positive constant, then*

$$\psi(x) < x\left[1+2aK_2(x)^{\frac{1}{2}}+\frac{2}{a^2}+\frac{9K_2(x)^{\frac{1}{2}}}{a}+21K_2(x)+18aK_2(x)\,K_2(x)^{\frac{1}{2}}\right].$$

Proof. By arguing as in the proof of Lemma 10, we have

$$\sum_{\rho} \frac{x^{\beta-1}}{|\gamma|^3} < 0{\cdot}0463 x^{-\frac{1}{2}} + \sum_{1468\leqslant\gamma\leqslant e^8} \frac{1}{\gamma^3}\, x^{-\frac{1}{152}} + \sum_{r=9}^{\infty}\left[\sum_{e^{r-1}\leqslant\gamma\leqslant e^r} \frac{1}{\gamma^3}\right] x^{-1/(19r)}.$$

However,
$$\sum_{e^{r-1} \leqslant \gamma \leqslant e^r} \frac{1}{\gamma^3} < \frac{1}{e^{r-1}} S_r.$$

By arguing as in the proof of Lemma 6, we get
$$\sum_{1468 \leqslant \gamma \leqslant e^8} \frac{1}{\gamma^3} < 0{\cdot}000{,}000{,}18.$$

Hence, for $x > e^{50}$,
$$0.0463 x^{-\frac{1}{2}} + \sum_{1468 \leqslant \gamma \leqslant e^8} \frac{1}{\gamma^3} x^{-\frac{1}{152}} < 0{\cdot}000{,}000{,}2 x^{-\frac{1}{152}}.$$

Thus
$$\sum_\rho \frac{x^{\beta-1}}{|\gamma|^3} < K_2(x).$$

Denote
$$\sum_\rho \frac{x^{\beta-1}}{|\gamma|^3} \text{ by } l(x), \quad 2axl(x)^{\frac{1}{2}} \text{ by } h(x), \quad \text{and} \quad axl(x)^{\frac{1}{2}} \text{ by } k(x),$$
and write $h(x)$, $k(x)$, and $l(x)$ as h, k, and l when no confusion will be caused thereby. Then

$$\int_x^{x+k} \left(\sum_\rho \frac{y^{\rho+1}}{\rho(\rho+1)} - \sum_\rho \frac{(y+h)^{\rho+1}}{\rho(\rho+1)} \right) dy$$
$$= \sum_\rho \frac{(x+k)^{\rho+2}}{\rho(\rho+1)(\rho+2)} - \sum_\rho \frac{x^{\rho+2}}{\rho(\rho+1)(\rho+2)} - \sum_\rho \frac{(x+h+k)^{\rho+2}}{\rho(\rho+1)(\rho+2)} + \sum_\rho \frac{(x+h)^{\rho+2}}{\rho(\rho+1)(\rho+2)}$$
$$< \sum_\rho \frac{x^{\beta+2}}{|\gamma|^3} + \sum_\rho \frac{(x+k)^{\beta+2}}{|\gamma|^3} + \sum_\rho \frac{(x+h)^{\beta+2}}{|\gamma|^3} + \sum_\rho \frac{(x+h+k)^{\beta+2}}{|\gamma|^3}.$$

However,
$$(x+a)^{\beta+2} = x^{\beta+2} \left(1 + \frac{a}{x}\right)^{\beta+2} < x^{\beta+2} \left(1 + \frac{a}{x}\right)^3.$$

Hence the integral is less than
$$\sum_\rho \frac{x^{\beta+2}}{|\gamma|^3} [1 + 1 + 3al^{\frac{1}{2}} + 3a^2 l + a^3 ll^{\frac{1}{2}} + 1 + 6al^{\frac{1}{2}} + 12a^2 l$$
$$+ 8a^3 ll^{\frac{1}{2}} + 1 + 9al^{\frac{1}{2}} + 27a^2 l + 27a^3 ll^{\frac{1}{2}}]$$
$$= x^3 l [4 + 18al^{\frac{1}{2}} + 42a^2 l + 36a^3 ll^{\frac{1}{2}}].$$

Let z be the value of y between x and $x+k$ for which

$$\sum_\rho \frac{y^{\rho+1}}{\rho(\rho+1)} - \sum_\rho \frac{(y+h)^{\rho+1}}{\rho(\rho+1)}$$

takes its minimum value. Then

$$k\left[\sum_\rho \frac{z^{\rho+1}}{\rho(\rho+1)} - \sum_\rho \frac{(z+h)^{\rho+1}}{\rho(\rho+1)}\right] \leqslant \int_x^{x+k} \left(\sum_\rho \frac{y^{\rho+1}}{\rho(\rho+1)} - \sum_\rho \frac{(y+h)^{\rho+1}}{\rho(\rho+1)}\right) dy.$$

Hence

$$\sum_\rho \frac{z^{\rho+1}}{\rho(\rho+1)} - \sum_\rho \frac{(z+h)^{\rho+1}}{\rho(\rho+1)} < x^2 l^{\frac{1}{2}} \left[\frac{4}{a} + 18 l^{\frac{1}{2}} + 42al + 36a^2 l l^{\frac{1}{2}}\right].$$

Now

$$\begin{aligned} h\psi(x) &\leqslant \int_z^{z+h} \psi(y)\,dy \\ &= \frac{(z+h)^2}{2} - \frac{z^2}{2} - \sum_\rho \frac{(z+h)^{\rho+1}}{\rho(\rho+1)} + \sum_\rho \frac{z^{\rho+1}}{\rho(\rho+1)} - \frac{\zeta'(0)}{\zeta(0)} h \\ &\qquad - \sum_{r=1}^{\infty} \frac{(z+h)^{1-2r}}{2r(2r-1)} + \sum_{r=1}^{\infty} \frac{z^{1-2r}}{2r(2r-1)}. \end{aligned}$$

However, since $h > 0$,

$$\begin{aligned} \frac{1}{z^{2r-1}} - \frac{1}{(z+h)^{2r-1}} &= \left(\frac{1}{z} - \frac{1}{z+h}\right)\left(\frac{1}{z^{2r-2}} + \ldots + \frac{1}{(z+h)^{2r-2}}\right) \\ &< \frac{h(2r-1)}{z^{2r}}. \end{aligned}$$

Since $\zeta'(0)/\zeta(0) = 1{\cdot}837{,}877$, we have

$$-\frac{\zeta'(0)}{\zeta(0)} h - \sum_{r=1}^{\infty} \frac{(z+h)^{1-2r}}{2r(2r-1)} + \sum_{r=1}^{\infty} \frac{z^{1-2r}}{2r(2r-1)} < 0$$

if $z \geqslant 2$. So, since $x > e^{50}$,

$$\psi(x) < z + \tfrac{1}{2}h + x\left[\frac{2}{a^2} + \frac{9 l^{\frac{1}{2}}}{a} + 21l + 18all^{\frac{1}{2}}\right].$$

However, $z \leqslant x+k = x + axl^{\frac{1}{2}}$ and $h = 2axl^{\frac{1}{2}}$, and so the lemma follows from the equality $l(x) < K_2(x)$.

LEMMA 12. *If* $m > 1000$, *and if* $p(n) > n \log n$ *for all* $n \leqslant m$, *and if* $H(x)$ *is a positive function such that* $x\left(1 + H(x)^{-1}\right) \geqslant \psi(x)$, *then*

$$p(m+1) > (m+1)\log(m+1)$$

if

$$\log m \leqslant H\big(p(m)\big)\left[\log\log m - 1 - \frac{1}{\log m} - \frac{2}{\log^2 m} - \frac{2\log m}{m}\right].$$

Proof.

$$\psi\big(p(m)\big) \geqslant \theta\big(p(m)\big) = \sum_{r=1}^{m} \log\big(p(r)\big)$$

$$= \log 2 + \log 3 + \log 5 + \log 7 + \log 11 + \sum_{r=6}^{m} \log\big(p(r)\big)$$

$$> 7{\cdot}745 + \sum_{r=6}^{m} \log(r\log r) > 7{\cdot}745 + \int_5^m (\log x + \log\log x)\,dx$$

$$= 7{\cdot}745 + m\log m - m - 5\log 5 + 5 + m\log\log m$$

$$-5\log\log 5 - \int_5^m \frac{dx}{\log x}$$

$$\geqslant 2{\cdot}318 + m\log m - m + m\log\log m - \frac{m}{\log m} + \frac{5}{\log 5} - \int_5^m \frac{dx}{\log^2 x}$$

$$\geqslant 5{\cdot}425 + m\log m - m + m\log\log m - \frac{m}{\log m} - \int_5^{15} \frac{dx}{\log^2 5}$$

$$-\int_{15}^{25} \frac{dx}{\log^2 15} - \int_{25}^{35} \frac{dx}{\log^2 25} - \int_{35}^{45} \frac{dx}{\log^2 35} - \int_{45}^{55} \frac{dx}{\log^2 45} - \int_{55}^{m} \frac{dx}{\log^2 x}.$$

Now, for $x \geqslant 55$, $\log x \geqslant 4$, and so

$$\frac{2}{\log^2 x} - \frac{4}{\log^3 x} \geqslant \frac{1}{\log^2 x}.$$

Hence

$$\psi\big(p(m)\big) > m\log m - m + m\log\log m - \frac{m}{\log m} - 2{\cdot}245$$

$$-\int_{55}^{m} \left[\frac{2}{\log^2 x} - \frac{4}{\log^3 x}\right] dx$$

$$= m\log m - m + m\log\log m - \frac{m}{\log m} - \frac{2m}{\log^2 m} - 2{\cdot}245 + \frac{110}{\log^2 55}$$

$$> m\log m - m + m\log\log m - \frac{m}{\log m} - \frac{2m}{\log^2 m}.$$

However, since $m > 1000$,

$$\psi\big(p(m)\big) > m\log m + m\log\log m - m - \frac{m}{\log m} - \frac{2m}{\log^2 m} - 2\log m + \log m + \log\log m + 2 + \frac{2\log\log m}{\log m}$$

$$> (m\log m + \log m + 2)\left[1 + \frac{\log\log m}{\log m} - \frac{1}{\log m} - \frac{1}{\log^2 m} - \frac{2}{\log^3 m} - \frac{2}{m}\right]$$

$$> (m+1)\log(m+1)\left[1 + \frac{\log\log m}{\log m} - \frac{1}{\log m} - \frac{1}{\log^2 m} - \frac{2}{\log^3 m} - \frac{2}{m}\right].$$

If $x\big(1 + H(x)^{-1}\big) \geqslant \psi(x)$, then

$$p(m)\left(1 + \frac{1}{H\big(p(m)\big)}\right) \geqslant \psi\big(p(m)\big)$$

$$> (m+1)\log(m+1)\left[1 + \frac{\log\log m}{\log m} - \frac{1}{\log m} - \frac{1}{\log^2 m} - \frac{2}{\log^3 m} - \frac{2}{m}\right].$$

Hence $p(m) > (m+1)\log(m+1)$ if

$$\frac{1}{H\big(p(m)\big)} \leqslant \frac{\log\log m}{\log m} - \frac{1}{\log m} - \frac{1}{\log^2 m} - \frac{2}{\log^3 m} - \frac{2}{m}.$$

The lemma follows on multiplying both sides by $\log m\, H\big(p(m)\big)$.

LEMMA 13. *If* $1000 < m_1 < m_2$, *and if* $p(n) > n\log n$ *for all* $n \leqslant m_1$, *and if* $H(x)$ *is a positive function such that* $x\big(1 + H(x)^{-1}\big) \geqslant \psi(x)$, *and if*

$$\log n \leqslant H\big(p(n)\big)\left[\log\log n - 1 - \frac{1}{\log n} - \frac{2}{\log^2 n} - \frac{2\log n}{n}\right]$$

for all n *between* $m_1 - 1$ *and* m_2, *then* $p(n) > n\log n$ *for all* $n \leqslant m_2$.

We take m to be $m_1, m_1+1, m_1+2, \ldots, m_2-1$ successively in Lemma 12.

LEMMA 14. $p(n) > n\log n$ *for all* $n \leqslant 1071$.

Proof. For $n \geqslant 1$, $p(n)$ and $n\log n$ are both increasing functions of n, so that from $p(26) > 29\log 29$ we infer that $p(26) > 26\log 26$, $p(27) > 27\log 27$, $p(28) > 28\log 28$, and $p(29) > 29\log 29$. Using this idea, the lemma follows from the inequalities stated below, which can be verified by computation. In making the computation, one must bear in mind that Lehmer gives 1 as the first prime, whereas in this paper 2 is counted as the first prime.

$$p(1) = 2 > 1 \log 1. \quad p(2) = 3 > 2 \log 2. \quad p(3) = 5 > 3 \log 3.$$

$$p(4) = 7 > 4 \log 4. \quad p(5) = 11 > 6 \log 6.$$

$$p(7) = 17 > 8 \log 8. \quad p(9) = 23 > 9 \log 9.$$

$$p(10) = 29 > 11 \log 11. \quad p(12) = 37 > 14 \log 14. \quad p(15) = 47 > 16 \log 16.$$

$$p(17) = 59 > 19 \log 19. \quad p(20) = 71 > 22 \log 22. \quad p(23) = 83 > 25 \log 25.$$

$$p(26) = 101 > 29 \log 29. \quad p(30) = 113 > 32 \log 32. \quad p(33) = 137 > 37 \log 37.$$

$$p(38) = 163 > 43 \log 43. \quad p(44) = 193 > 49 \log 49. \quad p(50) = 229 > 56 \log 56.$$

$$p(57) = 269 > 64 \log 64. \quad p(65) = 313 > 72 \log 72. \quad p(73) = 367 > 83 \log 83.$$

$$p(84) = 433 > 95 \log 95. \quad p(96) = 503 > 107 \log 107.$$

$$p(108) = 593 > 123 \log 123. \quad p(124) = 683 > 138 \log 138.$$

$$p(139) = 797 > 157 \log 157. \quad p(158) = 929 > 179 \log 179.$$

$$p(180) = 1069 > 201 \log 201. \quad p(202) = 1231 > 226 \log 226.$$

$$p(227) = 1433 > 258 \log 258. \quad p(259) = 1637 > 288 \log 288.$$

$$p(289) = 1879 > 324 \log 324. \quad p(325) = 2153 > 364 \log 364.$$

$$p(365) = 2467 > 410 \log 410. \quad p(411) = 2833 > 461 \log 461.$$

$$p(462) = 3271 > 522 \log 522. \quad p(523) = 3761 > 589 \log 589.$$

$$p(590) = 4297 > 661 \log 661. \quad p(662) = 4951 > 744 \log 744.$$

$$p(745) = 5657 > 840 \log 840. \quad p(841) = 6481 > 945 \log 945.$$

$$p(946) = 7477 > 1071 \log 1071.$$

3. *Theorems.*

THEOREM 1. *For each positive integer n, the n-th prime is greater than $n \log n$, if 2 is counted as the first prime.*

Proof. By Lemma 14, the theorem is true for $n \leqslant 1071$. We proceed at first by combining Lemma 13 and Lemma 10. For $m \geqslant 1071$, $p(m) \geqslant 8599$. Since the $K_1(x)$ of Lemma 10 is a decreasing function of x, we have $K_1\big(p(m)\big) \leqslant 0{\cdot}001{,}46$. Thus, if we take $H = 12{\cdot}5$, then

$$p(m)(1+H^{-1}) \geqslant \psi\big(p(m)\big)$$

for $m \geqslant 1071$. Now

$$\log\log n - 1 - \frac{1}{\log n} - \frac{2}{\log^2 n} - \frac{2\log n}{n}$$

is an increasing function of n, so that, for $1071 \leqslant n$,

$$0{\cdot}763 < \log\log 1071 - 1 - \frac{1}{\log 1071} - \frac{2}{\log^2 1071} - \frac{2\log 1071}{1071}$$

$$\leqslant \log\log n - 1 - \frac{1}{\log n} - \frac{2}{\log^2 n} - \frac{2\log n}{n}.$$

Hence, for $1071 \leqslant n \leqslant e^{9{\cdot}5}$,

$$\log n < (12{\cdot}5)(0{\cdot}763)$$

$$< H\big(p(n)\big)\left[\log\log n - 1 - \frac{1}{\log n} - \frac{2}{\log^2 n} - \frac{2\log n}{n}\right].$$

We can now put $n = e^{9{\cdot}5}$ in the right-hand side of the inequality and deduce that the inequality is satisfied for $e^{9{\cdot}5} \leqslant n \leqslant e^{14}$. Proceeding in this way, we show that the inequality is satisfied for $1071 \leqslant n \leqslant e^{28}$. Hence, by Lemma 13, $p(n) > n\log n$ for $n \leqslant e^{28}$.

For $m \geqslant e^{28}$, $p(m) > 28e^{28} > e^{31}$. So for $m \geqslant e^{28}$, $K_1\big(p(m)\big) \leqslant 0{\cdot}000{,}736$. So, if we take $H = 17{\cdot}8$, then $p(m)(1+H^{-1}) \geqslant \psi\big(p(m)\big)$ for $m \geqslant e^{28}$. Then we can show that

$$\log n \leqslant H\left[\log\log n - 1 - \frac{1}{\log n} - \frac{2}{\log^2 n} - \frac{2\log n}{n}\right]$$

for $e^{28} \leqslant n \leqslant e^{51}$. Hence $p(n) > n\log n$ for $n \leqslant e^{51}$.

We now use Lemma 11 and Lemma 13. The $K_2(x)$ of Lemma 11 is a decreasing function of x. If $m \geqslant e^{51}$, $p(m) \geqslant e^{54}$. Then

$$K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}232.$$

Take $a = 16$. Then if $H = 42$, $p(m)(1+H^{-1}) \geqslant \psi\big(p(m)\big)$ for $m \geqslant e^{51}$. Then

$$\log n \leqslant H\left[\log\log n - 1 - \frac{1}{\log n} - \frac{2}{\log^2 n} - \frac{2\log n}{n}\right]$$

for $e^{51} \leqslant n \leqslant e^{174}$. So $p(n) > n\log n$ for $n \leqslant e^{174}$.

If $m \geqslant e^{174}$, $p(m) \geqslant e^{179}$. So $K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}103$. Take $a = 18$. Then if $H = 55$, $p(m)(1+H^{-1}) \geqslant \psi\big(p(m)\big)$ for $m \geqslant e^{174}$. Then $p(n) > n\log n$ for $n \leqslant e^{247}$.

We continue this process, taking successively:

$$m \geqslant e^{247},\quad K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}065{,}2,\quad a = 20,\quad H = 65;$$

$$m \geqslant e^{306},\quad K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}045,\quad a = 21,\quad H = 73;$$

$$m \geqslant e^{355},\quad K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}033{,}1,\quad a = 22,\quad H = 81;$$

$$m \geqslant e^{404},\quad K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}024{,}3,\quad a = 23,\quad H = 90;$$

$$m \geqslant e^{461},\quad K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}017,\quad a = 25,\quad H = 102;$$

$$m \geqslant e^{539},\quad K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}010{,}5,\quad a = 27,\quad H = 119;$$

$$m \geqslant e^{651},\quad K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}005{,}27,\quad a = 30,\quad H = 151;$$

$$m \geqslant e^{870},\quad K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}001{,}44,\quad a = 37,\quad H = 233;$$

$$m \geqslant e^{1460},\quad K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}000{,}2,\quad a = 50,\quad H = 450.$$

We have then proved that $p(n) > n\log n$ for $n \leqslant e^{3000}$.

We now return to Lemma 10 and use the inequalities

$$S_8 < 0{\cdot}000{,}332 < 0{\cdot}274(8e^{-8}),$$

and, if $r > 8$,

$$S_r < 16{\cdot}6(r+9)e^{-2r} + 0{\cdot}274(r-2{\cdot}25)e^{-r} < 0{\cdot}274re^{-r}.$$

Hence, for $x \geqslant e^{3000}$,

$$K_1(x) < \sum_{r=8}^{\infty} 0{\cdot}3re^{-r}x^{-1/(19r)}.$$

However, for $r \geqslant 8$,

$$0{\cdot}2(\log x)^{\frac{1}{2}}e^{-2\sqrt{(\log x/19)}} \geqslant 0{\cdot}3re^{-r}x^{-1/(19r)}.$$

We can prove this by taking the logarithm of both sides and showing that

$$r + \frac{\log x}{19r} + \log\big(0{\cdot}2(\log x)^{\frac{1}{2}}\big) \geqslant 2\sqrt{(\log x/19)} + \log(0{\cdot}3r).$$

However, the minimum value of $r + \log x/19r$ is $2\sqrt{(\log x/19)}$, so that the inequality is satisfied for $r \leqslant \frac{2}{3}(\log x)^{\frac{1}{2}}$. If $r \geqslant \frac{2}{3}(\log x)^{\frac{1}{2}}$,

$$\begin{aligned} r &\geqslant 2\sqrt{(\log x/19)} + 0{\cdot}3r \\ &\geqslant 2\sqrt{(\log x/19)} + \log(0{\cdot}3r). \end{aligned}$$

So the sum of the first $(\log x)^{\frac{1}{2}}$ terms of the summation is less than

$$\tfrac{1}{5}\log x\, e^{-2\sqrt{(\log x/19)}}.$$

Since the summation starts with $r=8$, the sum of the remaining terms is less than

$$\int_{\sqrt{\log x}}^{\infty} re^{-r}\,dr = (\log x)^{\frac{1}{2}}\, e^{-\sqrt{\log x}} + e^{-\sqrt{\log x}}.$$

For $x \geqslant e^{3000}$, this is less than $0{\cdot}05 \log x\, e^{-2\sqrt{(\log x/19)}}$. So, for $x \geqslant e^{3000}$,

$$K_1(x) < \sum_{r=8}^{\infty} 0{\cdot}3r\, e^{-r} x^{-1/(19r)} < \tfrac{1}{4}\log x\, e^{-2\sqrt{(\log x/19)}}.$$

Now, for $m \geqslant e^{3000}$,

$$\log m \leqslant (\log m)^{-\frac{1}{2}}\, e^{\sqrt{(\log m/19)}}$$

and so

$$\log m \leqslant (\log m)^{-\frac{1}{2}}\, e^{\sqrt{(\log m/19)}} \left[\log\log m - 1 - \frac{1}{\log m} - \frac{2}{\log^2 m} - \frac{2\log m}{m}\right].$$

Hence we can conclude by Lemma 10 and Lemma 13 that, for all $n \geqslant e^{3000}$, $p(n) > n\log n$.

THEOREM 2. *If* $3 < n$, *then* $p(n) < n\log n + 2n\log\log n$.

Proof. Lemmas analogous to Lemmas 10–14 are proved and then this theorem is proved in a manner precisely analogous to the way Theorem 1 was proved.

Theorems 1 and 2 make more precise the ordinary theorem $p(n) \sim n\log n$ (Landau, p. 214). The sharper result $p(n) = n\bigl(\log n + \log\log n + O(1)\bigr)$ (Laudau, p. 215) is made more precise in Theorem 3. The still sharper result $p(n) = n\log n + n\log\log n - n + O(n\log\log n/\log n)$ (which can be proved by the method of Landau, p. 215) could also be made more precise by the methods of this paper, which accounts for the "$-n$" which occurs in Theorem 3.

THEOREM 3. *If* $n > 1$,

$$n\log n + n\log\log n - n - 9n < p(n) < n\log n + n\log\log n - n + 9n.$$

Proof. For $$1 < n \leqslant e^{e^{10}},$$

$n\log n + n\log\log n - n - 9n < p(n)$ by Theorem 1. For

$$e^{e^{10}} \leqslant n,$$

the method of proof used in Theorem 1 will work. The other half of the theorem is proved analogously.

Let B_n, C_n, and D_n be defined for $n > 1$ by

$$\sum_{m=1}^{n} \frac{1}{p(m)} = \log \log n + B_n, \qquad \prod_{m=1}^{n} \left(1 - \frac{1}{p(m)}\right) = \frac{C_n}{\log n},$$

and

$$\prod_{m=2}^{n} \left(1 - \frac{2}{p(m)}\right) = \frac{D_n}{\log^2 n}.$$

THEOREM 4. *For* $n > 1$, $B_n > B_{n+1}$, $C_n < C_{n+1}$, *and* $D_n < D_{n+1}$.

Proof.

$$\begin{aligned} B_{n+1} + \log \log (n+1) &= \sum_{m=1}^{n+1} \frac{1}{p(m)} = \sum_{m=1}^{n} \frac{1}{p(m)} + \frac{1}{p(n+1)} \\ &< B_n + \log \log n + \frac{1}{(n+1) \log (n+1)} \\ &< B_n + \log \log n + \int_n^{n+1} \frac{dx}{x \log x} \\ &< B_n + \log \log (n+1). \end{aligned}$$

Thus $B_{n+1} < B_n$.

$$\frac{C_n}{C_{n+1}} = \frac{\log n}{\log (n+1)\bigl(1 - 1/p(n+1)\bigr)} < \frac{\log (n+1) + \log \bigl(n/(n+1)\bigr)}{\log (n+1)\Bigl(1 - 1\big/\bigl((n+1) \log (n+1)\bigr)\Bigr)}$$

$$< \frac{\log (n+1) - 1/(n+1)}{\log (n+1)\Bigl(1 - 1\big/\bigl((n+1) \log (n+1)\bigr)\Bigr)} = 1.$$

Thus $C_n < C_{n+1}$.

$$\frac{D_n}{D_{n+1}} = \frac{\log^2 n}{\log^2 (n+1)\bigl(1 - 2/p(n+1)\bigr)} < \frac{\Bigl[\log (n+1) + \log \bigl(n/(n+1)\bigr)\Bigr]^2}{\log^2 (n+1)\Bigl(1 - 2\big/\bigl((n+1) \log (n+1)\bigr)\Bigr)}$$

$$< \frac{\Bigl[\log (n+1) - 1/(n+1) - 1\big/\bigl(2(n+1)^2\bigr)\Bigr]^2}{\log^2 (n+1)\Bigl(1 - 2\big/\bigl((n+1) \log (n+1)\bigr)\Bigr)}$$

$$< \frac{\log^2 (n+1) - 2 \log (n+1)/(n+1)}{\log^2 (n+1)\Bigl(1 - 2\big/\bigl((n+1) \log (n+1)\bigr)\Bigr)},$$

if $n > 2$. Thus $D_n/D_{n+1} < 1$ for $n > 2$. Moreover, $D_2 = 0{\cdot}16$ and $D_3 = 0{\cdot}24$.

Theorem 5.

$$\lim_{n\to\infty} B_n = 0{\cdot}261,497,212,8.$$

$$\lim_{n\to\infty} C_n = 0{\cdot}561,459,483,6.$$

$$\lim_{n\to\infty} D_n = 0{\cdot}832,429,065,7.$$

Proof.
$$\prod_{p<x}\left(1-\frac{1}{p}\right) = \frac{e^{-C}}{\log x} + O\left(\frac{1}{\log^2 x}\right)$$

(Landau, p. 139). Since $p(n) \sim n\log n$, it follows that

$$\lim_{n\to\infty} C_n = e^{-C} = \frac{1}{1{\cdot}781,072,417,990} = 0{\cdot}561,459,483,567.$$

Now, if $n > 6$,

$$\log C_n = \log\log n + \sum_{m=1}^{n} \log\left(1-\frac{1}{p(m)}\right)$$

$$= \log\log n + \log(1-\tfrac{1}{2}) + \log(1-\tfrac{1}{3}) + \log(1-\tfrac{1}{5}) + \log(1-\tfrac{1}{7})$$

$$+\log(1-\tfrac{1}{11}) + \log(1-\tfrac{1}{13}) - \sum_{m=7}^{n}\frac{1}{p(m)}$$

$$-\tfrac{1}{2}\sum_{m=7}^{n}\frac{1}{\big(p(m)\big)^2} - \dots - \frac{1}{r}\sum_{m=7}^{n}\frac{1}{\big(p(m)\big)^r} - \dots$$

$$= \log\log n - \sum_{m=1}^{n}\frac{1}{p(m)} + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} + \tfrac{1}{13} + 6\log 2$$

$$+\log 3 - \log 7 - \log 11 - \log 13$$

$$-\tfrac{1}{2}\sum_{m=7}^{n}\frac{1}{\big(p(m)\big)^2} - \dots - \frac{1}{r}\sum_{m=7}^{n}\frac{1}{\big(p(m)\big)^r} - \dots .$$

However, for $r > 1$,

$$\sum_{m=7}^{n}\frac{1}{\big(p(m)\big)^r} < \sum_{m=17}^{\infty}\frac{1}{m^r} < \frac{1}{(r-1)\,16^{r-1}}.$$

Thus we can conclude that

$$\lim_{n\to\infty} B_n = -\log\Big(\lim_{n\to\infty} C_n\Big)+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{11}+\tfrac{1}{13}$$

$$+6\log 2+\log 3-\log 7-\log 11-\log 13$$

$$-\tfrac{1}{2}\sum_{m=7}^{\infty}\frac{1}{\big(p(m)\big)^2}-\ldots-\frac{1}{r}\sum_{m=7}^{\infty}\frac{1}{\big(p(m)\big)^r}-\ldots.$$

However,

$$-\log\Big(\lim_{n\to\infty} C_n\Big) = C = 0{\cdot}577{,}215{,}664{,}901{,}329,$$

and

$$\sum_{m=7}^{\infty}\frac{1}{\big(p(m)\big)^r}$$

can be computed for $r>1$ from tables* of

$$\sum_{m=1}^{\infty}\frac{1}{\big(p(m)\big)^r}.$$

This gives $$\lim_{n\to\infty} B_n = 0{\cdot}261{,}497{,}212{,}847{,}643.$$

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* Such a table is given in the second volume of H. T. Davis's *Tables of the higher mathematical functions*, 249–250. The present author checked this table for $n \geqslant 23$ by the relation

$$\Sigma_n = 1+S_n-S_{2n}-S_{3n}-\frac{1}{6^n}S_n-\frac{1}{10^n}-\frac{1}{14^n}-\frac{1}{15^n}-\ldots,$$

and found that the last digits of the values given in the table for Σ_{30}, Σ_{32}, Σ_{38}, and Σ_{48} were all too large by one unit, and for Σ_{23}, Σ_{25}, Σ_{29}, Σ_{33}, Σ_{35}, and Σ_{37} were all too small by one unit. This would make it seem likely that, while the last digit is unreliable, it is probably never very much in error. As a further confirmation of this conclusion, it should be noted that the values check, except for the last digit, in the relation,

$$C-\lim_{x\to\infty}\Big(\sum_{p<x}\frac{1}{p}-\log\log x\Big) = \tfrac{1}{2}\log S_2+\tfrac{1}{3}\log S_3+\tfrac{1}{5}\log S_5-\tfrac{1}{6}\log S_6+\ldots$$

$$= \tfrac{1}{2}\Sigma_2+\tfrac{1}{3}\Sigma_3+\tfrac{1}{4}\Sigma_4+\tfrac{1}{5}\Sigma_5+\ldots.$$

This relation can be proved by combining modern results in the theory of primes with the argument given by J. W. Glaisher, "On the series $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+$etc.", *Quart. J. of Math.*, 25 (1891), 369–375. The numerical value of this constant works out to be 0·315,718,452,053,890,076,851,085 (last digit doubtful).

For $n > 6$,

$$\log D_n = 2 \log \log n + \sum_{m=2}^{n} \log \left(1 - \frac{2}{p(m)}\right)$$

$$= -2B_n - \log 7 + \log 9 - \log 13 + \tfrac{2}{2} + \tfrac{2}{3} + \tfrac{2}{5} + \tfrac{2}{7} + \tfrac{2}{11} + \tfrac{2}{13}$$

$$- \tfrac{4}{2} \sum_{m=7}^{n} \frac{1}{\left(p(m)\right)^2} - \ldots - \frac{2^r}{r} \sum_{m=7}^{n} \frac{1}{\left(p(m)\right)^r} - \ldots.$$

Letting n approach infinity, we get

$$\lim_{n \to \infty} \log D_n = -0 \cdot 183,407,267,169,9$$

(last digit doubtful). So

$$\lim_{n \to \infty} D_n = \frac{1}{1 \cdot 201,303,559,967} = 0 \cdot 832,429,065,662.$$

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ON THE EFFECTIVE RANGE OF GENERALIZED LIMIT PROCESSES

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1. The known methods of "summing" divergent sequences, *e.g.*, the means of Cesàro, Riesz, Borel, Lindelöf, are particular cases of the transformation of a sequence by a matrix.

The use of a matrix A for the transformation of a sequence z_k into

$$z_n' = \sum_{k=1}^{\infty} a_{n,k} z_k$$

is naturally restricted by the condition that, at least for a sufficiently large n, the series in question should be convergent. If this is the case, we say that the transformation A *applies* to the sequence z_k.

For instance, if A is a K-matrix[*], A applies to every bounded sequence. This leads to the ideas of "*range*" *of applicability*[†] of the transformation A, and of *lower and upper rank*[‡] of sequences.

If we employ A to make divergent sequences become convergent, we further restrict this range of applicability of K-matrices into what will be called the *effective range*; for K-matrices the effective range includes convergent (= regular) sequences.

The most important class of divergent sequences which we may wish to make convergent is furnished by the partial sums of Taylor series outside their circles of convergence, and in this case analytic continuation defines the "right" value to which the transform should converge. For obvious reasons, only T-matrices[§] are employed in actual practice.

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[*] P. Dienes, *The Taylor series* (Oxford, 1931), 388.

[†] *Ibid.*, 409, where it is simply called the "range".

[‡] *Ibid.*, 409.

[§] *Ibid.*, 389.

In the present paper, sufficient conditions are obtained for a matrix to be *inefficient** for (i) all divergent sequences, (ii) all bounded divergent sequences, (iii) all unbounded sequences, (iv) all Taylor series outside their circles of convergence.

From these results various problems arise, chiefly connected with reciprocals of matrices, which are dealt with in §§5–10.

In particular, we consider in §7 a curious phenomenon of summability by a matrix of a Taylor series at a *point* (or *isolated set of points*) outside the circle of convergence; this is a kind of intermediate stage between inefficiency at all points outside the circle, and efficiency in a *region* outside the circle. The results obtained in this paper obviously have some bearing on the allied question of finding sufficient conditions for a matrix to be *efficient* for some class of Taylor series in a region outside the circle of convergence; but this seems a more difficult topic, since the required conditions will depend not only on the matrix, but on the particular Taylor series chosen; *e.g.*, the Borel matrix sums Σz^n throughout the region $\mathfrak{R}(z) < 1$, but will not sum a Taylor series satisfying the conditions of Hadamard's gap theorem† anywhere outside the circle of convergence. This allied question of efficiency outside the circle of convergence is therefore left open for future investigation.

2. We begin by considering a class of K-matrices which are *totally inefficient*, *i.e.*, which make no divergent sequence become convergent.

Suppose that, for a given divergent sequence z_k, and for two given K-matrices A and B, we have

$$\sum_{i=1}^{\infty} b_{n,i}\left(\sum_{k=1}^{\infty} a_{i,k} z_k\right) = \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} b_{n,i} a_{i,k}\right) z_k; \tag{2.1}$$

i.e., suppose that in the case in question the order of the summations can be inverted without destroying the convergence or altering the sum.

If A transforms z_k into a convergent sequence z_i', then the left-hand side of (2.1) produces a convergent sequence for $n = 1, 2, \ldots$, since B is a K-matrix. Therefore the right-hand side also produces a convergent sequence; *i.e.*, B cannot spoil the regularity of the A-transform, and BA makes z_k convergent.

In particular, if B is a left-hand reciprocal ${}^{-1}A$ of A (not necessarily unique or two-sided), ${}^{-1}A$ being a K-matrix, the right-hand side of (2.1) becomes z_n, which is, by hypothesis, divergent; although, by the above argument, it should be convergent. Hence the premise "A transforms z_k into a convergent sequence" is untenable when ${}^{-1}A$ exists and is a K-matrix.

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* *I.e.*, the sequences in question are *not in the effective range of the matrix.*

† P. Dienes, *loc. cit.*, 231 and 361.

Thus, the problem of finding classes of totally inefficient matrices reduces in particular to that of determining classes of matrices to which the above argument applies.

Now (2.1) is satisfied unconditionally, *i.e.*, *for any sequence* z_k *whatsoever*, if both A and B are row-finite* (*e.g.*, lower semi-matrices). Also (2.1) is satisfied for any pair of K-matrices if z_k is bounded. Hence the two results:

(I). *If a row-finite K-matrix A has a left-hand reciprocal* ${}^{-1}A$ *which is a row-finite K-matrix, then A is totally inefficient*; *i.e., no divergent sequence is transformed into a convergent sequence by A.*

(II). *If the K-matrix A has a left-hand reciprocal* ${}^{-1}A$ *which is a K-matrix, then A is inefficient for all bounded divergent sequences*; *i.e., no bounded divergent sequence is transformed into a convergent sequence by A.*

The first result clearly applies to the partial sums of any Taylor series outside its circle of convergence as a particular case. We can slightly extend (II) to unbounded sequences by noticing that (2.1) is satisfied if z_k is within the range† of $B(A)$; we therefore obtain

(III). *If the K-matrix A has a left-hand reciprocal* ${}^{-1}A$ *which is a K-matrix, then A is inefficient for all sequences which lie within the range of* ${}^{-1}A(A)$.

3. We now proceed to extend the above results. Suppose that the row-finite matrix A has a left-hand row-finite reciprocal ${}^{-1}A$ which is a K_a-matrix‡.

We still have, for any sequence z_k,

$$\sum_{i=1}^{i_n} {}^{-1}a_{n,i} \sum_{k=1}^{k_i} a_{i,k} z_k = z_n. \tag{3.1}$$

Suppose also that A transforms the divergent sequence z_k into the convergent sequence $z_i' \to z'$, and denote the upper bound of $|z_i'|$ by b. By hypothesis, we have

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* P. Dienes, *loc. cit.*, 409.

† *Ibid.*, 413.

‡ *Ibid.*, 388.

$$\sum_{i=1}^{i_n} |{}^{-1}a_{n,i}| \leqslant M, \text{ independently of } n,$$

so that the modulus of the left-hand side of (3.1) is less than or equal to Mb. But this left-hand side is equal to z_n; therefore, if the upper bound of $|z_k|$ is greater than Mb, there is contradiction. This is certainly the case if the upper bound of $|z_k|$ is infinite. Also, if $s_n(z)$ is the n-th partial sum of any Taylor series, and z is outside its circle of convergence, then*

$$\overline{\lim_{n\to\infty}}\, |s_n(z)| = \infty.$$

Hence

(IV). *If a row-finite matrix A has a row-finite left-hand reciprocal ${}^{-1}A$ which is a K_a-matrix, the effective range of A does not include unbounded sequences; in particular, A is inefficient for any Taylor series outside its circle of convergence.*

4. We now consider the case of matrices having a left-hand reciprocal which is *not* a K_a-matrix.

For example, the arithmetic means have the unique, two-sided reciprocal

$$a_{n,k}^{-1} = \begin{pmatrix} 1, & 0, & 0, & 0, & 0, & \dots \\ -1, & 2, & 0, & 0, & 0, & \dots \\ 0, & -2, & 3, & 0, & 0, & \dots \\ 0, & 0, & -3, & 4, & 0, & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

This satisfies two of the three conditions for a semi-T-matrix, since $\lim_{n\to\infty} a_{n,k}^{-1} = 0$ for every fixed k, and $\lim_{n\to\infty} \sum_{k=1}^{n} a_{n,k}^{-1} = 1$; but it does not satisfy the third condition, since

$$\sum_{k=1}^{n} |a_{n,k}^{-1}| = (n-1)+n = 2n-1,$$

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[*] P. Dienes, *loc. cit.*, 353.

which is not bounded, independently of n; *i.e.*, the reciprocal is not a K_a-matrix.

Consider now a row-finite matrix A possessing a row-finite left-hand reciprocal ${}^{-1}A$ which is not a K_a-matrix; we restrict it by the condition that, for sufficiently large n,

$$\sum_{i=1}^{i_n} |{}^{-1}a_{n,i}| \leqslant \eta^n,$$

where η is *any* real number greater than unity; *i.e.*,

$$\varlimsup_{n\to\infty} \sqrt[n]{\left\{\sum_{i=1}^{i_n} |{}^{-1}a_{n,i}|\right\}} \leqslant 1. \tag{4.1}$$

It follows that

$$\varlimsup_{n\to\infty} \left| \sqrt[n]{\left\{\sum_{i=1}^{i_n} {}^{-1}a_{n,i} z_i'\right\}} \right| \leqslant 1,$$

where z_i' has the same meaning as in § 3, so that the upper bound of $|z_i'|$ is finite, and hence its n-th root tends to unity.

It then follows that

$$\varlimsup_{n\to\infty} |\sqrt[n]{z_n}| > 1 \tag{4.2}$$

creates a contradiction between the two sides of (3.1). We notice that (4.2) is satisfied by $s_n(z)$, the n-th partial sum of any Taylor series, when z is outside its circle of convergence*.

We consequently have

(V). *If a row-finite matrix A has a row-finite ${}^{-1}A$ which satisfies*

$$\varlimsup_{n\to\infty} \sqrt[n]{\left\{\sum_{i=1}^{i_n} |{}^{-1}a_{n,i}|\right\}} \leqslant 1,$$

then A is inefficient for sequences z_n satisfying

$$\varlimsup_{n\to\infty} |\sqrt[n]{z_n}| > 1;$$

in particular, A is inefficient for any Taylor series outside its circle of convergence.

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* P. Dienes, *loc. cit.*, 353 (4).

The conditions of (V) are satisfied by Cesàro means of any order $r > -1$.

5. It seems difficult to improve on the condition (4.1) for inefficiency of A for any Taylor series outside its circle of convergence. But the following result shows that if a rather less restrictive condition of similar type is assumed, A is inefficient for any Taylor series outside a larger concentric circle; so that we thereby *limit the region of possible efficiency.*

We shall assume that ${}^{-1}A$ does not satisfy the condition

$$\sum_{i=1}^{i_n} |{}^{-1}a_{n,i}| \leqslant \eta^n$$

for *every* $\eta > 1$ when $n > n_0$, but only for some *fixed* $\eta > 1$. Without loss of generality, we may assume the radius of convergence of the Taylor series to be unity. We then prove

(VI). *If a row-finite matrix A has a row-finite ${}^{-1}A$ which satisfies*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\left\{\sum_{i=1}^{i_n} |{}^{-1}a_{n,i}|\right\}} \leqslant \eta,$$

for some fixed $\eta > 1$, then A is inefficient for any Taylor series whose radius of convergence is 1 for $|z| > \eta$.

We first give an inequality for $\overline{\lim\limits_{n\to\infty}} |\sqrt[n]{\{s_n(z)\}}|$ when $|z| > 1$, which is an improvement on that given in § 4*.

LEMMA. *If $s_n(z)$ is the n-th partial sum of a Taylor series whose radius of convergence is 1, and $|z| > \eta > 1$, then*

$$\overline{\lim_{n\to\infty}} |\sqrt[n]{\{s_n(z)\}}| > \eta.$$

For suppose if possible that

$$\overline{\lim_{n\to\infty}} |\sqrt[n]{\{s_n(z)\}}| \leqslant \eta \quad \text{when} \quad |z| > \eta > 1.$$

Then, for every given positive σ, there is an $n(\sigma)$ such that, for $n > n(\sigma)$,

$$|s_n(z)| \leqslant (\eta+\sigma)^n, \quad |s_{n+1}(z)| \leqslant (\eta+\sigma)^{n+1};$$

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* *I.e.*, P. Dienes, *loc. cit.*, 353 (4).

i.e., $$|c_{n+1}z^{n+1}| = |s_{n+1}(z) - s_n(z)| \leqslant 2(\eta+\sigma)^{n+1},$$

the Taylor series being $\sum_{n=0}^{\infty} c_n z^n$.

Now put $|z| = \eta+\epsilon$, and take $\sigma = \epsilon/\eta$. Then

$$|c_{n+1}|\,(\eta+\epsilon)^{n+1} \leqslant 2\left(\eta+\frac{\epsilon}{\eta}\right)^{n+1},$$

i.e. $$\overline{\lim_{n\to\infty}}\, |\sqrt[n+1]{c_{n+1}}| \leqslant \frac{\eta+\epsilon/\eta}{\eta+\epsilon}, \quad i.e., \quad 1 \leqslant \frac{\eta+\epsilon/\eta}{\eta+\epsilon},$$

so that $1 \leqslant 1/\eta$, which is a contradiction.

This proves the lemma.

Proof of theorem. We have

$$\sum_{i=1}^{in} {}^{-1}a_{n,i} \sum_{k=1}^{ki} a_{i,k}\, s_k(z) = s_n(z). \tag{5.1}$$

If A transforms the divergent sequence $s_k(z)$ ($|z| > \eta > 1$) into the convergent sequence $s_n'(z) \to f(z)$, then $|s_n'(z)|$ has a fixed upper bound for a fixed z.

Then, since

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\left\{\sum_{i=1}^{in} |{}^{-1}a_{n,i}|\right\}} \leqslant \eta,$$

we have $$\overline{\lim_{n\to\infty}} \left| \sqrt[n]{\left\{\sum_{i=1}^{in} {}^{-1}a_{n,i}\, s'_i(z)\right\}} \right| \leqslant \eta,$$

for the given fixed $\eta > 1$.

It follows that

$$\overline{\lim_{n\to\infty}} |\sqrt[n]{\{s_n(z)\}}| > \eta \quad \text{for} \quad |z| > \eta$$

creates a contradiction between the two sides of (5.1).

Hence the result is proved.

6. In this section we consider *lower semi-T-matrices* only.

A matrix having no left-hand reciprocal will not come under any of the above theorems, so the question arises whether such a matrix can be modified so as to possess a left-hand reciprocal, without affecting its efficiency or inefficiency for a Taylor series. For a lower semi-matrix A where, for every n, $a_{n,n} \neq 0$, we have* a unique right-hand reciprocal, which is a lower semi-matrix with $a^{-1}_{n,n} = 1/a_{n,n}$; and this right-hand reciprocal is also a left-hand reciprocal, and is the only two-sided reciprocal among lower semi-matrices.

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* P. Dienes, "Notes on linear equations in infinite matrices", *Quart. J. of Math.* (Oxford), 3 (1932), 253–268 (261).

Thus a lower semi-matrix has a left-hand reciprocal unless

(*a*) a finite number of the $a_{n,n}$ are zero,

or (*b*) an infinity of the $a_{n,n}$ are zero.

Case (*a*) is covered by the following result*.

(VII). *If two T-matrices differ in at most a finite number of elements, they are absolutely equivalent*† *for all sequences.*

For if the matrices are A and A', we have, for any sequence z_k,

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k} - a'_{n,k}||z_k| = 0,$$

which proves the result‡.

COROLLARY. *If A, A' are lower semi-T-matrices, such that a finite number of the $a_{n,n}$ are zero, but none of the $a'_{n,n}$ are zero, then A' has a left-hand reciprocal, and A, A' are both efficient or both inefficient for any Taylor series in the same region.*

Also, case (*b*) is covered by the following result:

(VIII). *Let A be a lower semi-T-matrix with infinitely many zero elements in its leading diagonal, and let A' be the lower semi-T-matrix obtained by replacing these zero elements by a sequence of non-zero elements $\epsilon_1, \epsilon_2, \ldots$, such that $\epsilon_n \to 0$. Then A and A' are absolutely equivalent for all bounded sequences, and for unbounded sequences z_n such that $\epsilon_n z_n \to 0$.*

For

(i) $$\overline{\lim_{n\to\infty}} \sum_{k=1}^{n} |a_{n,k} - a'_{n,k}| = \lim_{n\to\infty} |\epsilon_n| = 0,$$

and

(ii) $$\overline{\lim_{n\to\infty}} \sum_{k=1}^{n} |a_{n,k} - a'_{n,k}||z_k| = \lim_{n\to\infty} |\epsilon_n z_n| = 0,$$

which proves the result.

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* We have to thank Dr. L. S. Bosanquet for this result, which is an improvement on our original result.

† R. G. Cooke, "On mutual consistency and regular T-limits", *Proc. London Math. Soc.* (2), 41 (1936), 113–125 (114).

‡ *Ibid.*, 114 (Theorem I), and 120 (Theorem III).

Now, for $n > n_0$, we have $|\epsilon_n| < 1/|z_n|$, and, for some values of n,

$$^{-1}a'_{n,n} = 1/\epsilon_n,$$

$^{-1}a'_{n,k}$ being the unique left-hand lower semi-matrix reciprocal of $a'_{n,k}$; hence, for such values of $n > n_0$, we have

$$\sum_{i=1}^{n} |^{-1}a'_{n,i}| \geqslant \frac{1}{|\epsilon_n|} > |z_n|.$$

It follows that if z_n is the n-th partial sum of a Taylor series at a point z outside its circle of convergence (assumed of unit radius) for which $|z| > \eta$, then for such values of $n > n_0$,

$$\sum_{i=1}^{n} |^{-1}a'_{n,i}| > \eta^n,$$

i.e.

$$\varlimsup_{n\to\infty} \sqrt[n]{\left\{\sum_{i=1}^{n} |^{-1}a'_{n,i}|\right\}} > \eta > 1,$$

by the Lemma to (VI).

Thus A' cannot be deduced, from any theorem so far obtained, to be inefficient for any region of the z-plane when applied to $s_n(z)$, while it is at the same time absolutely equivalent to A for partial sums of all Taylor series.

7. We now turn attention to the curious phenomenon mentioned in § 1, namely that of certain types of matrices being *efficient at a point, or isolated set of points* (as opposed to a *region*) for Taylor series outside the circle of convergence.

(i) Let $a_{n,n} = \lambda$, $a_{n,n-1} = 1-\lambda$, $a_{n,k} = 0$ $(1 \leqslant k < n-1)$ be a lower semi-matrix, λ being any (complex) number, not zero. In order that A, which is a T-matrix, may sum Σz^n at z to its "right" value $1/(1-z)$, it is necessary and sufficient that

$$\sigma_n(z) = \sum_{k=1}^{n} a_{n,k} z^k \to 0 \quad \text{as} \quad n \to \infty.$$

Here $\sigma_n(z) = z^{n-1}\{\lambda z + (1-\lambda)\} = 0$ at the point $z = 1 - 1/\lambda$, for every $n > 1$; and $z = 1 - 1/\lambda$ is outside $|z| = 1$ if $|\lambda - 1| > |\lambda|$, *i.e.*, if $\mathfrak{R}(\lambda) < \frac{1}{2}$.

Hence, if $\mathfrak{R}(\lambda) < \frac{1}{2}$ and $\lambda \neq 0$, A sums Σz^n to its "right" value $1/(1-z)$ at $z = 1 - 1/\lambda$, which is outside the circle of convergence. If λ is real, and $0 < \lambda < 1$, A is a positive lower semi-T-matrix.

Let us examine this in the light of (V) and (VI).

The matrix A has a two-sided reciprocal, which is a lower semi-matrix with $a^{-1}_{n,n} = 1/a_{n,n}$, and is the only two-sided reciprocal among lower semi-matrices; the right-hand reciprocal is unique, but there are infinitely many left-hand reciprocals, in consequence of the fact that the elements of the first column of a left-hand reciprocal of this matrix A are arbitrary* for $n > 1$. The two-sided reciprocal is given by

$$
{}^{-1}a_{n,i} = \begin{pmatrix}
\frac{1}{\lambda}, & 0, & 0, & 0, & 0, & \ldots \\
\frac{\lambda-1}{\lambda^2}, & \frac{1}{\lambda}, & 0, & 0, & 0, & \ldots \\
\frac{(\lambda-1)^2}{\lambda^3}, & \frac{\lambda-1}{\lambda^2}, & \frac{1}{\lambda}, & 0, & 0, & \ldots \\
\frac{(\lambda-1)^3}{\lambda^4}, & \frac{(\lambda-1)^2}{\lambda^3}, & \frac{\lambda-1}{\lambda^2}, & \frac{1}{\lambda}, & 0, & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots\ \ldots & \ldots
\end{pmatrix},
$$

and
$$\sum_{i=1}^{n} |{}^{-1}a_{n,i}| = \sum_{i=1}^{n} \frac{|\lambda-1|^{i-1}}{|\lambda|^{i}} = \frac{1}{|\lambda|} \sum_{i=0}^{n-1} \eta^{i},$$

where $\eta = |1-1/\lambda|$. Thus

$$\sum_{i=1}^{n} |{}^{-1}a_{n,i}| = \frac{1}{|\lambda|}\,\frac{\eta^{n}-1}{\eta-1} \leqslant K\eta^{n},$$

and $\eta > 1$ if $\mathfrak{R}(\lambda) < \frac{1}{2}$.

Hence, by (VI), when $\mathfrak{R}(\lambda) < \frac{1}{2}$, A is inefficient for *any* Taylor series whose radius of convergence is 1 for $|z| > |1-1/\lambda|$, and not merely for Σz^{n}.

Also
$$\overline{\lim_{n\to\infty}} \sqrt[n]{\left\{\sum_{i=1}^{n} |{}^{-1}a_{n,i}|\right\}} > 1$$

when $\eta > 1$†, *i.e.*, when $\mathfrak{R}(\lambda) < \frac{1}{2}$; so that, as we should expect, (V) does not apply in this case; when $\mathfrak{R}(\lambda) > \frac{1}{2}$ (so that $\eta < 1$, and $1-1/\lambda$ is inside $|z| = 1$), (V) does apply.

(ii) If m is any positive integer (independent of n), we can construct a row-finite T-matrix which sums Σz^{n} to $1/(1-z)$ at m points outside $|z| = 1$.

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* P. Dienes, "Notes on linear equations in infinite matrices," *Quart. J. of Math.* (Oxford), 3 (1932), 253–268 (261).

† This is also true if the two-sided reciprocal is replaced by any one of the other infinitely many left-hand reciprocals, whose first columns are arbitrary for $n > 1$.

For take $a_{n,n}=1-\lambda$, $a_{n,n+m}=\lambda$, $a_{n,k}=0$ $(k\neq n$ or $n+m)$, $\lambda\neq 0$. Then $\sigma_n(z)=z^n\{\lambda z^m+(1-\lambda)\}=0$ when $z=(1-1/\lambda)^{1/m}$, thus giving m values of z which are all outside $|z|=1$ if $\mathfrak{R}(\lambda)<\frac{1}{2}$. As a special case, take $\lambda=-(e^{\mu m}-1)^{-1}$, $\mu>0$ [so that $\mathfrak{R}(\lambda)<\frac{1}{2}$ is satisfied]; then

$$z=\left(1-\frac{1}{\lambda}\right)^{1/m}=(e^{\mu m})^{1/m},$$

so that $|z|=e^\mu>1$. We thus obtain m points z equally spaced round the circle $|z|=e^\mu$, *i.e.*, round *any* circle with centre at the origin and radius greater than 1.

Conversely, given the m points $z=e^\mu$, $e^{\mu+2\pi i/m}$, $e^{\mu+4\pi i/m}$, ... $(\mu>0)$, there is a corresponding row-finite T-matrix

$$a_{n,n}=-\frac{e^{\mu m}}{1-e^{\mu m}},\quad a_{n,n+m}=\frac{1}{1-e^{\mu m}},\quad a_{n,k}=0\quad (k\neq n \text{ or } n+m),$$

which sums Σz^n to $1/(1-z)$ at each of the given points.

Thus, given any point whatsoever outside $|z|=1$, a row-finite T-matrix can (by a suitable choice of λ) be constructed to sum Σz^n to $1/(1-z)$ at this point; and the same matrix, being a T-matrix, sums Σz^n to $1/(1-z)$ at all points inside $|z|=1$.

(iii) We now generalize from Σz^n to a whole class of Taylor series.

(IX). *Given a Taylor series* $\sum_{n=0}^{\infty} c_n z^n$ *with radius of convergence* 1, *satisfying the conditions*

(*a*) $|c_n z_1{}^n|>\delta>0$ *for every* n *and a fixed* $|z_1|>1$,

(*b*) $\dfrac{1}{|c_n|}\max\limits_{0\leqslant s\leqslant n}|c_s|<M$, *independently of* n,

there exists a lower semi-T*-matrix* A, *such that* A *is efficient for the Taylor series at the point* z_1 *outside* $|z|=1$.

Let $f(z)=\sum_{n=0}^{\infty}c_n z^n$ inside $|z|=1$, $s_k(z)=\sum_{n=0}^{k}c_n z^n$, $R_k(z)=f(z)-s_k(z)$, the definition of $R_k(z)$ holding whether z is inside, on, or outside $|z|=1$, so long as $f(z)$ is defined at the point in question.

Let A be a lower semi-matrix for which $a_{n,k}=0$ except when $k=n-1$ or n, and consider

$$\begin{aligned}\sigma_n(z)&=a_{n,n}R_n(z)+a_{n,n-1}R_{n-1}(z)\\&=(a_{n,n}+a_{n,n-1})f(z)-\{a_{n,n}s_n(z)+a_{n,n-1}s_{n-1}(z)\}.\end{aligned}$$

Let $a_{n,n-1} = 1 - a_{n,n}$ $(n \geqslant 1)$, $a_{0,0} = 1$; then since

$$s_n(z) = s_{n-1}(z) + c_n z^n,$$

we have
$$\sigma_n(z) = f(z) - s_{n-1}(z) - a_{n,n} c_n z^n.$$

Now suppose z_1 to be any point outside $|z| = 1$ at which $f(z)$ is defined; then, for $n \geqslant 1$, $\sigma_n(z) = 0$ at this point (and therefore A is efficient for the Taylor series at this point) if we choose

$$a_{n,n} = \left\{ f(z_1) - \sum_{s=0}^{n-1} c_s z_1^s \right\} \Big/ c_n z_1^n,$$

and we may suppose that z_1 is also chosen so that, for every n, $a_{n,n} \neq 0$.

We now show that, under the conditions (*a*) and (*b*) of the theorem, A is a T-matrix.

For $\lim_{n \to \infty} a_{n,k} = 0$ for every fixed k, and since

$$a_{n,n-1} = 1 - a_{n,n} = \left\{ \sum_{s=0}^{n} c_s z_1^s - f(z_1) \right\} \Big/ c_n z_1^n,$$

we have

$$\sum_{k=1}^{n} |a_{n,k}| \leqslant 2 \left| \frac{f(z_1)}{c_n z_1^n} \right| + \frac{2}{|c_n|} \sum_{s=0}^{n} |c_s| |z_1|^{s-n}$$

$$< \frac{2M'}{\delta} + 2M \sum_{p=0}^{n} |z_1|^{-p} < K$$

independently of n, as $|z_1| > 1$.

Also $\sum_{k=1}^{n} a_{n,k} = 1$; so that the proof that A is a T-matrix is complete.

Thus A sums all Taylor series satisfying the conditions of the theorem to $f(z)$ at all points z inside $|z| = 1$, and at the given point z_1 outside $|z| = 1$, A sums the series to $f(z_1)$.

Condition (*b*) is satisfied if, *e.g.*, $|c_s|$ $(s = 0, 1, 2, \ldots)$ is a monotonic increasing sequence; or if the c_s are all bounded and none of them is zero, while $c_n \to a \neq 0$ as $n \to \infty$. As in Ex. (i), A has an infinity of left-hand reciprocals, all of which satisfy

$$\overline{\lim_{n \to \infty}} \sqrt[n]{\left\{ \sum_{i-1}^{i_n} |{}^{-1}a_{n,i}| \right\}} > 1.$$

Thus a T-matrix may be:

(1) *inefficient at all points outside C, the circle of convergence of any Taylor series*; or

(2) *efficient only at one or more isolated points outside C*; or

(3) *efficient in a region outside C (the nature of the region depending on the Taylor series).*

8. It may be as well to point out that there is nothing contradictory between the above results and the property of *overconvergence* of a Taylor series.

Let $f(z) = \sum_{r=0}^{\infty} c_r z^r$ satisfy the conditions of Ostrowski's theorem on overconvergence; *i.e.*, suppose the radius of convergence to be 1, and that there are sequences p_n, q_n, such that $c_r = 0$ for $p_n < r < q_n$, where

$$q_n \geqslant (1+\vartheta) p_n \quad (n = 1, 2, \ldots, \infty),$$

ϑ being fixed and positive. Overconvergence may be regarded as the transformation by a row-finite T-matrix of a divergent sequence of complete partial sums of the Taylor series into a suitably selected convergent subsequence of these partial sums.

Thus $\sum_{k=0}^{\infty} a_{n,k} s_k(z) = s_{p_n}(z)$, where $a_{n,k} = 0 \ (k \neq p_n)$, $a_{n,p_n} = 1$.

In a domain of overconvergence (outside the circle of convergence), $s_{p_n}(z)$ tends to a limit, namely $f(z)$, as $n \to \infty$; but $s_k(z)$ diverges. That this does not contradict our results is seen at once from the fact that $a_{n,k}$ has no left-hand reciprocal; for owing to the hypothesis concerning gaps, $a_{n,k}$ has an infinity of zero columns*.

This example shows that an *upper* semi-T-matrix may be efficient for certain types of Taylor series in a *region* outside the circle of convergence.

9. The following result shows that inefficiency of a matrix A for Taylor series outside the circle of convergence is not necessarily due to the existence of a left-hand reciprocal of A satisfying suitable conditions.

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* Dienes, *loc. cit.*, 260.

(X). *If a row-finite K-matrix A, for which* $a_{n,k}=0$ *when* $k>k_n$, *satisfies the conditions*

$$(a) \qquad \overline{\lim_{n\to\infty}} \frac{1}{|a_{n,k_n}|} \sum_{i=1}^{k_n-1} |a_{n,i}-a_{n,i+1}| \leqslant 1,$$

and

$$(b) \qquad \lim_{n\to\infty} |\sqrt[k_n]{a_{n,k_n}}| \geqslant 1,$$

then A is inefficient for Σz^n *outside* $|z|=1$.

If $s_k(z)=\sum_{n=0}^{k-1} z^n$, then

$$z_n'=\sum_{k=1}^{k_n} a_{n,k} s_k(z)=\frac{A_n}{1-z}-\frac{1}{1-z}\sum_{k=1}^{k_n} a_{n,k} z^k,$$

where
$$A_n=\sum_{k=1}^{k_n} a_{n,k}.$$

If A is a K-matrix, A_n tends to a limit α as $n\to\infty$; and $\alpha=1$ if A is a T-matrix. Thus a necessary and sufficient condition for A to be efficient in this case is that

$$z_n''=\sum_{k=1}^{k_n} a_{n,k} z^k$$

should converge for some z such that $|z|\geqslant 1$.

A dominating last term may make this impossible.

By the partial summation formula, we have, if $z\neq 1$,

$$z_n''=\sum_{i=1}^{k_n-1} z\left(\frac{1-z^i}{1-z}\right)(a_{n,i}-a_{n,i+1})+z\left(\frac{1-z^{k_n}}{1-z}\right)a_{n,k_n};$$

i.e.,
$$\frac{z-1}{z} z_n''=a_{n,k_n} z^{k_n}+\sum_{i=1}^{k_n-1}(a_{n,i}-a_{n,i+1})z^i-a_{n,1}, \tag{9.1}$$

where $a_{n,1}\to\alpha_1$ if A is a K-matrix, and $\alpha_1=0$ if A is a T-matrix.

By condition (*a*) of the theorem, if $|z|>1$,

$$\overline{\lim_{n\to\infty}} \left| \frac{1}{a_{n,k_n} z^{k_n}} \sum_{i=1}^{k_n-1} (a_{n,i}-a_{n,i+1}) z^i \right| \leqslant \frac{1}{|z|}<1.$$

Now choose a positive ϵ so small that $(1-\epsilon)|z|>1$; by condition (*b*), $|\sqrt[k_n]{a_{n,k_n}}|>1-\epsilon$ for infinitely many n; and therefore, for this set of values of n, $|a_{n,k_n} z^{k_n}|\to\infty$.

For this set, the square bracket in

$$\frac{z-1}{z} z_n'' = a_{n,k_n} z^{k_n} \left[1 + \frac{1}{a_{n,k_n} z^{k_n}} \sum_{i=1}^{k_n-1} (a_{n,i} - a_{n,i+1}) z^i \right] - a_{n,1},$$

i.e., (9.1), cannot counterbalance the first factor on the right, and hence z_n'' (and thus also z_n') is divergent when $|z| > 1$.

This proves the result.

(XI). *If a row-finite T-matrix A, for which* $a_{n,k} = 0$ *for* $k > k_n$, *satisfies the conditions*

$$(a') \quad \overline{\lim_{n\to\infty}} \frac{1}{|a_{n,k_n}|} \sum_{i=1}^{k_n-1} |a_{n,i} - a_{n,i+1}| < 1, \quad \textit{and} \quad (b') \quad \underline{\lim_{n\to\infty}} |a_{n,k_n}| > 0,$$

then A is inefficient for Σz^n *outside* $|z| = 1$, *and does not limit its partial sums to the right value anywhere on the unit circle.*

For z_n'' must tend to 0 for efficiency at z when A is a T-matrix; the proof is then as in (X).

The following two results are special cases of (X).

(XII) *If, in a positive lower semi-K-matrix A, infinitely many rows form non-decreasing sequences such that*

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_{n,k} = \alpha > 0,$$

then A is inefficient for Σz^n *outside* $|z| = 1$.

Let n denote any one of the rows which satisfy the given conditions.

Then the conditions (a) and (b) of (X) are satisfied.

For, since $a_{n,k} \leqslant a_{n,k+1}$ ($k = 1, 2, \ldots, n-1$), and $a_{n,k} > 0$, (a) reduces to

$$\sum_{k=1}^{n-1} (a_{n,k+1} - a_{n,k}) \leqslant a_{n,n}(1+\epsilon_n),$$

where $\epsilon_n > 0$, $\epsilon_n \to 0$; *i.e.*, $a_{n,n} - a_{n,1} \leqslant a_{n,n}(1+\epsilon_n)$, which is true.

Also
$$\sum_{k=1}^{n} a_{n,k} = \alpha + \eta_n \leqslant n a_{n,n},$$

where $\eta_n \to 0$; hence $a_{n,n} \geqslant (\alpha+\eta_n)/n$, and thus ($b$) is satisfied.

Cesàro means of order r, where $0 < r \leqslant 1$, form a special case.

(XIII). *If, in a positive lower semi-K-matrix A, infinitely many rows form non-increasing sequences such that*

(i) $$\lim_{n\to\infty} \sum_{k=1}^{n} a_{n,k} = \alpha > 0,$$

and

(ii) $$\overline{\lim_{n\to\infty}} \frac{a_{n,1}}{a_{n,n}} \leqslant 2,$$

then A is inefficient for Σz^n *outside* $|z| = 1$.

For, since $a_{n,k} \geqslant a_{n,k+1}$ $(k = 1, 2, \ldots, n-1)$, and $a_{n,k} > 0$, (a) reduces to $a_{n,1} - a_{n,n} \leqslant a_{n,n}\,(1+\epsilon_n)$, $\epsilon_n > 0$, $\epsilon_n \to 0$; whence

$$\frac{a_{n,1}}{a_{n,n}} - 1 \leqslant 1 + \epsilon_n,$$

which is true, by condition (ii).

Again, from $n a_{n,1} \geqslant \sum_{k=1}^{n} a_{n,k} = \alpha + \eta_n$, we have

$$a_{n,n} \geqslant \frac{\alpha+\eta_n}{n(2+\epsilon_n)},$$

where $\eta_n \to 0$; *i.e.*, (b) is satisfied.

10. In this final section, we consider some questions connected with reciprocals of matrices, which evidently play a prominent part, not only in the preceding theory, but in most work in which matrices are involved. In the first place, if a T-matrix has a left-hand K-reciprocal, some examples suggest that this K-reciprocal is in reality a T-reciprocal. This is proved below to be always the case, and this leads in turn to the consideration of the nature of the product matrices AB and BA, where A is a T-matrix, and B is a K-matrix.

(XIV). *If A is a T-matrix and B is a K-matrix, the characteristic numbers* of AB are those of B.*

Denote the characteristic numbers of B by β, β_k, and those of AB by γ, γ_k, and write

$$c_{n,i} \equiv \sum_k a_{n,k} b_{k,i}, \quad C_n \equiv \sum_i c_{n,i} = \sum_i \sum_k a_{n,k} b_{k,i}. \tag{10.1}$$

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* P. Dienes, *The Taylor series*, 388.

From $$|C_n| \leqslant \sum_i \sum_k |a_{n,k}|\,|b_{k,i}| \leqslant \sum_k |a_{n,k}| N \leqslant MN,$$

we see that, in (10.1), the order of summation can be altered.

Thus we obtain

$$C_n = \sum_k a_{n,k} B_k, \quad \text{where} \quad B_k = \sum_i b_{k,i}, \tag{10.2}$$

which shows that the sequence C_n is the A-transform of the sequence B_k.

Since, by hypothesis, A is a T-matrix, and $\lim\limits_{k\to\infty} B_k = \beta$, we have

$$\gamma \equiv \lim_{n\to\infty} C_n = \beta. \tag{10.3}$$

Also, from the first equation of (10.1), for a fixed i, $c_{n,i}$ is the A-transform of $b_{k,i}$; and, by hypothesis, $\lim\limits_{k\to\infty} b_{k,i} = \beta_i$. Thus

$$\gamma_i \equiv \lim_{n\to\infty} c_{n,i} = \beta_i. \tag{10.4}$$

(10.3) and (10.4) prove the result.

We now try to apply the same argument to BA. Thus let

$$c'_{n,i} \equiv \sum_k b_{n,k}\, a_{k,i}, \quad C_n' \equiv \sum_i c'_{n,i} = \sum_i \sum_k b_{n,k}\, a_{k,i};$$

then by the above argument

$$C_n' = \sum_k b_{n,k} A_k, \quad \text{where} \quad A_k = \sum_i a_{k,i},$$

so that C_n' is the B-transform of $A_k \to 1$. Hence*

$$\gamma' \equiv \lim_{n\to\infty} C_n' = \beta \,.\, 1 + \sum_k \beta_k (A_k - 1). \tag{10.5}$$

If, as in most well-known examples, $A_k = 1$, then $\gamma' = \beta$, and so, if $\beta \neq 1$, BA is not a T-matrix.

Also, for a fixed i, $c'_{n,i}$ is the B-transform of $a_{k,i} \to 0$ as $k \to \infty$; *i.e.*,

$$\gamma_i' \equiv \lim_{n\to\infty} c'_{n,i} = \beta \,.\, 0 + \sum_k \beta_k (a_{k,i} - 0),$$

i.e. $$\gamma_i' = \sum_k a_{k,i} \beta_k. \tag{10.6}$$

(10.5) and (10.6) suggest that, in general, the characteristic numbers of BA are *not* those of B, so that there is a distinction between the properties of AB and BA.

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* P. Dienes, *loc. cit.*, 385.

BA will be a T-matrix if $\gamma' = 1$, $\gamma_i' = 0$ for every i; as an example of this, let A be the lower semi-T-matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \dots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \dots \\ \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & \end{pmatrix},$$

and take $\beta = \beta_1 = 1$, $\beta_2 = -1$, $\beta_k = 0$ for $k > 2$.

Then $A_k = 1$, and (10.5) gives $\gamma' = \beta = 1$, while (10.6) becomes $\gamma_i' = a_{1,i}\beta_1 + a_{2,i}\beta_2$, so that $\gamma_i' = 0$ for $i > 2$, since A is a lower semi-matrix; also $\gamma_1' = 1.1 + 1.(-1) = 0$, $\gamma_2' = 0.1 + 0.(-1) = 0$. Thus $\gamma_i' = 0$ for every i. We may take for B the matrix $b_{n,1} = 1$, $b_{n,2} = -1$ for every n, $b_{n,k} = 1/n$ $(3 \leqslant k \leqslant n+2)$, $b_{n,k} = 0$ $(k > n+2)$; *i.e.*,

$$\begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 & \dots \\ 1 & -1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ 1 & -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \dots \\ 1 & -1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \dots \\ \dots & \dots & \dots & \dots & \dots & & \\ \dots & \dots & \dots & \dots & \dots & & \end{pmatrix}.$$

A case in which we can make a more precise statement about the nature of BA is that in which A has a left-hand K-reciprocal, as is shown by the following result:

(XV). *If A is a T-matrix, B is a K-(but not a T-) matrix, and if A has a left-hand K-reciprocal, then BA is a K-, but not a T-, matrix.*

For any convergent z_k, we have

$$\sum_i b_{n,i} \sum_k a_{i,k} z_k = \sum_k \left(\sum_i b_{n,i} a_{i,k} \right) z_k.$$

If $\sum_k a_{i,k} z_k = z_i'$, then z_i' is convergent.

Now, by hypothesis, there is a convergent $z_i' \to z'$ such that

$$\sum_i b_{n,i} z_i' \to z'' \neq z'.$$

We therefore have to prove that this particular z_i' is the A-transform of a convergent sequence z_k; *i.e.*, given $a_{i,k}$ and z_i', we have to show that the system of equations

$$\sum_k a_{i,k} z_k = z_i' \tag{10.7}$$

has a convergent solution in z_k.

This is the case, since A has a left-hand K-reciprocal ${}^{-1}A$; for then, since A is a T-matrix,

$$\sum_i {}^{-1}a_{n,i} z_i' = z_n;$$

and since z_i' is convergent, so is its K-transform z_n; *i.e.*, (10.7) admits of a convergent solution.

COROLLARY. *Any K-reciprocal of a T-matrix is a T-matrix.*

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DIFFERENTIAL INVARIANTS FOR HIGHER PATH SPACES

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This paper is a continuation of a note by D. D. Kosambi*. The results, method, and notation of that memoir are used to deduce the basis from which all the differential invariants of the space K_n^σ associated with the differential equations

$$x^{(\sigma+1)i}+\alpha^i(x, x^{(1)}, \ldots, x^{(\sigma)}, t)=0 \quad (i=1, 2, \ldots, n;\ \sigma \geqslant 2) \tag{1}$$

are to be derived. Following Kosambi we use, besides the tensor summation convention for Latin indices, the following notation:

$$u^{(0)i}=u^i, \quad u^{(\gamma)i}=\frac{du^{(\gamma-1)i}}{dt},$$

$$f_{,i}=f_{(0)i}=\frac{\partial f}{\partial x^i}, \quad f_{(\gamma)i}=\frac{\partial f}{\partial x^{(\gamma)i}}.$$

The principal differential operators are defined by

$$\left.\begin{aligned} Du^i &= u^{(1)i}+\gamma_r{}^i u^r \\ Dv_i &= v_i^{(1)}-\gamma_i{}^r v_r \end{aligned}\right\} \text{ where } \gamma_r{}^i=\frac{1}{\sigma+1}\, a^i_{(\sigma)r}, \tag{2}$$

$$\underset{\gamma}{\nabla}_j=\frac{\partial}{\partial x^{(\gamma)j}}+\sum_{\rho=\gamma+1}^{\sigma} \overset{\rho}{\underset{\gamma}{Q}}{}_j{}^r \underset{\rho}{\nabla}_r \quad (\gamma>0), \tag{3}$$

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* D. D. Kosambi, "Path-spaces of higher order", *Quart. J. of Math.* 7 (1936), 97–104. I express my gratitude to Prof. Kosambi for his suggestions, guidance and help throughout the course of this investigation.

where the coefficients are given by

$$\underset{\rho}{\overset{\rho+1}{Q}}{}_j{}^i = -(\rho+1)\gamma_j{}^i, \tag{3a}$$

$$\underset{\gamma}{\overset{\rho}{Q}}{}_j{}^i = \underset{\gamma+1}{\overset{\rho+1}{Q}}{}_j{}^i - \left(\frac{d}{dt}\underset{\gamma+1}{\overset{\rho}{Q}}{}_j{}^i + \gamma_r{}^i \underset{\gamma+1}{\overset{\rho}{Q}}{}_j{}^r\right) \qquad (\gamma+1<\rho<\sigma), \tag{3b}$$

$$\underset{\gamma}{\overset{\sigma}{Q}}{}_j{}^i = -\alpha^i_{(\gamma+1)j} - \left(\frac{d}{dt}\underset{\gamma+1}{\overset{\sigma}{Q}}{}_j{}^i + \gamma_r{}^i \underset{\gamma+1}{\overset{\sigma}{Q}}{}_j{}^r\right) \quad (\gamma+1<\sigma). \tag{3c}$$

The operator $\underset{0}{\nabla}$ is defined by

$$\underset{0}{\nabla}{}_j u^i = u^i_{(0)j} + \sum_{\rho=1}^{\sigma} \underset{0}{\overset{\rho}{Q}}{}_j{}^r \underset{\rho}{\nabla}{}_r u^i + \left(\underset{1}{\nabla}{}_j \gamma_r{}^i\right) u^r. \tag{4}$$

Among the above Kosambian operators, it is only (4) that corresponds to a fully covariant derivative in as much as a summation for each index is present.

The "primary" tensors $\underset{\rho}{P}{}_j{}^i$ are given by the formula

$$\underset{\rho}{P}{}_j{}^i = \binom{\sigma+1}{\rho}\theta^{\sigma-\rho}(-\gamma_j{}^i) + \sum_{\gamma=\rho}^{\sigma}\binom{\gamma}{\rho}\alpha^i_{(\gamma)r}\theta^{\gamma-\rho-1}(-\gamma_j{}^r), \tag{5}$$

where the operator θ is defined by

$$\theta\beta_j{}^i = \frac{d}{dt}\beta_j{}^i - \beta_r{}^i\gamma_j{}^r, \tag{5a}$$

$$\theta^{-1}(-\gamma_r{}^i) = \delta_r{}^i \tag{5b}$$

and

$$\binom{\gamma}{\mu} = \frac{\gamma!}{\mu!(\gamma-\mu)!}. \tag{5c}$$

Throughout, the transformation group is taken to be

$$\bar{x}^i = F^i(x^1, x^2, \ldots, x^n), \quad \bar{t} = t, \tag{T}$$

$$F_j{}^i = \frac{\partial \bar{x}^i}{\partial x^j}, \qquad |F_j{}^i| \neq 0.$$

Using these equations, we can put (1) in the invariantive form

$$D^\sigma x^{(1)i} + \sum_{\rho=1}^{\sigma-1}\frac{\rho}{\sigma+1}\underset{\rho}{P}{}_r{}^i(D^{\sigma-1}x^{(1)r}) + \epsilon^i = 0, \tag{I}$$

where

$$\epsilon^i = a^i - \sum_{\rho=1}^{\sigma} \frac{\rho}{\sigma+1}\, a^i_{(\rho)r}\, x^{(\rho)r} \tag{6}$$

form the components of a vector.

This shows that the primary tensors $\underset{\gamma}{P}{}_j^i$ except $\underset{0}{P}{}_j^i$ are of the nature of torsion tensors, rather than of curvature tensors proper.

The main purpose of this note is to prove a theorem of Kosambi*.

THEOREM. *The differential invariants of the* K_n^σ *defined by equations* (1) *or* (I) *with respect to the transformation group* T *can all be generated from*

$$x^{(1)i}, \quad \underset{\gamma}{P}{}_j^i \;(\gamma = 0, 1, \ldots, \sigma-1), \quad \gamma^i_{j(\sigma)k}, \quad \text{and} \quad \underset{1}{\nabla}_k \underset{1}{\nabla}_j \gamma_r^i.$$

by the usual algebraic processes and the (complete) set of differential operators D, $\underset{\gamma}{\nabla}_k$.

Incidentally, it is to be noted that the operator $\partial/\partial t$ is not independent, but is derivable from D and $\underset{\gamma}{\nabla}_k$, as can be seen from the formula

$$\begin{aligned}
\frac{\partial u^i}{\partial t} = {} & D u^i - x^{(1)k}\left[\underset{0}{\nabla}_k u^i - \sum_{\rho=1}^{\sigma}\left\{\underset{\rho}{\nabla}_l u^i \sum_{r=0}^{\rho-2}(-1)^{r+1}\binom{\sigma-\rho+r}{r} D^r \underset{\sigma-\rho+r+1}{P}{}_k^l\right\}\right] \\
& - \sum_{\gamma=1}^{\sigma-1} D^\gamma x^{(1)k}\left[\underset{\gamma}{\nabla}_k u^i - \sum_{\rho=\gamma+1}^{\sigma} \underset{\rho}{\nabla}_l u^i \sum_{r=0}^{\rho-\gamma-2}(-1)^{r+1}\binom{\sigma-\rho+r}{r} D^r \underset{\sigma-\rho+\gamma+r+1}{P}{}_k^l\right] \\
& + u^i_{(\sigma)k}\left[\epsilon^k + \sum_{\rho=1}^{\sigma-1}\frac{\rho}{\sigma+1} \underset{\rho}{P}{}_l^k D^{\rho-1} x^{(1)l}\right] \\
& + x^{(1)k}\left[\underset{1}{\nabla}_k \gamma_r^i - \underset{1}{\nabla}_r \gamma_k^i\right] + \left[\underset{0}{\nabla}_r x^{(1)i}\right] u^r.
\end{aligned} \tag{7}$$

In the reduction formulae necessary for the proof we make use of the following devices.

We choose a special coordinate system in which $\gamma_j^i = 0$ along some portion of a path curve. From the law of transformation of γ_j^i it is seen that $\bar{\gamma}_k{}^i$ all vanish provided that the

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* Stated without proof by Prof. Kosambi in a letter to the author dated March 14, 1937.

$$\gamma_k{}^i \bar{F}_j{}^k + \frac{d}{dt} \bar{F}_j{}^i = 0.$$

This shows that a sufficient condition for the reducibility in question is the existence of an n-uple of vectors $\lambda^i_{(j)}$ that are all bi-parallel along the base and are equal to the coefficients of the inverse transformation to $F_j{}^i$, *i.e.* we must have

$$\lambda^i_{(j)} = \bar{F}_j{}^i, \quad D\lambda^i_{(j)} = 0.$$

For the transformations that we are considering this is always the case*.

Next, we use the method of indeterminate coefficients. When alternating all the differential operators taken in pairs, we first group together terms that can be expressed in terms of known tensors. The residual terms, which are not obviously expressible in terms of known tensors, but which together should necessarily be tensorial, we express in the form

$$l_1 T^1 + l_2 T^2 + l_3 T^3 + \ldots,$$

where l_1, l_2, ... are constants to be determined and T^1, T^2, ... are known groups of non-tensorial terms that occur in the residual expression. Then, making use of the particular combinations of T^1, T^2, ... which are expressible in terms of known tensors, we show that $l_1 T^1 + l_2 T^2 + \ldots$ can be expressed as a combination of known tensors.

1. The first differential invariant is $x^{(1)i}$. All the others are to be those that contain partial derivatives of α^i of various orders. The tensors $\underset{\gamma}{P}_j{}^i$ and $\gamma^i_{j(\sigma)k}$ in normal coordinates reduce to $\alpha^i_{(\gamma)j}$, $\gamma < \sigma$, and $\alpha^i_{(\sigma)j(\sigma)k}$. They contain derivatives of α^i of different orders and hence are independent. Since α^i contains only $x, x^{(1)}, \ldots, x^{\sigma}, t$, the $\underset{\gamma}{P}_j{}^i$ exhaust all invariants of the second rank. Corresponding to t, we have, of course, the vector $\partial\alpha^i/\partial t$, which will be shown later (*vide* 2.16) to depend primarily upon $D\epsilon^i$. As for the invariants of the third rank, the first is $\gamma^i_{j(\sigma)k}$. The $\underset{\gamma}{\nabla}_k \gamma_j{}^i$, for $\gamma \geqslant 2$, are also tensors†. But these are expressible in terms of the primary tensors and

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* D. D. Kosambi, " Parallelism and path-spaces ", *Math. Zeitschrift*, 37 (1933), 608–618.

† D. D. Kosambi, " Path-spaces of higher order ", *Quart. J. of Math.*, 7 (1936), 101.

$\gamma^i_{j(\sigma)k}$, as can be seen from

$$
\begin{aligned}
(1.1)\quad & \left[(\sigma+1)-\frac{1}{\sigma+1}\binom{\sigma+1}{\mu}^2\right]\underset{\mu}{\nabla}_k\gamma_j^{\,i} \quad (\mu>1) \\
&= \underset{\mu}{P}{}^i_{k(\sigma)j}+\frac{1}{\sigma+1}\binom{\sigma+1}{\mu}\underset{\mu}{P}{}^i_{j(\sigma)k} \\
&\quad +(\sigma+1)\sum_{\rho=\mu+1}^{\sigma}\underset{\rho}{\nabla}_r\gamma_j^{\,i}\sum_{r=0}^{\rho-\mu-2}(-1)^{r+1}\binom{\sigma-\rho+r}{r}D^r\underset{(\sigma-\rho+\mu+r+1)}{P_k^{\,r}} \\
&\quad +\binom{\sigma+1}{\mu}\sum_{\rho=\mu+1}^{\sigma}\underset{\rho}{\nabla}_r\gamma_k^{\,i}\sum_{r=0}^{\rho\ \mu-2}(-1)^{r+1}\binom{\sigma-\rho+r}{r}D^r\underset{(\sigma-\rho+\mu+r+1)}{P_j^{\,r}} \\
&\quad +\binom{\sigma+1}{\mu}\left[\sum_{r=1}^{\sigma-\mu}\binom{\sigma-\mu}{r}D^r\underset{(\mu+r)}{\nabla_j}\gamma_k^{\,i}\right. \\
&\qquad\qquad\qquad \left.+\frac{1}{\sigma+1}\binom{\sigma+1}{\mu}\sum_{r=1}^{\sigma-\mu}\binom{\sigma-\mu}{r}D^r\underset{(\mu+r)}{\nabla_k}\gamma_j^{\,i}\right] \\
&\quad +\sum_{\gamma=\mu+1}^{\sigma-1}\binom{\gamma}{\mu}\underset{\gamma}{P}{}_r^{\,i}\left[\sum_{r=0}^{\gamma-\mu-1}\binom{\gamma-\mu-1}{r}D^r\underset{(\sigma-\gamma+\mu+1+r)}{\nabla_j}\gamma_k^{\,r}\right. \\
&\qquad\qquad\qquad \left.+\frac{1}{\sigma+1}\binom{\sigma+1}{\mu}\sum_{r=0}^{\gamma-\mu-1}\binom{\gamma-\mu-1}{r}D^r\underset{(\sigma-\gamma+\mu+1+r)}{\nabla_k}\gamma_j^{\,r}\right].
\end{aligned}
$$

But $\underset{1}{\nabla}_k\gamma_j^{\,i}$ is of the nature of the Christoffel symbols of the Riemannian space and is not a tensor. It has the law of transformation

$$
(1.2)\qquad \underset{1}{\overline{\nabla}}_r\overline{\gamma}_l^{\,m}=\left(\underset{1}{\nabla}_k\gamma_j^{\,i}\right)\frac{\partial x^k}{\partial\overline{x}^r}\frac{\partial\overline{x}^m}{\partial x^i}\frac{\partial x^j}{\partial\overline{x}^l}+\frac{\partial^2\overline{x}^m}{\partial x^j\partial x^i}\frac{\partial x^i}{\partial\overline{x}^r}\frac{\partial x^j}{\partial\overline{x}^l}.
$$

From this it is clear that $\underset{1}{\nabla}_k\gamma_j^{\,i}-\underset{1}{\nabla}_j\gamma_k^{\,i}$ is a tensor. This is the usual torsion tensor in the affine geometry. We see from the following that even this can be expressed in terms of $\underset{\gamma}{P}{}_j^{\,i}$ and $\gamma^i_{j(\sigma)k}$.

$$
\begin{aligned}
(1.3)\quad & \underset{1}{\nabla}_j\gamma_k^{\,i}-\underset{1}{\nabla}_k\gamma_j^{\,i} \\
&= \frac{1}{\sigma+1}\underset{1}{P}{}^i_{j(\sigma)k}+\sum_{\rho=1}^{\sigma-1}\binom{\sigma-1}{\rho}D^\rho\underset{\rho+1}{\nabla}_k\gamma_j^{\,i} \\
&\qquad +\sum_{\rho=2}^{\sigma}\underset{\rho}{\nabla}_l\gamma_k^{\,i}\left[\sum_{r=0}^{\rho-3}(-1)^{r+1}\binom{\sigma-\gamma+r}{r}D^r\underset{\sigma-\rho+r+2}{P_j^{\,l}}\right] \\
&\qquad +\sum_{\gamma=2}^{\sigma-1}\frac{1}{\sigma+1}\gamma P_l^i\left[\sum_{r=0}^{\gamma-2}\binom{\gamma-2}{r}D^r\underset{\sigma-\gamma+2+r}{\nabla_k}\gamma_j^{\,l}\right].
\end{aligned}
$$

When we evaluate $\underset{\gamma}{\nabla}_p \underset{1}{\nabla}_k \gamma_j^i$, for $\gamma \geqslant 1$, it is seen that the last term in the right-hand side of (1.2), being merely a point function, vanishes under the operation $\underset{\gamma}{\nabla}_p$ and hence $\underset{\gamma}{\nabla}_p \underset{1}{\nabla}_k \gamma_j^i$ is a tensor. It is shown in formula (2.14) that $\underset{\gamma}{\nabla}_p \underset{1}{\nabla}_k \gamma_j^i$ can be expressed in terms of $\underset{\gamma}{P}{}_j^i$ and $\gamma^i_{j(\sigma)k}$ whenever $\gamma \geqslant 2$. But we shall show that $\underset{1}{\nabla}_k \underset{1}{\nabla}_j \gamma_r^i$ is a new tensor. If this is to depend upon the known tensors, it should primarily consist of $\underset{1}{\nabla}_k \underset{\sigma}{\nabla}_r \underset{1}{P}{}_j^i$ and some other known invariants. On actual evaluation of $\underset{1}{\nabla}_k \underset{\sigma}{\nabla}_r \underset{1}{P}{}_j^i$ we find

$$
\begin{aligned}
(1.4)\quad \underset{1}{\nabla}_k \underset{\sigma}{\nabla}_r \underset{1}{P}{}_j^i = {} & (\sigma+1)\left(\underset{1}{\nabla}_k \underset{1}{\nabla}_j \gamma_r^i - \underset{1}{\nabla}_k \underset{1}{\nabla}_r \gamma_j^i\right) \\
& -(\sigma+1)\sum_{\rho=1}^{\sigma-1}\binom{\sigma-1}{\rho}\underset{1}{\nabla}_k\left(D^\rho \underset{\rho+1}{\nabla}_r \gamma_j^i\right) \\
& -(\sigma+1)\sum_{\rho=2}^{\sigma}\underset{1}{\nabla}_k \underset{\rho}{\nabla}_l \gamma_r^i \sum_{r=0}^{\rho-3}(-1)^{r+1}\binom{\sigma-\gamma+r}{r} D^r \underset{\sigma-\rho+r+2}{P}{}_j^l\Big) \\
& -(\sigma+1)\sum_{\rho=2}^{\sigma}\underset{\rho}{\nabla}_l \gamma_r^i \sum_{r=0}^{\rho-3}(-1)^{r+1}\binom{\sigma-\gamma+r}{r}\underset{1}{\nabla}_k\left(D^r \underset{\sigma-\rho+r+2}{P}{}_j^l\right) \\
& -\sum_{\gamma=2}^{\sigma-1}\gamma \underset{1}{\nabla}_k \underset{\gamma}{P}{}_l^i\left[\sum_{\rho=0}^{\gamma-2}\binom{\gamma-2}{\rho} D^\rho \underset{\sigma-\gamma+2+\rho}{\nabla}_r \gamma_j^l\right] \\
& -\sum_{\gamma=2}^{\sigma-1}\gamma \underset{\gamma}{P}{}_l^i\left[\sum_{\rho=0}^{\gamma-2}\binom{\gamma-2}{\rho}\underset{1}{\nabla}_k\left(D^\rho \underset{\sigma-\gamma+2+\rho}{\nabla}_r \gamma_j^l\right)\right].
\end{aligned}
$$

In this, since $\underset{1}{\nabla}_k \underset{1}{\nabla}_j \gamma_r^i$ and $-\underset{1}{\nabla}_k \underset{1}{\nabla}_r \gamma_j^i$ occur with equal coefficients, it is impossible to eliminate either and solve for the other. Hence $\underset{1}{\nabla}_k \underset{1}{\nabla}_j \gamma_r^i$ cannot be expressed in terms of the basic tensors.

2. We now make use of the classical procedure of alternating all the fundamental operations that are tensor invariant. All the differential invariants should occur as coefficients on the right-hand side. We show that all these are derivable from the basis.

We require the following formulae:

$$
\begin{aligned}
(2.1)^*\quad \overset{\gamma}{\underset{0}{Q}}{}_j^i = {} & \sum_{\mu=0}^{\gamma-2}(-1)^\mu \binom{\sigma-\gamma+\mu-1}{\mu} D^\mu \overset{\sigma}{\underset{\sigma-\gamma+\mu}{Q}}{}_j^i + F(\gamma_j^i,\ D\gamma_j^i,\ \ldots,\ D^{\gamma-1}\gamma_j^i) \\
& + \phi\left(D^r \overset{\sigma}{\underset{\sigma-\mu}{Q}}{}_l^i\right)\psi(\gamma_j^l,\ D\gamma_j^l,\ \ldots,\ D^{\gamma-3}\gamma_j^l) \\
& (r=0,\ 1,\ \ldots,\ \gamma-\mu-1;\ \mu=2,\ 3,\ \ldots,\ \gamma-1),
\end{aligned}
$$

and the functions F and ψ vanish in the special coordinate system.

$$\begin{aligned}(2.2)^* \quad \overset{\sigma}{\underset{\sigma-\gamma}{Q}}{}_j^i = \sum_{r=1}^{\gamma-1} (-1)^r D^{r-1} \underset{\sigma-\gamma+r}{P}{}_j^i + F(\gamma_j^i,\ D\gamma_j^i,\ \ldots,\ D^{\gamma-1}\gamma_j^i) \\ +\psi\Big(D^r \underset{\mu}{P}{}_l^i\Big)\, \phi(\gamma_j^l,\ D\gamma_j^l,\ \ldots,\ D^{\gamma-3}\gamma_j^l) \\ (\mu = 1,\ 2,\ \ldots,\ \gamma-2;\ r = 0,\ 1,\ \ldots,\ \gamma-\mu-2),\end{aligned}$$

and hence from (2.1) and (2.2) we deduce

$$\begin{aligned}(2.3) \quad \overset{\gamma}{\underset{0}{Q}}{}_j^i = \sum_{r=0}^{\gamma-2} (-1)^{r+1} \binom{\sigma-\gamma+r}{r} D^r \underset{(\sigma-\gamma+r+1)}{P}{}_j^i + F(\gamma_j^i,\ D\gamma_j^i,\ \ldots,\ D^{\gamma-1}\gamma_j^i) \\ +\phi\Big(D^\rho \underset{\mu}{P}{}_l^i)\, \psi(\gamma_j^l,\ D\gamma_j^l,\ \ldots,\ D^{\gamma-3}\gamma_j^l\Big),\end{aligned}$$

$$\begin{aligned}(2.4)^* \quad \overset{\rho}{\underset{\mu}{Q}}{}_j^i = \sum_{r=0}^{\rho-\mu-2} \binom{\mu+r-1}{r} D^r \overset{\rho-\mu-r}{\underset{0}{Q}}{}_j^i + F(\gamma_j^i,\ D\gamma_j^i,\ \ldots,\ D^{\rho-\mu-1}\gamma_j^i) \\ +\phi\Big(D^r \overset{s}{\underset{0}{Q}}{}_r^i\Big)\, \psi(\gamma_j^r,\ D\gamma_j^r,\ \ldots,\ D^{\rho-\mu-3}\gamma_j^r) \\ (r = 0,\ 1,\ \ldots,\ \rho-\mu-s-1;\ s = 2,\ 3,\ \ldots,\ \rho-\mu-1).\end{aligned}$$

From (2.3) and (2.4) we get

$$\begin{aligned}(2.5) \quad \overset{\rho}{\underset{\mu}{Q}}{}_j^i = & \sum_{r=0}^{\rho-\mu-2} (-1)^{r+1} \binom{\sigma-\rho+r}{r} D^r \underset{(\sigma-\rho+\mu+r+1)}{P}{}_j^i \\ & + F(\gamma_j^i,\ D\gamma_j^i,\ \ldots,\ D^{\rho-\mu-1}\gamma_j^i) \\ & +\phi_1\Big(D^\alpha \underset{\beta}{P}{}_r^i\Big)\, \phi_2(\gamma_j^r,\ D\gamma_j^r,\ \ldots,\ D^{\rho-\mu-3}\gamma_j^r) \\ & +\psi_1\Big(D^\alpha \underset{\beta}{P}{}_l^i\Big)\, \psi_2(\gamma_r^l,\ D\gamma_r^l,\ \ldots,\ D^{s+r-1}\gamma_r^l)\, \psi_3(\gamma_j^r,\ D\gamma_j^r,\ \ldots,\ D^{\rho-\mu-3}\gamma_j^r) \\ & +\psi_4(\gamma_r^i,\ D\gamma_r^i,\ \ldots,\ D^{s+1-r}\gamma_r^i)\, \psi_5(\gamma_j^r,\ D\gamma_j^r,\ \ldots,\ D^{\rho-\mu-3}\gamma_j^r),\end{aligned}$$

$$(2.6)^* \qquad \underset{\mu}{\nabla}_k D^r \gamma_j^i \Big|_0 = \sum_{r=0}^{\gamma} \binom{\gamma}{r} D^r \underset{\mu-\gamma+r}{\nabla}_k\ \gamma_j^i.$$

A bar and a subscript 0 denote the value in the special coordinate system.

---

* All formulae with an asterisk are established by the method of induction.

Differentiating (2.5) and making use of (2.6), we get

$$(2.7)\quad \underset{\gamma}{\nabla}_k \underset{\mu}{\overset{\rho}{Q}}{}_j^i \Big|_0 = \sum_{r=0}^{\rho-\mu-2} (-1)^{r+1} \binom{\sigma-\rho+r}{r} \underset{\gamma}{\nabla}_k D^r \underset{(\sigma-\rho+\mu+r+1)}{P_j^i}$$

$$+F\Big[\underset{\gamma}{\nabla}_k \gamma_j^i,\ \ldots,\ \sum_{r=0}^{\rho-\mu-1} \binom{\rho-\mu-1}{r} D^r \underset{(\gamma-\rho+\mu+1+r)}{\nabla_k} \gamma_j^i\Big]$$

$$+\phi\Big(D^\alpha \underset{\beta}{P}{}_r^i\Big)\Big[\underset{\gamma}{\nabla}_k \gamma_j^r,\ \ldots,\ \sum_{r=0}^{\rho-\mu-3} \binom{\rho-\mu-3}{r} D^r \underset{\gamma-\rho+\mu+3+r}{\nabla_k} \gamma_j^r\Big].$$

$$(2.8)\quad \Big[\underset{0}{\nabla}_j \gamma_k^i - \underset{0}{\nabla}_k \gamma_j^i\Big] + \gamma_r^i \Big[\underset{1}{\nabla}_k \gamma_j^r - \underset{1}{\nabla}_j \gamma_k^r\Big] + \gamma_k^r \underset{1}{\nabla}_j \gamma_r^i - \gamma_j^r \underset{1}{\nabla}_k \gamma_r^i$$

$$= \frac{1}{\sigma+2}\Big[\underset{0}{P}{}_j^i(\sigma)\,k - \underset{0}{P}{}_k^i(\sigma)\,j\Big] + \frac{1}{\sigma+2} \sum_{\rho=1}^{\sigma-1} \binom{\sigma}{\rho} D^\rho \Big[\underset{\rho}{\nabla}_k \gamma_j^i - \underset{\rho}{\nabla}_j \gamma_k^i\Big]$$

$$+ \frac{1}{\sigma+2} \sum_{\gamma=1}^{\sigma-1} \underset{\gamma}{P}{}_l^i \Big[\sum_{\rho=0}^{\gamma-2} \binom{\gamma-1}{\rho} D^\rho \Big(\underset{\sigma-\gamma+1+\rho}{\nabla_k} \gamma_j^l - \underset{\sigma-\gamma+1+\rho}{\nabla_j} \gamma_k^l\Big)\Big]$$

$$+ \frac{\sigma+1}{\sigma+2} \sum_{\rho=2}^{\sigma} \Big[\underset{\rho}{\nabla}_l \gamma_k^r \Big\{\sum_{r=0}^{\rho-2} (-1)^{r+1} \binom{\sigma-\rho+r}{r} D^r \underset{\sigma-\rho+r+1}{P_j^l}\Big\}$$

$$- \underset{\rho}{\nabla}_l \gamma_j^r \Big\{\sum_{r=0}^{\rho-2} (-1)^{r+1} \binom{\sigma-\rho+r}{r} D^r \underset{\sigma-\rho+r+1}{P^l{}_k}\Big\}\Big],$$

$$(2.9)\quad \frac{(\sigma+1)(\sigma^2-4)}{2\sigma} B^i_{jk}$$

$$= \Big(\underset{\sigma-1}{\nabla}_j \underset{1}{P}{}_k^i + \frac{2}{\sigma} \underset{\sigma-1}{\nabla}_k \underset{1}{P}{}_j^i\Big) - \Big(\underset{1}{\nabla}_k \underset{\sigma-1}{P}{}_j^i + \frac{2}{\sigma} \underset{1}{\nabla}_j \underset{\sigma-1}{P}{}_k^i\Big)$$

$$+ (\sigma+1)(\sigma-2)\, D\Big(\underset{1}{\nabla}_j \gamma_k^i - \underset{1}{\nabla}_k \gamma_j^i\Big)$$

$$+ \sum_{\rho=3}^{\sigma} \Big[\underset{\rho}{\nabla}_r \underset{\sigma-1}{P}{}_j^i + \frac{\sigma(\sigma+1)}{2} \Big(\underset{\rho-1}{\nabla}_r \gamma_j^i + D \underset{\rho}{\nabla}_r \gamma_j^i\Big)\Big]$$

$$\times \Big[\sum_{r=0}^{\rho-3} (-1)^{r+1} \binom{\sigma-\rho+r}{r} D^r \underset{\sigma-\rho+r+2}{P_k^r}\Big]$$

$$+ \frac{2}{\sigma} \sum_{\rho=3}^{\sigma} \Big[\underset{\rho}{\nabla}_r \underset{\sigma-1}{P}{}_k^i + \frac{\sigma(\sigma+1)}{2} \Big(\underset{\rho-1}{\nabla}_r \gamma_k^i + D \underset{\rho}{\nabla}_r \gamma_k^i\Big)\Big]$$

$$\times \Big[\sum_{r=0}^{\rho-3} (-1)^{r+1} \binom{\sigma-\rho+r}{r} D^r \underset{\sigma-\rho+r+2}{P_j^r}\Big]$$

$$+ (\sigma+1) \sum_{\rho=2}^{\sigma-1} \binom{\sigma-1}{\rho} D^\rho \underset{\rho}{\nabla}_j \gamma_k^i + \frac{2}{\sigma} (\sigma+1) \sum_{\rho=2}^{\sigma-1} \binom{\sigma-1}{\rho} D^\rho \underset{\rho}{\nabla}_k \gamma_j^i$$

$$+ \sum_{\gamma=2}^{\sigma-1} \gamma \underset{\gamma}{P}{}_r^i \Big[\sum_{\rho=0}^{\gamma-2} \binom{\gamma-2}{\rho} D^\rho \underset{(\sigma-\gamma+1+\rho)}{\nabla_j} \gamma_k^r$$

$$+ \frac{2}{\sigma} \sum_{\rho=0}^{\gamma-2} \binom{\gamma-2}{\rho} D^\rho \underset{\sigma-\gamma+1+\rho}{\nabla_k} \gamma_j^r\Big],$$

where

$$B^i_{jk} = \underset{0}{\nabla}_k \gamma_j^{\ i} - D\underset{1}{\nabla}_k \gamma_j^{\ i} - \left(\underset{1}{\nabla}_k \gamma_r^{\ i}\right)\gamma_j^{\ r} + \left(\underset{1}{\nabla}_k \gamma_j^{\ r}\right)\gamma_r^{\ i}.$$

Now, with the aid of the above formulae, let us proceed with our alternations. We always evaluate in the special coordinate system.

$$(2.10)\quad \underset{0}{\nabla}_k Du^i - D\underset{0}{\nabla}_k u^i$$

$$= \sum_{\rho=1}^{\sigma} \underset{\rho}{\nabla}_r u^i \left[\sum_{r=0}^{\rho-1} (-1)^{r+1} \binom{\sigma-\rho+r}{r} D^r \underset{\sigma-\rho+r}{P_k^{\ r}}\right] + u^r\,[B^i_{rk}].$$

Hence this contains only the $\underset{\gamma}{P_j^{\ i}}$ and $\gamma^i_{j(\sigma)k}$.

$$(2.11)\quad \underset{\sigma}{\nabla}_k \underset{\mu}{\nabla}_j u^i - \underset{\mu}{\nabla}_j \underset{\sigma}{\nabla}_k u^i \quad (\mu > 0)$$

$$= \sum_{\rho=\mu+1}^{\sigma} \overset{\rho}{\underset{\mu}{Q}}{}_j^{\ r} \left[\underset{\sigma}{\nabla}_k \underset{\rho}{\nabla}_r u^i - \underset{\rho}{\nabla}_r \underset{\sigma}{\nabla}_k u^i\right] + \sum_{\rho=\mu+1}^{\sigma} \underset{\rho}{\nabla}_r u^i \left[\underset{\sigma}{\nabla}_k \overset{\rho}{\underset{\mu}{Q}}{}_j^{\ r}\right],$$

$$\overset{\rho}{\underset{\mu}{Q}}{}_j^{\ r}\Big|_0 = \sum_{r=0}^{\rho-\mu-2} (-1)^{r+1} \binom{\sigma-\rho+r}{r} D^r \underset{(\sigma-\rho+\mu+r+1)}{P_j^{\ r}}.$$

In $\underset{\sigma}{\nabla}_k \overset{\rho}{\underset{\mu}{Q}}{}_j^{\ r}\Big|_0$ we get, besides the curvature tensors, terms of the form $\underset{\phi}{\nabla}_k \gamma_j^{\ r}$ (*vide* Formula 2.7), where $\phi \geqslant \sigma-\rho+\mu+1$. Here $\rho < \sigma$ and $\mu \geqslant 1$ and hence $\sigma-\rho+\mu+1 \geqslant 2$.

Thus we get only terms of the form $\underset{2}{\nabla}_k \gamma_j^{\ i}$, $\underset{3}{\nabla}_k \gamma_j^{\ i}$, ..., and these are known tensors.

$$(2.12)\quad \underset{\sigma}{\nabla}_k \underset{0}{\nabla}_j u^i - \underset{0}{\nabla}_j \underset{\sigma}{\nabla}_k u^i$$

$$= \sum_{\rho=1}^{\sigma} \overset{\rho}{\underset{0}{Q}}{}_j^{\ r} \left[\underset{\sigma}{\nabla}_k \underset{\rho}{\nabla}_r u^i - \underset{\rho}{\nabla}_r \underset{\sigma}{\nabla}_k u^i\right] + \sum_{\rho=1}^{\sigma-1} \underset{\rho}{\nabla}_r u^i \left[\underset{\sigma}{\nabla}_k \overset{\rho}{\underset{0}{Q}}{}_j^{\ r}\right]$$

$$+ \underset{\sigma}{\nabla}_r u^i \left[\underset{\sigma}{\nabla}_k \overset{\sigma}{\underset{0}{Q}}{}_j^{\ r} + \underset{1}{\nabla}_j \gamma_k^{\ r}\right] + \left[\underset{\sigma}{\nabla}_k \underset{1}{\nabla}_j \gamma_r^{\ i}\right] u^r.$$

$\underset{\sigma}{\nabla}_k \overset{\rho}{\underset{0}{Q}}{}_j^{\ r}\Big|_0$ contains terms of the form $\underset{\phi}{\nabla}_k \gamma_j^{\ r}$, where $\phi \geqslant \sigma-\rho+1$. Here $\rho \leqslant \sigma-1$ and hence $\phi \geqslant 2$. In $\underset{\sigma}{\nabla}_k \overset{\sigma}{\underset{0}{Q}}{}_j^{\ r}\Big|_0$, $\phi \geqslant 1$, and hence there is one term of the form $lD^{\alpha}\underset{1}{\nabla}_k\gamma_j^r$. The rest are known tensors. Hence we should have

$$lD^{\alpha}\underset{1}{\nabla}_k\gamma_j^r+\underset{1}{\nabla}_j\gamma_k^r=\text{tensor}.$$

But $\underset{1}{\nabla}_k\gamma_j^r-\underset{1}{\nabla}_j\gamma_k^r$ is a tensor, by formula (1.3). Adding, we see that $(lD^{\alpha}+1)\underset{1}{\nabla}_k\gamma_j^r$ is a tensor. This is impossible unless $\alpha=0$ and $l=-1$, *i.e.* in $\underset{\sigma}{\nabla}_k\underset{0}{\overset{\sigma}{Q}}{}_j^r\Big|_0$ we get, besides known tensors, the tensor $\underset{1}{\nabla}_j\gamma_k^r-\underset{1}{\nabla}_k\gamma_j^r$, but this is expressed in terms of known tensors by formula (1.3).

Next

$$\underset{\sigma}{\nabla}_k\underset{1}{\nabla}_j\gamma_r^i-\underset{1}{\nabla}_j\underset{\sigma}{\nabla}_k\gamma_r^i=\sum_{\rho=2}^{\sigma}\underset{1}{\overset{\rho}{Q}}{}_j^l\left[\underset{\rho}{\nabla}_l\underset{\sigma}{\nabla}_k\gamma_r^i-\underset{\sigma}{\nabla}_k\underset{\rho}{\nabla}_l\gamma_r^i\right]+\sum_{\rho=2}^{\sigma}\underset{\rho}{\nabla}_l\gamma_r^i\left[\underset{\sigma}{\nabla}_k\underset{1}{\overset{\rho}{Q}}{}_j^l\right].$$

From this it is easily seen that $\underset{\sigma}{\nabla}_k\underset{1}{\nabla}_j\gamma_r^i$ is expressible in terms of known tensors.

$$(2.13)\quad \underset{\gamma}{\nabla}_k\underset{\mu}{\nabla}_j u^i-\underset{\mu}{\nabla}_j\underset{\gamma}{\nabla}_k u^i\quad(\gamma\geqq\mu\geqq1)$$

$$=\sum_{\rho=\gamma+1}^{\sigma}\underset{\gamma}{\overset{\rho}{Q}}{}_k^r\left[\underset{\rho}{\nabla}_r\underset{\mu}{\nabla}_j u^i-\underset{\mu}{\nabla}_j\underset{\rho}{\nabla}_r u^i\right]+\sum_{\rho=\mu+1}^{\sigma}\underset{\mu}{\overset{\rho}{Q}}{}_j^r\left[\underset{\gamma}{\nabla}_k\underset{\rho}{\nabla}_r u^i-\underset{\rho}{\nabla}_r\underset{\gamma}{\nabla}_k u^i\right]$$

$$-\sum_{\rho=\gamma+1}^{\sigma}\underset{\gamma}{\overset{\rho}{Q}}{}_k^r\sum_{\delta=\mu+1}^{\sigma}\underset{\mu}{\overset{\delta}{Q}}{}_j^l\left[\underset{\rho}{\nabla}_r\underset{\delta}{\nabla}_l u^i-\underset{\delta}{\nabla}_l\underset{\rho}{\nabla}_r u^i\right]$$

$$+\sum_{\rho=\mu+1}^{\gamma}\underset{\rho}{\nabla}_r u^i\overset{(A)}{\left[\underset{\gamma}{\nabla}_k\underset{\mu}{\overset{\rho}{Q}}{}_j^r-\sum_{\delta=\gamma+1}^{\sigma}\underset{\gamma}{\overset{\delta}{Q}}{}_k^l\underset{\delta}{\nabla}_l\underset{\mu}{\overset{\rho}{Q}}{}_j^r\right]}$$

$$+\sum_{\rho=\gamma+1}^{\sigma}\underset{\rho}{\nabla}_r u^i\overset{(B)}{\left[\underset{\gamma}{\nabla}_k\underset{\mu}{\overset{\rho}{Q}}{}_j^r-\underset{\mu}{\nabla}_j\underset{\gamma}{\overset{\rho}{Q}}{}_k^r+\sum_{\delta=\mu+1}^{\sigma}\underset{\mu}{\overset{\delta}{Q}}{}_j^l\underset{\delta}{\nabla}_l\underset{\gamma}{\overset{\rho}{Q}}{}_k^r\right.}$$

$$\left.-\sum_{\delta=\gamma+1}^{\sigma}\underset{\gamma}{\overset{\delta}{Q}}{}_k^l\underset{\delta}{\nabla}_l\underset{\mu}{\overset{\rho}{Q}}{}_j^r\right],$$

where (A) and (B) denote the whole expressions within the brackets over which they stand.

In the expression (A), $\underset{\gamma}{\nabla}_k\underset{\mu}{\overset{\rho}{Q}}{}_j^r\Big|_0$ contains only known tensors. For we get terms of the form $\underset{\phi}{\nabla}_k \gamma_j^r$, where $\phi \geqslant \gamma - \rho + \mu + 1$. But $\rho \leqslant \gamma$, $\mu \geqslant 1$, and hence $\phi \geqslant 2$.

Similarly in the term $\underset{\delta}{\nabla}_l \overset{\rho}{\underset{\mu}{Q}}{}_j^r \Big|_0$. For here $\phi \geqslant \delta - \rho + \mu + 1$. But $\delta \geqslant \gamma + 1$, $\rho \leqslant \gamma$, and hence $\phi \geqslant 3$.

Let us now consider the expression (B). In this also terms of the form $\underset{\phi}{\nabla}_k \gamma_j^i$ appear; but we are not able to discover the lower limit for ϕ.

All the non-tensorial terms in $\underset{\gamma}{\nabla}_k \overset{\rho}{\underset{\mu}{Q}}{}_j^r \Big|_0$ can be classified as

$$\begin{aligned} & l_1 D^{\alpha_1} \underset{1}{\nabla}_k \gamma_j^r + m_1 D^{\beta_1} \underset{0}{\nabla}_k \gamma_j^r \\ & \quad + l_2 \left(D^{\alpha_2} \underset{0}{\nabla}_l \gamma_j^r \right) \underset{\alpha_2}{T}{}_k^l + m_2 \left(D^{\beta_2} \underset{1}{\nabla}_l \gamma_j^r \right) \underset{\beta_2}{\bar{T}}{}_k^l \\ & \quad + \phi \left(D^r \underset{\beta}{P}{}_l^r \right) \Big[l_3 D^{\alpha_3} \underset{1}{\nabla}_k \gamma_j^l + m_3 D^{\beta_3} \underset{0}{\nabla}_k \gamma_j^l \\ & \qquad\qquad + l_4 \left(D^{\alpha_4} \underset{1}{\nabla}_m \gamma_j^l \right) \underset{\alpha_4}{T}{}_k^m + m_4 \left(D^{\beta_4} \underset{0}{\nabla}_m \gamma_j^l \right) \underset{\beta_4}{\bar{T}}{}_k^m \Big]. \end{aligned}$$

Similarly for $\underset{\mu}{\nabla}_j \overset{\rho}{\underset{\gamma}{Q}}{}_k^r \Big|_0$.

Non-tensorial terms in $\overset{\delta}{\underset{\mu}{Q}}{}_j^l \underset{\delta}{\nabla}_l \overset{\rho}{\underset{\gamma}{Q}}{}_k^r \Big|_0$ are

$$\begin{aligned} & \phi \left(D^{\alpha} \underset{\beta}{P}{}_j^l \right) \Big[p_1 D^{\epsilon_1} \underset{1}{\nabla}_l \gamma_k^r + p_2 D^{\epsilon_2} \underset{0}{\nabla}_l \gamma_k^r \\ & \qquad + p_3 \left(D^{\epsilon_3} \underset{0}{\nabla}_m \gamma_k^r \right) \underset{\epsilon_3}{T}{}_l^m + p_4 \left(D^{\epsilon_4} \underset{1}{\nabla}_m \gamma_k^r \right) \underset{\epsilon_4}{\bar{T}}{}_l^m \\ & \qquad + \psi \left(D^r \underset{s}{P}{}_m^r \right) \Big\{ p_1' D^{\epsilon_1'} \underset{1}{\nabla}_l \gamma_k^m + p_2' D^{\epsilon_2'} \underset{0}{\nabla}_l \gamma_k^m \\ & \qquad\qquad + p_3' \left(D^{\epsilon_3'} \underset{1}{\nabla}_p \gamma_k^m \right) \underset{\epsilon_3'}{T}{}_l^p + p_4' \left(D^{\epsilon_4'} \underset{0}{\nabla}_p \gamma_k^m \right) \underset{\epsilon_4'}{\bar{T}}{}_l^p \Big\} \Big]. \end{aligned}$$

Similarly for $\overset{\delta}{\underset{\gamma}{Q}}{}_k^l \underset{\delta}{\nabla}_l \overset{\rho}{\underset{\mu}{Q}}{}_j^r \Big|_0$.

Here T_k^l stands for a combination of the curvature tensors $\underset{\gamma}{P}{}_k^l$.

In these, we find that all the non-tensorial terms are of the form $\underset{1}{\nabla}_k \gamma_j^i$, $\underset{0}{\nabla}_k \gamma_j^i$, $\underset{1}{\nabla}_j \gamma_k^i$, $\underset{0}{\nabla}_j \gamma_k^i$ and the most general combination of these is

$$lD^\alpha \underset{1}{\nabla}_k \gamma_j^i + mD^\beta \underset{0}{\nabla}_k \gamma_j^i + pD^\gamma \underset{1}{\nabla}_j \gamma_k^i + qD^\delta \underset{0}{\nabla}_j \gamma_k^i.$$

This must necessarily be a tensor.

We know that

$$\underset{1}{\nabla}_k \gamma_j^i - \underset{1}{\nabla}_j \gamma_k^i$$

is a (known) tensor, by formula (1.3). Therefore

$$lD^\alpha \underset{1}{\nabla}_k \gamma_j^i - lD^\alpha \underset{1}{\nabla}_j \gamma_k^i = \text{tensor.}$$

Also
$$mD^\beta \underset{0}{\nabla}_k \gamma_j^i - mD^\beta \underset{0}{\nabla}_j \gamma_k^i = \text{tensor.}$$

since $\underset{0}{\nabla}_k \gamma_j^i - \underset{0}{\nabla}_j \gamma_k^i$ is a tensor from formula (2.8).

Adding the two we get

$$lD^\alpha \underset{1}{\nabla}_k \gamma_j^i + mD^\beta \underset{0}{\nabla}_k \gamma_j^i - lD^\alpha \underset{1}{\nabla}_j \gamma_k^i - mD^\beta \underset{0}{\nabla}_j \gamma_k^i = \text{tensor.}$$

But, by hypothesis,

$$lD^\alpha \underset{1}{\nabla}_k \gamma_j^i + mD^\beta \underset{0}{\nabla}_k \gamma_j^i + pD^\gamma \underset{1}{\nabla}_j \gamma_k^i + qD^\delta \underset{0}{\nabla}_j \gamma_k^i = \text{tensor.}$$

Therefore
$$(pD^\gamma + lD^\alpha) \underset{1}{\nabla}_j \gamma_k^i + (q\,D^\delta + mD^\beta) \underset{0}{\nabla}_j \gamma_k^i = \text{tensor.}$$

But
$$-D \underset{1}{\nabla}_j \gamma_k^i + \underset{0}{\nabla}_j \gamma_k^i$$

is a (known) tensor by formula (2.9). This is impossible unless

$$\left.\begin{aligned} qD^\delta + mD^\beta &= tD^r, \\ pD^\gamma + lD^\alpha &= -tD^{r+1}, \end{aligned}\right\}$$

i.e.
$$\left.\begin{aligned} \delta = \beta = r \quad &\text{and} \quad q + m = t, \\ \gamma = \alpha = r+1 \quad &\text{and} \quad p + l = -t. \end{aligned}\right\}$$

The most general combination reduces to

$$lD^{r+1} \underset{1}{\nabla}_k \gamma_j^i + mD^r \underset{0}{\nabla}_k \gamma_j^i - (l+t)\,D^{r+1} \underset{1}{\nabla}_j \gamma_k^i + (t-m)\,D^r \underset{0}{\nabla}_j \gamma_k^i,$$

i.e. to

$$lD^{r+1}\left[\underset{1}{\nabla}_k \gamma_j{}^i - \underset{1}{\nabla}_j \gamma_k{}^i\right] + mD^r\left[\underset{0}{\nabla}_k \gamma_j{}^i - \underset{0}{\nabla}_j \gamma_k{}^i\right] + tD^r\left[\underset{0}{\nabla}_j \gamma_k{}^i - D\underset{1}{\nabla}_j \gamma_k{}^i\right],$$

and this is a combination of known tensors. Hence we find that even the expression (B) contains only known tensors.

$$\begin{aligned}
(2.14)\quad & \underset{\gamma}{\nabla}_k \underset{0}{\nabla}_j u^i - \underset{0}{\nabla}_j \underset{\gamma}{\nabla}_k u^i \\
&= \sum_{\rho=\gamma+1}^{\sigma} \overset{\rho}{\underset{\gamma}{Q}}{}_k{}^r \left[\underset{\rho}{\nabla}_r \underset{0}{\nabla}_j u^i - \underset{0}{\nabla}_j \underset{\rho}{\nabla}_r u^i\right] + \sum_{\rho=1}^{\sigma} \overset{\rho}{\underset{0}{Q}}{}_j{}^r \left[\underset{\gamma}{\nabla}_k \underset{\rho}{\nabla}_r u^i - \underset{\rho}{\nabla}_r \underset{\gamma}{\nabla}_k u^i\right] \\
&\quad - \sum_{\rho=\gamma+1}^{\sigma} \overset{\rho}{\underset{\gamma}{Q}}{}_k{}^r \sum_{\delta=1}^{\sigma} \overset{\delta}{\underset{0}{Q}}{}_j{}^l \left[\underset{\rho}{\nabla}_r \underset{\delta}{\nabla}_l u^i - \underset{\delta}{\nabla}_l \underset{\rho}{\nabla}_r u^i\right] \\
&\quad + \sum_{\rho=1}^{\gamma-1} \underset{\rho}{\nabla}_r u^i \overset{(A)}{\left[\underset{\gamma}{\nabla}_k \overset{\rho}{\underset{0}{Q}}{}_j{}^r - \sum_{\delta=\gamma+1}^{\sigma} \overset{\delta}{\underset{\gamma}{Q}}{}_k{}^l \underset{\delta}{\nabla}_l \overset{\rho}{\underset{0}{Q}}{}_j{}^r\right]} \\
&\quad + \underset{\gamma}{\nabla}_r u^i \overset{(B)}{\left[\underset{\gamma}{\nabla}_k \overset{\gamma}{\underset{0}{Q}}{}_j{}^r + \underset{1}{\nabla}_j \gamma_k{}^r - \sum_{\delta=\gamma+1}^{\sigma} \overset{\delta}{\underset{\gamma}{Q}}{}_k{}^l \underset{\delta}{\nabla}_l \overset{\gamma}{\underset{0}{Q}}{}_j{}^r\right]} \\
&\quad - \sum_{\rho=\gamma+1}^{\sigma} \underset{\rho}{\nabla}_r u^i \overset{(C)}{\left[\underset{0}{\nabla}_j \overset{\rho}{\underset{\gamma}{Q}}{}_k{}^r - \underset{\gamma}{\nabla}_k \overset{\rho}{\underset{0}{Q}}{}_j{}^r + \sum_{\delta=\gamma+1}^{\sigma} \overset{\delta}{Q}{}_k{}^l \nabla_l \overset{\rho}{\underset{0}{Q}}{}_j{}^r \right.} \\
&\qquad\qquad \left. - \sum_{\delta=1}^{\sigma} \overset{\delta}{\underset{0}{Q}}{}_j{}^l \underset{\delta}{\nabla}_l \overset{\rho}{\underset{\gamma}{Q}}{}_k{}^r + \left(\underset{1}{\nabla}_j \gamma_k{}^l\right) \overset{\rho}{\underset{\gamma}{Q}}{}_l{}^r\right] \\
&\quad + u^r \overset{(D)}{\left[\underset{\gamma}{\nabla}_k \underset{1}{\nabla}_j \gamma_r{}^i - \sum_{\rho=\gamma+1}^{\sigma} \overset{\rho}{\underset{\gamma}{Q}}{}_k{}^l \underset{\rho}{\nabla}_l \underset{1}{\nabla}_j \gamma_r{}^i\right]},
\end{aligned}$$

where, as previously, (A), (B), (C), (D) denote the whole expressions within the brackets over which they stand. The expressions (A), (B), (C), (D) have to be tensors.

Expression (A).

In $\underset{\gamma}{\nabla}_k \overset{\rho}{\underset{0}{Q}}{}_j{}^r \Big|_0$ terms of the form $\underset{\phi}{\nabla}_k \gamma_j{}^r$ appear, where $\phi \geqslant \gamma-\rho+1$. But $\rho<\gamma-1$ and hence $\phi \geqslant 2$. In $\underset{\delta}{\nabla}_l \overset{\rho}{\underset{0}{Q}}{}_j{}^r \Big|_0$, $\phi \geqslant \delta-\rho+1$. But $\delta \geqslant \gamma+1$, $\rho \leqslant \gamma-1$ and hence $\phi \geqslant 3$.

So (A) consists only of known tensors.

Expression (*B*).

In $\underset{\delta}{\nabla}_l \overset{\gamma}{\underset{0}{Q}}{}_j^r \Big|_0$, $\phi \geqslant \delta - \gamma + 1$, $\delta > \gamma + 1$ and hence $\phi \geqslant 2$. $\underset{\gamma}{\nabla}_k \overset{\gamma}{\underset{0}{Q}}{}_j^r + \underset{1}{\nabla}_j \gamma_k^r \Big|_0$ should, therefore, form a tensor. In $\underset{\gamma}{\nabla}_k \overset{\gamma}{\underset{0}{Q}}{}_j^r \Big|_0$, $\phi \geqslant 1$, *i.e.* there is one term of the form $lD^\alpha \underset{1}{\nabla}_k \gamma_j^r$, *i.e.* $lD^\alpha \underset{1}{\nabla}_k \gamma_j^r + \underset{1}{\nabla}_j \gamma_k^r$ should be a tensor. By arguments used already it will be seen that this is simply $\underset{1}{\nabla}_j \gamma_k^r - \underset{1}{\nabla}_k \gamma_j^r$, and this is expressible in terms of known tensors.

Expression (*C*).

This is of the same form as (*B*) in (2.13) with $\mu = 0$. The additional term $\left(\underset{1}{\nabla}_j \gamma_k^l\right) \overset{\rho}{\underset{\gamma}{Q}}{}_l^r$ is present because of the presence of $\left(\underset{1}{\nabla}_j \gamma_k^l\right)$ in $\underset{0}{\nabla}$, but this does not affect our argument. So, using the same argument, we may assert that it contains only known tensors.

Expression (*D*).

$$\left[\underset{\gamma}{\nabla}_k \underset{1}{\nabla}_j \gamma_r^i - \underset{1}{\nabla}_j \underset{\gamma}{\nabla}_k \gamma_r^i\right]$$

$$= \sum_{\rho=\gamma+1}^{\sigma} \overset{\rho}{\underset{\gamma}{Q}}{}_k^l \left[\underset{\rho}{\nabla}_l \underset{1}{\nabla}_j \gamma_r^i - \underset{1}{\nabla}_j \underset{\rho}{\nabla}_l \gamma_r^i\right] + \sum_{\rho-2}^{\sigma} \overset{\rho}{\underset{1}{Q}}{}_j^l \left[\underset{\gamma}{\nabla}_k \underset{\rho}{\nabla}_l \gamma_r^i - \underset{\rho}{\nabla}_l \underset{\gamma}{\nabla}_k \gamma_r^i\right]$$

$$- \sum_{\rho=\gamma+1}^{\sigma} \overset{\rho}{\underset{\gamma}{Q}}{}_k^m \sum_{\delta=2}^{\sigma} \overset{\delta}{\underset{1}{Q}}{}_j^l \left[\underset{\rho}{\nabla}_m \underset{\delta}{\nabla}_l \gamma_r^i - \underset{\delta}{\nabla}_l \underset{\rho}{\nabla}_m \gamma_r^i\right]$$

$$+ \sum_{\rho=2}^{\gamma} \underset{\rho}{\nabla}_l \gamma_r^i \left[\underset{\gamma}{\nabla}_k \overset{\rho}{\underset{1}{Q}}{}_j^l - \sum_{\delta=\gamma+1}^{\sigma} \overset{\delta}{\underset{\gamma}{Q}}{}_k^m \underset{\delta}{\nabla}_m \overset{\rho}{\underset{1}{Q}}{}_j^l\right]$$

$$+ \sum_{\rho-\gamma+1}^{\sigma} \underset{\rho}{\nabla}_l \gamma_r^i \left[\underset{\gamma}{\nabla}_k \underset{1}{Q}{}_j^l - \underset{1}{\nabla}_j \overset{\rho}{\underset{\gamma}{Q}}{}_k^l + \sum_{\delta=2}^{\sigma} \overset{\delta}{\underset{1}{Q}}{}_j^m \underset{\delta}{\nabla}_m \overset{\rho}{\underset{\gamma}{Q}}{}_k^l - \sum_{\delta=\gamma+1}^{\sigma} \overset{\delta}{\underset{\gamma}{Q}}{}_k^m \underset{\delta}{\nabla}_m \overset{\rho}{\underset{1}{Q}}{}_j^l\right].$$

The coefficients here are the same as those in (2.13) with $\mu = 1$. So they are known tensors. Thus it is seen that, whenever $\gamma > 1$, $\underset{\gamma}{\nabla}_k \underset{1}{\nabla}_j \gamma_r^i$ is expressible in terms of known tensors. But $\underset{1}{\nabla}_k \underset{1}{\nabla}_j \gamma_r^i$ is not so expressible and hence is a new tensor (*cf.* formula 1.4)

$$
\begin{aligned}
(2.15)\quad & \left[\underset{0}{\nabla}_k \underset{0}{\nabla}_j u^i - \underset{0}{\nabla}_j \underset{0}{\nabla}_k u^i\right] \\
&= \sum_{\rho=1}^{\sigma} \overset{\rho}{\underset{0}{Q}}{}_k^r \left[\underset{\rho}{\nabla}_r \underset{0}{\nabla}_j u^i - \underset{0}{\nabla}_j \underset{\rho}{\nabla}_r u^i\right] - \sum_{\rho=1}^{\sigma} \overset{\rho}{\underset{0}{Q}}{}_j^r \left[\underset{\rho}{\nabla}_r \underset{0}{\nabla}_k u^i - \underset{0}{\nabla}_k \underset{\rho}{\nabla}_r u^i\right] \\
&\quad - \sum_{\rho=1}^{\sigma} \overset{\rho}{\underset{0}{Q}}{}_k^r \sum_{\delta=1}^{\sigma} \overset{\delta}{\underset{0}{Q}}{}_j^l \left[\underset{\rho}{\nabla}_r \underset{\delta}{\nabla}_l u^i - \underset{\delta}{\nabla}_l \underset{\rho}{\nabla}_r u^i\right] \\
&\quad - \sum_{\rho=1}^{\sigma} \underset{\rho}{\nabla}_r u^i \bigg[\underset{0}{\nabla}_j \overset{\rho}{\underset{0}{Q}}{}_k^r - \underset{\delta}{\nabla}_k \overset{\rho}{\underset{0}{Q}}{}_j^r + \sum_{\delta=1}^{\sigma} \overset{\delta}{\underset{0}{Q}}{}_k^l \underset{\delta}{\nabla}_l \overset{\rho}{\underset{0}{Q}}{}_j^r - \sum_{\delta=1}^{\sigma} \overset{\delta}{\underset{0}{Q}}{}_j^l \underset{\delta}{\nabla}_l \overset{\rho}{\underset{0}{Q}}{}_k^r \\
&\qquad\qquad + \overset{\rho}{\underset{0}{Q}}{}_l^r \left(\underset{1}{\nabla}_j \gamma_k^l - \underset{1}{\nabla}_k \gamma_j^l\right)\bigg] \\
&\quad + \underset{0}{\nabla}_r u^i \left[\underset{1}{\nabla}_j \gamma_k^r - \underset{1}{\nabla}_k \gamma_j^r\right] \\
&\quad - u^r \bigg[\underset{0}{\nabla}_j \underset{1}{\nabla}_k \gamma_r^i - \underset{0}{\nabla}_k \underset{1}{\nabla}_j \gamma_r^i - \left(\underset{1}{\nabla}_j \gamma_l^i\right)\left(\underset{1}{\nabla}_k \gamma_r^l\right) + \left(\underset{1}{\nabla}_k \gamma_l^i\right)\left(\underset{1}{\nabla}_j \gamma_r^l\right) \\
&\qquad + \underset{1}{\nabla}_l \gamma_r^i \left(\underset{1}{\nabla}_j \gamma_k^l - \underset{1}{\nabla}_k \gamma_j^l\right) + \sum_{\rho=1}^{\sigma} \overset{\rho}{\underset{0}{Q}}{}_k^l \underset{\rho}{\nabla}_l \underset{1}{\nabla}_j \gamma_r^i - \sum_{\rho-1}^{\sigma} \overset{\rho}{\underset{0}{Q}}{}_j^l \underset{\rho}{\nabla}_l \underset{1}{\nabla}_k \gamma_r^i\bigg].
\end{aligned}
$$

The coefficient of $\underset{\rho}{\nabla}_r u^i$ is of the same form as the expression B in (2.13) with $\gamma = \mu = 0$. Following the same arguments, we may assert that it contains no new tensors.

Coefficient of u^r.

If $$A^i_{jk} = \underset{1}{\nabla}_k (a_{\sigma-1\,j}) - \sigma(\sigma+1)\left[D \underset{1}{\nabla}_k \gamma_j^i + \gamma_j^l \underset{1}{\nabla}_k \gamma_l^i\right],$$

we have $$B^i_{jk} = \frac{2}{\sigma(\sigma+1)}\left[A^i_{jk} - \underset{1}{\nabla}_k \underset{\sigma-1}{P}{}_j^i\right],$$

and so A^i_{jk} is an invariant.

$$
\begin{aligned}
&\sigma(\sigma+1)\Big[\underset{0}{\nabla}_j\underset{1}{\nabla}_k\gamma_r^{\,i}-\underset{0}{\nabla}_k\underset{1}{\nabla}_j\gamma_r^{\,i}+\Big(\underset{1}{\nabla}_k\gamma_l^{\,i}\Big)\Big(\underset{1}{\nabla}_j\gamma_r^{\,l}\Big)-\Big(\underset{1}{\nabla}_j\gamma_l^{\,\prime}\Big)\Big(\underset{1}{\nabla}_k\gamma_r^{\,l}\Big)\Big]\\
&\quad=\Big(\underset{1}{\nabla}_k A^i_{rj}-\underset{1}{\nabla}_j A^i_{rk}\Big)-\sigma(\sigma+1)D\Big[\underset{1}{\nabla}_j\underset{1}{\nabla}_k\gamma_r^{\,i}-\underset{1}{\nabla}_k\underset{1}{\nabla}_j\gamma_r^{\,i}\Big]\\
&\qquad+\sigma(\sigma+1)\underset{1}{\nabla}_l\gamma_r^{\,i}\Big(\underset{1}{\nabla}_k\gamma_j^{\,l}-\underset{1}{\nabla}_j\gamma_k^{\,l}\Big)\\
&\qquad+2\Big(\underset{1}{\nabla}_k\gamma_j^{\,l}-\underset{1}{\nabla}_j\gamma_k^{\,l}\Big)\Big[\underset{2}{\nabla}_l\underset{\sigma-1}{P}{}_r^{\,i}+\frac{\sigma(\sigma+1)}{2}D\underset{2}{\nabla}_l\gamma_r^{\,i}\Big]\\
&\qquad+\sum_{\gamma=3}^{\sigma}\Big[\underset{\gamma}{\nabla}_l\underset{\sigma-1}{P}{}_r^{\,i}+\frac{\sigma(\sigma+1)}{2}\Big(D\underset{\gamma}{\nabla}_l\gamma_r^{\,i}+\underset{\gamma-1}{\nabla}_l\gamma_r^{\,i}\Big)\Big]\Big(\underset{1}{\nabla}_j\overset{\gamma}{\underset{1}{Q}}{}_k^{\,l}-\underset{1}{\nabla}_k\overset{\gamma}{\underset{1}{Q}}{}_j^{\,l}\Big)\\
&\qquad+\sum_{\rho=2}^{\sigma}\overset{\rho}{\underset{1}{Q}}{}_k^{\,l}\Big[\underset{1}{\nabla}_j\underset{\rho}{\nabla}_l\underset{\sigma-1}{P}{}_r^{\,i}+\frac{\sigma(\sigma+1)}{2}\Big(D\underset{1}{\nabla}_j\underset{\rho}{\nabla}_l\gamma_r^{\,i}+\underset{0}{\nabla}_j\underset{\rho}{\nabla}_l\gamma_r^{\,i}+\underset{1}{\nabla}_j\underset{\rho-1}{\nabla}_l\gamma_r^{\,i}\Big)\Big]\\
&\qquad-\sum_{\rho=2}^{\sigma}\overset{\rho}{\underset{1}{Q}}{}_j^{\,l}\Big[\underset{1}{\nabla}_k\underset{\rho}{\nabla}_l\underset{\sigma-1}{P}{}_r^{\,i}+\frac{\sigma(\sigma+1)}{2}\Big(D\underset{1}{\nabla}_k\underset{\rho}{\nabla}_l\gamma_r^{\,i}+\underset{0}{\nabla}_k\underset{\rho}{\nabla}_l\gamma_r^{\,i}+\underset{1}{\nabla}_k\underset{\rho-1}{\nabla}_l\gamma_r^{\,i}\Big)\Big]\\
&\qquad+\sum_{\rho=2}^{\sigma}\overset{\rho}{\underset{1}{Q}}{}_j^{\,l}\Big[\underset{\rho}{\nabla}_l(\alpha^i_{\sigma-1\,r(1)\,k})-\sigma(\sigma+1)\Big(\underset{\rho}{\nabla}_l\gamma_m^{\,i}\Big)\underset{1}{\nabla}_k\gamma_r^{\,m}\Big]\\
&\qquad\qquad-\sum_{\rho=2}^{\sigma}\overset{\rho}{\underset{1}{Q}}{}_k^{\,l}\Big[\underset{\rho}{\nabla}_l(\alpha^i_{\sigma-1\,r(1)\,j})-\sigma(\sigma+1)\Big(\underset{\rho}{\nabla}_l\gamma_m^{\,i}\Big)\underset{1}{\nabla}_j\gamma_r^{\,m}\Big].
\end{aligned}
$$

On actual evaluation it is found that

$$
\begin{aligned}
\underset{\rho}{\nabla}_l(\alpha^i_{\sigma-1\,r(1)\,k})\Big|_0&=\underset{\rho}{\nabla}_l\underset{1}{\nabla}_k\underset{\sigma-1}{P}{}_r^{\,i}-\sum_{\gamma=3}^{\sigma}\overset{\gamma}{\underset{1}{Q}}{}_k^{\,l}\underset{\rho}{\nabla}_l\underset{\gamma}{\nabla}_m\underset{\sigma-1}{P}{}_r^{\,i}\\
&\quad-\frac{\sigma(\sigma+1)}{2}\sum_{\gamma=3}^{\sigma}\overset{\gamma}{\underset{1}{Q}}{}_k^{\,l}\Big[D\underset{\rho}{\nabla}_l\underset{\gamma}{\nabla}_m\gamma_r^{\,i}+\underset{\rho-1}{\nabla}_l\underset{\gamma}{\nabla}_m\gamma_r^{\,i}+\Big(\underset{\rho}{\nabla}_l\gamma_p^{\,i}\Big)\Big(\underset{\gamma}{\nabla}_m\gamma_r^{\,p}\Big)\\
&\qquad\qquad+\underset{\gamma}{\nabla}_m\gamma_p^{\,i}\Big(\underset{\rho}{\nabla}_l\gamma_r^{\,p}\Big)-\Big(\underset{\gamma}{\nabla}_p\gamma_r^{\,i}\Big)\Big(\underset{\rho}{\nabla}_l\gamma_m^{\,p}\Big)+\underset{\rho}{\nabla}_l\underset{\gamma-1}{\nabla}_m\gamma_r^{\,i}\Big]\\
&\quad+\frac{\sigma(\sigma+1)}{2}\Big[2D\underset{\rho}{\nabla}_l\underset{1}{\nabla}_k\gamma_r^{\,i}+2\underset{\rho-1}{\nabla}_l\underset{1}{\nabla}_k\gamma_r^{\,i}+\underset{\rho}{\nabla}_l B^i_{rk}\\
&\qquad\qquad+2\Big(D\underset{2}{\nabla}_m\gamma_r^{\,i}\Big)\underset{\rho}{\nabla}_l\gamma_k^{\,m}\Big]\\
&\quad+2\Big(\underset{2}{\nabla}_m\underset{\sigma-1}{P}{}_r^{\,i}\Big)\Big(\underset{\rho}{\nabla}_l\gamma_k^{\,m}\Big)+\sigma(\sigma+1)\Big(\underset{\rho}{\nabla}_l\gamma_m^{\,i}\Big)\Big(\underset{1}{\nabla}_k\gamma_r^{\,m}\Big)\\
&\quad-\sum_{\gamma=3}^{\sigma}\Big[\underset{\gamma}{\nabla}_m\underset{\sigma-1}{P}{}_r^{\,i}+\frac{\sigma(\sigma+1)}{2}\Big(D\underset{\gamma}{\nabla}_m\gamma_r^{\,i}+\underset{\gamma-1}{\nabla}_m\gamma_r^{\,i}\Big)\Big]\underset{\rho}{\nabla}_l\overset{\gamma}{\underset{1}{Q}}{}_k^{\,m}.
\end{aligned}
$$

Similarly for $\underset{\rho}{\nabla}_l(\alpha^i_{\sigma-1\,r(1)\,j})$.

Hence we find that

$$\sigma(\sigma+1)\Big[\underset{0}{\nabla}_j\underset{1}{\nabla}_k\gamma_r{}^i-\underset{0}{\nabla}_k\underset{1}{\nabla}_j\gamma_r{}^i+\Big(\underset{1}{\nabla}_k\gamma_l{}^i\Big)\Big(\underset{1}{\nabla}_j\gamma_r{}^l\Big)-\Big(\underset{1}{\nabla}_j\gamma_l{}^i\Big)\Big(\underset{1}{\nabla}_k\gamma_r{}^l\Big)$$

$$+\underset{1}{\nabla}_l\gamma_r{}^i\Big(\underset{1}{\nabla}_j\gamma_k{}^l-\underset{1}{\nabla}_k\gamma_j{}^l\Big)\Big]$$

$$=\text{known tensors}$$

$$+\sum_{\gamma=3}^{\sigma}\Big[\underset{\gamma}{\nabla}_l\underset{\sigma-1}{P}_r{}^i+\frac{\sigma(\sigma+1)}{2}\Big(D\underset{\gamma}{\nabla}_l\gamma_r{}^i+\underset{\gamma-1}{\nabla}_l\gamma_r{}^i\Big)\Big]$$

$$\times\Big[\underset{1}{\nabla}_j\overset{\gamma}{\underset{1}{Q}}{}_k{}^l-\underset{1}{\nabla}_k\overset{\gamma}{\underset{1}{Q}}{}_j{}^l-\sum_{\rho=3}^{\sigma}\overset{\rho}{\underset{1}{Q}}{}_j{}^m\underset{\rho}{\nabla}_m\overset{\gamma}{\underset{1}{Q}}{}_k{}^l+\sum_{\rho=3}^{\sigma}\overset{\rho}{\underset{1}{Q}}{}_k{}^m\underset{\rho}{\nabla}_m\overset{\gamma}{\underset{1}{Q}}{}_j{}^l\Big].$$

But this has to be a tensor. Now it is of the same form as the expression (B) in (2.13) with $\gamma=\mu=1$. Hence it follows that it is expressible in terms of known tensors.

We have omitted the alternations with the operator $\partial/\partial t$, since we have already shown that it is dependent upon the others (*vide* formula 7).

But even performing those alternations we find that we get only the vector $\partial a^i/\partial t$ and its derivatives. From the formula given below it will be clear that $\partial a^i/\partial t$ is not a new vector, but can be expressed in terms of the basis.

$$(2.16)^* \quad \frac{\partial a^i}{\partial t}=D\epsilon^i-\underset{0}{P}_r{}^i x^{(1)r}+\sum_{\rho=1}^{\sigma-1}\frac{\rho}{\sigma+1}\Big(D\underset{\rho}{P}_r{}^i\Big)D^{\rho-1}x^{(1)r}$$

$$+\sum_{\rho=1}^{\sigma-1}\Big(\frac{\rho}{\sigma+1}-1\Big)\underset{\rho}{P}_r{}^i(D^\rho x^{(1)r}).$$

Thus we have shown that when alternating all the differential operators taken two at a time we get only the tensors $\underset{\gamma}{P}_j{}^i$, $\gamma^i_{j(\sigma)k}$, $\underset{1}{\nabla}_k\underset{1}{\nabla}_j\gamma_r{}^i$ and their biderivates and covariant derivatives as coefficients on the right-hand side. The primary tensors $\underset{\gamma}{P}_j{}^i$ occur in the equations of variation of the paths, while $\gamma^i_{j(\sigma)k}$ appears as the coefficient of u^k in alternating $\underset{\sigma}{\nabla}$ and D, and $\underset{1}{\nabla}_k\underset{1}{\nabla}_j\gamma_r{}^i$ occurs as one of the terms in the coefficient of u^r when alternating

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[*] This is a direct generalisation of the corresponding formula in D. D. Kosambi, "Problem of differential invariants", *Journal Indian Math. Soc.*, 20 (1934), 185–188, and has been suggested by Prof. Kosambi.

$\underset{1}{\nabla}_k$ and $\underset{0}{\nabla}_j$. The vector ϵ^i is a differential invariant of the space, but can be derived from $x^{(1)i}$. (*Vide* Formula I.)

Hence the theorem of Kosambi is established.

3. Let us now derive the tensorial form of the Euler equations, *i.e.* the equations that give the metric for our space if one exists. The equations are

$$(3.1)\qquad \delta_i f \equiv f_{(0)i} - f^{(1)}_{(1)i} + f^{(2)}_{(2)i} - f^{(3)}_{(3)i} + \ldots + (-1)^\lambda f^{\lambda}_{\lambda i} = 0 \quad \left(\lambda = \left[\frac{\sigma+1}{2}\right]\right).$$

Let us choose the usual special coordinates. Then

$$f_{(\gamma)i} = \underset{\gamma}{\nabla}_i f - \sum_{\mu=\gamma+2}^{\lambda} \overset{\mu}{\underset{\gamma}{Q}}{}^l{}_i \underset{\mu}{\nabla}_l f,$$

$$(-1)^\gamma f^{(\gamma)}_{(\gamma)i} = (-1)^\gamma D^\gamma \left[\underset{\gamma}{\nabla}_i f - \sum_{\mu=\gamma+2}^{\lambda} \overset{\mu}{\underset{\gamma}{Q}}{}_i^l \underset{\mu}{\nabla}_l f\right],$$

and therefore

$$\delta_i f = \sum_{\gamma=0}^{\lambda} (-1)^\gamma D^\gamma \underset{\gamma}{\nabla}_i f + \sum_{\gamma=0}^{\lambda} \sum_{\mu=\gamma+2}^{\lambda} (-1)^{\gamma+1} \sum_{r=0}^{\gamma} \binom{\gamma}{r} \left(D^r \underset{\mu}{\nabla}_l f\right) \left(D^{\gamma-r} \overset{\mu}{\underset{\gamma}{Q}}{}_i^l\right)$$

$$= \sum_{\gamma=0}^{\lambda} (-1)^\gamma D^\gamma \underset{\gamma}{\nabla}_i f + \sum_{\mu=\gamma+2}^{\lambda} \sum_{r=0}^{\gamma} D^r \underset{\mu}{\nabla}_l f \left[\sum_{\gamma=0}^{\lambda} (-1)^{r+1} \binom{\gamma}{r} D^{\gamma-r} \overset{\mu}{\underset{\gamma}{Q}}{}_i^l\right].$$

In all these the limits of summation can always be assumed to be constant, say 0 and λ, by adopting the conventions that $\overset{\mu}{\underset{\gamma}{Q}}{}_j^i = 0$ whenever $\mu < \gamma + 1$ and that $\binom{\gamma}{r} = 0$ if $\gamma < r$. And hence, also, the summations can be performed in any order.

Now we have the formula

$$(3.2)\qquad \sum_{\gamma=r}^{\mu-2} (-1)^{\gamma+1} \binom{\gamma}{r} D^{\gamma-r} \overset{\mu}{\underset{\gamma}{Q}}{}_i^l \Big|_0$$

$$= (-1)^{r+1} \left(\overset{\mu-r}{\underset{0}{Q}}{}_i^l - D \overset{\mu-r-1}{\underset{0}{Q}}{}_i^l\right)$$

$$= (-1)^{r+1} \sum_{\rho=0}^{\mu-r-2} (-1)^{\rho+1} \binom{\sigma+\rho-\mu+r+1}{\rho} D^\rho \underset{\sigma+\rho-\mu+r+1}{P_i^l},$$

and hence

$$(3.3)\quad \delta_i f = \sum_{\gamma=0}^{\lambda} (-1)^\gamma D^\gamma \underset{\gamma}{\nabla_i} f + \sum_{\mu=2}^{\lambda} \sum_{r=0}^{\lambda} D^r \underset{\mu}{\nabla_l} f \Big[(-1)^{r+1} \sum_{\rho=0}^{\mu-r-2} (-1)^{\rho+1} \times \binom{\sigma+\rho-\mu+r+1}{\rho} D^\rho \underset{\sigma+\rho-\mu+r+1}{P^l_i} \Big].$$

This can be expanded and rearranged in the form*

$$(3.4)\quad \delta_i f = \sum_{\gamma=0}^{\lambda} (-1)^\gamma D^\gamma \underset{\gamma}{\nabla_i} f + \sum_{\gamma=1}^{\lambda-1} \Big[\Big\{ \sum_{r=0}^{\gamma-1} (-1)^r \binom{\sigma-\gamma+r}{r} D^r \underset{\sigma-\gamma+r}{P^l_i} \Big\} \times \Big\{ \sum_{\rho=0}^{\lambda-\gamma-1} (-1)^\rho D^\rho \underset{\rho+\gamma+1}{\nabla_l} f \Big\} \Big] = 0.$$

If we substitute an arbitrary function $\phi(f)$ for f, then we have

$$(3.5)\quad \delta_i \phi(f) = \sum_{\gamma=0}^{\lambda} (-1)^\gamma \Big\{ \phi(f) \Big\}^{(\gamma)}_{(\gamma)i} = \sum_{\gamma=0}^{\lambda} (-1)^\gamma \Big\{ \frac{d\phi}{df} f_{(\gamma)i} \Big\}^{(\gamma)}.$$

Expanding by Leibniz's theorem, we get for the sum on the right

$$\sum_{\gamma=0}^{\lambda} (-1)^\gamma \sum_{r=0}^{\gamma} \binom{\gamma}{r} f^{\gamma-r}_{(\gamma)i} \left(\frac{d\phi}{df}\right)^{(r)}$$

$$= \sum_{r=0}^{\lambda} \left(\frac{d\phi}{df}\right)^{(r)} \sum_{\gamma=r}^{\lambda} (-1)^\gamma \binom{\gamma}{r} f^{(\gamma-r)}_{(\gamma)i}$$

$$= \sum_{r=0}^{\lambda} \left(\frac{d\phi}{df}\right)^{(r)} \overset{r}{E}_i,$$

where

$$(3.6)\quad \overset{r}{E}_i = \sum_{\gamma=r}^{\lambda} (-1)^\gamma \binom{\gamma}{r} f^{(\gamma-r)}_{(\gamma)i}$$

are the Synge-Craig vectors of the r-th kind†. It is easily seen that $\overset{0}{E}_i$, is the Euler vector.

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* *Cf* (4.9) of D. D. Kosambi, "Path-spaces of higher order", *Quart. J. of Math.*, 7 (1936), 97–104.

† J. L. Synge, " Some intrinsic and derived vectors in a Kawaguchi space ", *American J. of Math.*, 57 (1935), 684. H. V. Craig, "On a generalised tangent-vector, II", *American J. of Math.*, 58 (1936), 833.

To express $\overset{r}{E}_i$ in an invariantive form, we can either follow the same method as for the Euler vector or we can substitute $\phi(f)$ for f in (3.4). Following the latter method we get

$$
\begin{aligned}
\delta_i \phi(f) = {} & \sum_{\gamma=0}^{\lambda} (-1)^\gamma D^\gamma \Big[\underset{\gamma}{\nabla_i} \phi(f) \Big] \\
& + \sum_{\gamma=1}^{\lambda-1} \Bigg[\Bigg\{ \sum_{r=0}^{\gamma-1} (-1)^r \binom{\sigma-\gamma+r}{r} D^r \underset{\sigma-\gamma+r}{P_i^l} \Bigg\} \\
& \qquad \times \Bigg\{ \sum_{\rho=0}^{\lambda-\gamma-1} (-1)^\rho D^\rho \Big(\underset{\rho+\gamma+1}{\nabla_l} \phi(f) \Big) \Bigg\} \Bigg] \\
= {} & \sum_{\gamma=0}^{\lambda} (-1)^\gamma D^\gamma \Big[\frac{d\phi}{df} \underset{\gamma}{\nabla_i} f \Big] \\
& + \sum_{\gamma=1}^{\lambda-1} \Bigg[\Bigg\{ \sum_{r=0}^{\gamma-1} (-1)^r \binom{\sigma-\gamma+r}{r} D^r \underset{\sigma-\gamma+r}{P_i^l} \Bigg\} \\
& \qquad \times \Bigg\{ \sum_{\rho=0}^{\lambda-\gamma-1} (-1)^\rho D^\rho \Big(\frac{d\phi}{df} \underset{\rho+\gamma+1}{\nabla_l} f \Big) \Bigg\} \Bigg] \\
= {} & \sum_{\gamma=0}^{\lambda} (-1)^\gamma \sum_{r=0}^{\gamma} \binom{\gamma}{r} \Big(\frac{d\phi}{df} \Big)^{(r)} D^{\gamma-r} \underset{\gamma}{\nabla_i} f \\
& + \sum_{\gamma=1}^{\lambda-1} \Bigg[\Bigg\{ \sum_{s=0}^{\gamma-1} (-1)^s \binom{\sigma-\gamma+s}{s} D^s \underset{\sigma-\gamma+s}{P_i^l} \Bigg\} \\
& \qquad \times \Bigg\{ \sum_{\rho=0}^{\lambda-\gamma-1} (-1)^\rho \sum_{r=0}^{\rho} \binom{\rho}{r} \Big(\frac{d\phi}{df} \Big)^{(r)} D^{\rho-r} \underset{\rho+\gamma+1}{\nabla_l} f \Bigg\} \Bigg] \\
= {} & \sum_{r=0}^{\lambda} \Big(\frac{d\phi}{df} \Big)^{(r)} \sum_{\gamma=r}^{\lambda} (-1)^\gamma \binom{\gamma}{r} D^{\gamma-r} \underset{\gamma}{\nabla_i} f \\
& + \sum_{r=0}^{\lambda} \Big(\frac{d\phi}{df} \Big)^{r} \Bigg[\sum_{\gamma=1}^{\lambda-r-1} \Bigg\{ \sum_{s=0}^{\gamma-1} (-1)^s \binom{\sigma-r+s}{s} D^s \underset{\sigma-\gamma+s}{P_i^l} \Bigg\} \\
& \qquad \times \Bigg\{ \sum_{\rho=0}^{\lambda-\gamma-1} (-1)^\rho \binom{\rho}{r} D^{\rho-r} \underset{\rho+\gamma+1}{\nabla_l} f \Bigg\} \Bigg] \\
= {} & \sum_{r=0}^{\lambda} \Big(\frac{d\phi}{df} \Big)^{r} \overset{r}{E}_i,
\end{aligned}
$$

by (3.4), and hence

$$
\begin{aligned}
\text{(3.7)} \quad \overset{r}{E}_i = {} & \sum_{\gamma=r}^{\lambda} (-1)^\gamma \binom{\gamma}{r} D^{\gamma-r} \underset{\gamma}{\nabla_i} f \\
& + \sum_{\gamma=1}^{\lambda-r-1} \Bigg[\Bigg\{ \sum_{s=0}^{\gamma-1} (-1)^s \binom{\sigma-\gamma+s}{s} D^s \underset{\sigma-\gamma+s}{P_i} \Bigg\} \\
& \qquad \times \Bigg\{ \sum_{\rho=0}^{\lambda-\gamma-1} (-1)^\rho \binom{\rho}{r} D^{\rho-r} \underset{\rho+\gamma+1}{\nabla_l} f \Bigg\} \Bigg].
\end{aligned}
$$

In all these it must be remembered that alternative forms are always possible, since we can write $\underset{\gamma}{\nabla_k} D - D \underset{\gamma}{\nabla_k} = \underset{\gamma-1}{\nabla_k} + \left(\underset{\gamma}{\nabla_k} \gamma\right) + \dots$. In fact, we have

$$
\begin{aligned}
(3.8) \quad \overset{0}{E}_i = {} & \sum_{\mu=0}^{\lambda} \sum_{r=0}^{\mu} (-1)^{\mu+r} \binom{\mu}{r} \underset{\mu-r}{\nabla_i} (D^{\mu-r} f) \\
& + \sum_{\gamma=1}^{\lambda-1} \left[\left\{ \sum_{r=0}^{\gamma-1} (-1)^r \binom{\sigma-\gamma+r}{r} D^r \underset{\sigma-\gamma+r}{P_i^l} \right\} \right. \\
& \left. \times \left\{ \sum_{\rho=0}^{\lambda-\gamma-1} \sum_{s=0}^{\rho} (-1)^{\rho+s} \binom{\rho}{s} \underset{\rho+\gamma+1-s}{\nabla_l} (D^{\rho-s} f) \right\} \right] = 0
\end{aligned}
$$

as an alternative form of the Euler equations.

4. Next, we shall show that the following equations of Zermelo[*], expressing the various conditions of invariance of *fdt* under change of parameter, are tensor invariant. The equations are

$$(4.1) \qquad (x'^r T)^{(\gamma)} f_{(\gamma)r} = (fT)' \quad (\gamma = 0, 1, \dots, \lambda);$$

$$(4.2) \qquad \sum_{\gamma=\mu}^{\lambda} \binom{\gamma}{\mu} x'^{r(\gamma-\mu)} f_{(\gamma)r} = \delta_\mu{}^1 f \quad (\mu \neq 0);$$

$$(4.3) \qquad \sum_{\gamma=1}^{\lambda} \binom{\gamma}{1} x'^{r(\gamma-1)} Z_{\gamma r} = f;$$

$$(4.4) \qquad Z_{Ar} = \frac{1}{A} \sum_{\beta=A}^{\lambda} (-1)^{\beta-A} f_{(\beta)r}^{(\beta-A)}.$$

Equations (4.1) can be expanded and written in the form

$$(4.5) \qquad \sum_{\mu=1}^{\gamma} T^{(\mu)} \left[\binom{\gamma}{\mu} x'^{r(\gamma-\mu)} f_{(\gamma)r} - \delta_\mu{}^1 f \right] = 0.$$

From this it is clear that (4.2) are merely the coefficients of the various derivatives of T in (4.1) equated to zero.

Kawaguchi[†] has proved that, if T_i is any covariant vector of order λ,

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[*] H. V. Craig, " On tensors relative to the extended point transformation ", *American J. of Math.*, 59 (1937), 770.

[†] A. Kawaguchi, " Some intrinsic derivations in a generalised space ", *Proc. Imp. Academy, Tokio*, 12 (1936), 149.

i.e. dependent upon t, $x^{(0)i}$, ..., $x^{\lambda i}$, and X^i a contravariant vector of any order, then the n quantities

$$\sum_{\beta=\rho}^{\lambda} \binom{\beta}{\rho} T_{i(\beta)j} X^{j(\beta-\rho)} \tag{4.6}$$

are the components of a covariant vector.

With the same analysis that Kawaguchi has employed, it can be easily shown that, if $T_{ijk\ldots m}$ is any covariant tensor of rank r and if X^i is a contravariant vector, then

$$\sum_{\beta=\rho}^{\lambda} \binom{\beta}{\rho} T_{ijk\ldots m(\beta)l} X^{l(\beta-\rho)} \tag{4.7}$$

are the components of a covariant tensor of rank r.

Also, if f is any invariant function, then

$$\sum_{\beta=\rho}^{\lambda} \binom{\beta}{\rho} f_{(\beta)j} X^{j(\beta-\rho)} \tag{4.8}$$

is an invariant.

Comparing this with (4.5) it is seen that (4.5) is an invariant. For X^i we have x'^i.

Equations (4.3) and (4.4) can be combined into one and written in the form

$$\sum_{\beta=1}^{\lambda} x'^{i(\beta-1)} \sum_{\rho=\beta}^{\lambda} (-1)^{\rho-\beta} f_{(\rho)i}^{(\rho-\beta)} = f. \tag{4.9}$$

By suitable substitutions this can be transformed into

$$\sum_{r=0}^{\lambda-1} x'^{i(r)} \sum_{p=1}^{\lambda-p} (-1)^{p-1} f_{(r+p)i}^{(p-1)} = f. \tag{4.10}$$

This is of the same form as (2.17) of Synge* and hence is an invariant. We express (4.10) in terms of the E-vectors, so that their invariantive character will become more apparent.

In fact, we prove that

$$\sum_{r=0}^{\lambda-1} X^{i(r)} \sum_{p=1}^{\lambda-r} (-1)^{p-1} f_{(p+r)i}^{(p-1)} = \sum_{r=0}^{\lambda-1} (-1)^{r+1} \left(X^i \overset{r+1}{E}_i \right)^{(r)}. \tag{4.11}$$

---

* J. L. Synge, *loc. cit.*, 683.

Let us expand the expression on the right-hand side. It is equal to

$$\sum_{r=0}^{\lambda-1}(-1)^{r+1}\left[\sum_{p=0}^{r}\binom{r}{p}X^{i(r-p)}\left(\overset{r+1}{E}_i\right)^{(p)}\right]$$

$$=\sum_{r=0}^{\lambda-1}(-1)^{r+p+1}\sum_{p}\binom{r+p}{p}X^{i(r)}\left(\overset{r+p+1}{E}_i\right)^{(p)}$$

$$=\sum_{r=0}^{\lambda-1}X^{i(r)}\left[(-1)^{r+1}\overset{r+1}{E}_i+(-1)^{r+2}\binom{r+1}{1}\left(\overset{r+2}{E}_i\right)^{(1)}+(-1)^{r+3}\binom{r+2}{2}\left(\overset{r+3}{E}_i\right)^{(2)}+\dots\right]$$

$$=\sum_{r=0}^{\lambda-1}X^{i(r)}\Bigg[f_{(r+1)i}-\binom{r+2}{1}f^{(1)}_{(r+2)i}+\binom{r+3}{2}f^{(2)}_{(r+3)i}-\binom{r+4}{3}f^{(3)}_{(r+4)i}+\dots$$
$$+\binom{r+1}{1}f^{(1)}_{(r+2)i}-\binom{r+1}{1}\binom{r+3}{1}f^{(2)}_{(r+3)i}+\binom{r+1}{1}\binom{r+4}{2}f^{(3)}_{(r+4)i}-\dots$$
$$+\binom{r+2}{2}f^{(2)}_{(r+3)i}-\binom{r+2}{2}\binom{r+4}{1}f^{(3)}_{(r+4)i}+\dots$$
$$+\binom{r+3}{3}f^{(3)}_{(r+4)i}-\dots\Bigg].$$

The general term here is

$$(-1)^s f^{\cdot\cdot(s)}_{(r+s+1)i}\Bigg[\binom{r+s+1}{s}-\binom{r+s+1}{s-1}\binom{r+1}{1}+\binom{r+s+1}{s-2}\binom{r+2}{2}-\dots$$
$$+(-1)^s\binom{r+s+1}{0}\binom{r+s}{s}\Bigg].$$

Making use of the formula

$$\binom{n}{\rho}-\binom{n}{\rho-1}\binom{a}{1}+\binom{n}{\rho-2}\binom{a+1}{2}-\binom{n}{\rho-3}\binom{a+2}{3}+\dots$$
$$+(-1)^{\rho}\binom{n}{0}\binom{a+\rho-1}{\rho}=\binom{n-a}{\rho}$$

we find that the general term is merely

$$(-1)^s f^{\cdot\cdot(s)}_{(r+s+1)i}.$$

Hence, the right-hand side is equal to

$$\sum_{r=0}^{\lambda-1}X^{i(r)}\sum_{s=0}^{\lambda-r-1}(-1)^s f^{\cdot\cdot(s)}_{(r+s+1)i}=\sum_{r=0}^{\lambda-1}X^{i(r)}\sum_{p=1}^{\lambda-r}(-1)^{p-1}f^{\cdot\cdot(p-1)}_{(p+r)i},$$

and hence our result (4.11) is established.

Putting x'^i for X^i in this we get the desired form of (4.10).

Kawaguchi* has applied his covariant derivatives referred to in (4.6) of this paper to the generalized path equations of Kosambi, viz.

$$\phi^i \equiv x^{(\sigma+1)i} + a^i(t, x, x^{(1)}, \ldots, x^{\sigma}) = 0. \tag{1}$$

Since the left-hand side ϕ^i of this equation has the vector property, he has shown *in his own notation* that

$$(4.12) \qquad \overset{\sigma+1-\rho}{D_j^i}(\phi)X^j \equiv \sum_{\gamma=\rho}^{\sigma+1} \binom{\gamma}{\rho} \phi^i_{\gamma(j)} X^{j(\gamma-\rho)}$$

$$= \binom{\sigma+1}{\rho} X^{i(\sigma+1-\rho)} + \sum_{\gamma=\rho}^{\sigma} \binom{\gamma}{\rho} a_i^{\gamma(j)} X^{j(\gamma-\rho}$$

are the components of a vector.

Dividing by the constant $\binom{\sigma+1}{\rho}$ he gets the result

$$(4.13) \qquad \overset{\sigma+1-\rho}{\Delta_j^i}(\phi)X^j = X^{i(\sigma+1-\rho)} + \binom{\sigma+1}{\rho}^{-1} \sum_{\gamma=\rho}^{\sigma} \binom{\gamma}{\rho} a^i_{(\gamma)j} X^{j(\gamma-\rho)}.$$

We see that the above can be expressed in the invariantive form

$$(4.14) \qquad \overset{(\sigma+1-\rho)}{\Delta_j^i}(\phi)X^j = D^{\sigma+1-\rho}X^i + \binom{\sigma+1}{\rho}^{-1} \sum_{\gamma=\rho}^{\sigma} \binom{\gamma}{\rho} \underset{\gamma}{P_j^i} D^{\gamma-\rho} X^j,$$

where D, $\underset{\gamma}{P_j^i}$ are the D operator and the primary tensors of Kosambi. Hence we do not get any new operators. It has been observed by Kawaguchi himself that

$$(4.15) \qquad \overset{1}{\Delta_j^i}(\phi)X^j = DX^i.$$

We get the full set of primary tensors only in

$$(4.16) \qquad \overset{\sigma+1}{\Delta_j^i}(\phi)X^j = D^{\sigma+1}X^i + \sum_{\gamma=0}^{\sigma} \underset{\gamma}{P_j^i} D^{\gamma} X^j.$$

This equated to zero gives us exactly the equations of variation of the paths, if X^i is taken to be the infinitesimal vector displacement $(x'^i - x^i)/\delta t$.

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* A. Kawaguchi, *loc. cit.*, 151.

THE ABSOLUTE CONVERGENCE OF SERIES OF INTEGRALS

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1. *Introduction.*

The present paper has its origin in two problems. The first concerns a question raised by one of the authors (**1**), namely whether the summability $|C,\ 1|$ of a Fourier series at a point is or is not a local property of the generating function*. The other problem is as follows: S. Saks has shown in (**4**) that, if $\{S_n(x)\}$ is a sequence of functions summable (*i.e.* having a finite integral in the Lebesgue sense) over a finite interval $(a,\ b)$ and such that

$$\overline{\lim_{n\to\infty}} \int_a^b |S_n(x)|\,dx = \infty,$$

then $(a,\ b)$ contains a set K such that

$$\overline{\lim_{n\to\infty}} \left| \int_K S_n(x)\,dx \right| = \infty.$$

Taking $S_n(x)$ to be the n-th partial sum of a series of functions $\{f_n(x)\}$ summable over $(a,\ b)$ we have the result that, if

$$\overline{\lim_{n\to\infty}} \int_a^b \left| \sum_{r=1}^{n} f_r(x) \right| dx = \infty,$$

then there is a function $\phi(x)$ measurable in $(a,\ b)$ and assuming only the values 0 and 1 and such that

$$\overline{\lim_{n\to\infty}} \left| \sum_{r=1}^{n} \int_a^b f_r(x)\,\phi(x)\,dx \right| = \infty.$$

---

[*] $\sum_{\nu=0}^{\infty} a_\nu$ is said to be summable $|C,1|$ if $\Sigma|\sigma_n-\sigma_{n-1}|<\infty$, where $\sigma_n = \dfrac{1}{n+1}\sum_{\nu=0}^{n}(n+1-\nu)\,a_\nu$.

This raises the question whether the assumption

$$\text{(A)} \qquad \sum_{n=1}^{\infty} \int_a^b |f_n(x)|\,dx = \infty, \quad \textit{i.e.} \quad \int_a^b \sum_{n=1}^{\infty} |f_n(x)|\,dx = \infty$$

mplies the existence of a function $\phi(x)$ such that

$$\text{(B)} \quad \phi(x) \text{ is summable over } (a, b) \text{ and } \sum_{n=1}^{\infty} \left| \int_a^b f_n(x)\phi(x)\,dx \right| = \infty.$$

Evidently $\phi(x)$ will not always exist to satisfy (B) and at the same time be *bounded* in (a, b); for example

$$\Sigma \int_0^{2\pi} \left| \frac{1}{n} \sin nx \right| dx = \infty$$

and, for every $\phi(x)$ summable over $(0, 2\pi)$,

$$\Sigma \left| \int_0^{2\pi} \frac{\sin nx}{n} \phi(x)\,dx \right| = \Sigma \frac{|b_n|}{n},$$

where b_n is the n-th Fourier sine coefficient of $\phi(x)$; the last series will certainly not diverge if $\phi(x)$ is bounded since, by Parseval's theorem, $\Sigma b_n^2 < \infty$. (In fact $\Sigma |b_n|/n < \infty$ whenever $\phi(x) \in L^k(0, 2\pi)$ for some $k > 1$.)

In order that $f_n(x)\phi(x)$ may be summable over (a, b) for every $\phi(x)$ summable over (a, b) it is necessary for $f_n(x)$ to be essentially bounded* in (a, b) (Lemma 1). There will, therefore, be no loss of generality if we suppose that this condition is satisfied for every n. In these circumstances we show in § 2 that *a necessary and sufficient condition for the existence of a* $\phi(x)$ *to satisfy* (B) *is that* $\sum_{n=1}^{\infty} |f_n(x)|$ *is not essentially bounded in* (a, b) (Theorem 1).

From this result we can at once elucidate the logical relation between (A) and (B). For let us consider the relation between (A) and the proposition

$$\text{(C)} \qquad \text{“}\sum_{n=1}^{\infty} |f_n(x)| \text{ is not essentially bounded in } (a, b)\text{”.}$$

---

[*] $f(x)$ is said to be *essentially bounded* in (a, b) if there is a finite number M such that $|f(x)| \leqslant M$ for almost all x in (a, b).

Then it is easy to see that (A) *formally implies* (*C*) *if* $b-a<\infty$. But if $b-a=\infty$ this is not so, *e.g.* if $\Sigma|f_n(x)|=1/x$, $a=1$, $b=\infty$. Thus "$\sum_{n=1}^{\infty}|f_n(x)|$ is essentially bounded in (a, b)" *is consistent with* (A) if $b-a=\infty$. On the other hand (C) *does not formally imply* (A), even if $f_n(x)$ is bounded for $n=1, 2, \ldots$, *e.g.* if $f_n(x)=\{(n\sqrt{x}+1)(\overline{n-1}\sqrt{x}+1)\}^{-1}$, $\sum_{n=1}^{\infty}|f_n(x)|=x^{-\frac{1}{2}}$, $a=0$, $b=1$. Thus "$\int_a^b \sum_{n=1}^{\infty}|f_n(x)|\,dx<\infty$" *is consistent with* (C).

We can therefore assert that, *if* $f_n(x)$ *is measurable and essentially bounded in* (a, b) *for* $n=1, 2, \ldots$, *then*

(i) *if* $b-a<\infty$, (A) *is a sufficient condition for the existence of a* $\phi(x)$ *satisfying* (B), *but this is false if* $b-a=\infty$,

(ii) (A) *is not necessary for the existence of such a* $\phi(x)$.

In §3 we apply Theorem 1 to show that the summability $|C, 1|$ of a Fourier series is *not* a local property of the function (Theorem 2). In fact, the summability $|C, 1|$ at a point $t=x$ may be destroyed by altering the function in any interval (α, β) not containing a point $t\equiv x \pmod{2\pi}$. In §4 we solve the same problem in a different way by constructing an explicit example. Finally in §5 we give a general theorem concerning functionals defined in an abstract space (Theorem 3), which is in effect a restatement of the method of §2.

2. We need the following lemma*.

LEMMA 1. *Suppose* $f(x)$ *is measurable in* (a, b), *where* $b-a\leqslant\infty$. *Then a necessary and sufficient condition for* $f(x)\phi(x)$ *to be summable over* (a, b) *for every function* $\phi(x)$ *summable over* (a, b) *is that* $f(x)$ *is essentially bounded in* (a, b).

Proof. The sufficiency of the condition is trivial. To prove its necessity let us suppose that $f(x)$ is not essentially bounded in (a, b), and dismiss the trivial case where $|f(x)|=\infty$ in a set of positive measure. Then there is a sequence $\{S_n\}$ of mutually exclusive sub-sets of (a, b), each of positive measure and such that $|f(x)|\geqslant n$ in S_n for $n=1, 2, \ldots$. Let $\phi(x)$ be defined in (a, b) so that

$$\phi(x)=(n^2 mS_n)^{-1} \text{ in } S_n, \text{ for } n=1, 2, \ldots, \text{ and } \phi(x)=0 \text{ elsewhere,}$$

---

[*] See Banach (2), 85, where the lemma is deduced from a general theorem concerning the form of a linear functional.

where mS_n denotes the measure of S_n. Then $\phi(x)$ is measurable in (a, b) and

$$\int_a^b |\phi(x)|\,dx = \sum_{n=1}^{\infty} \int_{S_n} |\phi(x)|\,dx = \sum_{n=1}^{\infty} n^{-2} < \infty,$$

while
$$\int_a^b |f(x)\phi(x)|\,dx = \sum_{n=1}^{\infty} \int_{S_n} |f(x)\phi(x)|\,dx \geqslant \sum_{n=1}^{\infty} 1/n = \infty,$$

which is contrary to hypothesis. Hence $f(x)$ must be essentially bounded in (a, b).

We now prove our main result.

THEOREM 1. *Suppose $f_n(x)$ to be measurable in (a, b), where $b-a \leqslant \infty$, for $n = 1, 2, \ldots$. Then a necessary and sufficient condition that, for every function $\phi(x)$ summable over (a, b), the functions $f_n(x)\phi(x)$ should be summable over (a, b) and*

$$\sum_{n=1}^{\infty} \left| \int_a^b f_n(x)\phi(x)\,dx \right| < \infty, \tag{1}$$

is that $\sum_{n=1}^{\infty} |f_n(x)|$ is essentially bounded in (a, b).

Proof. The sufficiency of the condition is trivial, and we omit the proof. Before proving its necessity we observe at once that, if, for $n = 1, 2, \ldots, f_n(x)\phi(x)$ is summable over (a, b) whenever $\phi(x)$ is, then, by Lemma 1,

(2) $f_n(x)$ is essentially bounded in (a, b) $(n = 1, 2, \ldots)$.

Now let E be the aggregate of all functions summable over (a, b). If $y(t)$ is summable over (a, b) we denote $y(t)$ by y, and

$$\int_a^b |y(t)|\,dt \quad \text{by} \quad \|y\|.$$

If y_1 and y_2 belong to E the function $y_1(t)+y_2(t)$ is denoted by y_1+y_2. We also put

$$W_n(y) = \sum_{r=1}^{n} \left| \int_a^b f_r(t)y(t)\,dt \right| \quad \text{and} \quad W(y) = \sum_{r=1}^{\infty} \left| \int_a^b f_r(t)y(t)\,dt \right|.$$

We first show* that, if $W(y) < \infty$ for every y of E, there is a member

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* Alternatively we may use the corollary to Theorem 3, taking

$$U_n(x) = \int_a^b f_n(t)x(t)\,dt, \quad \|U_n(x)\| = |U_n(x)|, \quad \|x\| = \int_a^b |x(t)|\,dt,$$

and obtain (6) at once.

y_0 of E and a positive number δ such that

$$(3) \qquad W(y) \text{ is bounded for } \|y-y_0\| \leqslant \delta.$$

Now if z belongs to E and $W(z) > k$ there is an integer n for which $W_n(z) > k$, and consequently, by (2),

$$W(y) \geqslant W_n(y) > k \text{ provided that } \|y-z\| \text{ is small enough.}$$

Hence, if (3) is *false* for every positive number δ and every member y_0 of E, it is easily seen that there is a sequence of members $y_1, y_2, \ldots$ of E and a sequence of numbers $\{\epsilon_n\}$ tending to zero such that, for $n = 1, 2, \ldots,$

$$(4) \qquad W(y) > n \quad \text{if} \quad \|y-y_n\| \leqslant \epsilon_n,$$

and

$$(5) \qquad 0 < \epsilon_{n+1} < \epsilon_n - \|y_n - y_{n+1}\|.$$

By (5), since

$$\|z_1-z_2\| \leqslant \|z_1-z\| + \|z-z_2\|,$$

we have $$\|y_n-y_{n+m}\| < \epsilon_n \quad (n=1, 2, \ldots; \; m=1, 2, \ldots),$$

and so the sequence of functions $\{y_n(t)\}$ converges *in mean*, *i.e.* there is a member y of E such that $\lim_{n\to\infty} \|y-y_n\| = 0$; hence, and since

$$\|y-y_p\| \leqslant \|y-y_n\| + \|y_n-y_p\| < \|y-y_n\| + \epsilon_p \quad \text{for all} \quad n > p,$$

it follows that $\|y-y_p\| \leqslant \epsilon_p$ $(p=1, 2, \ldots)$, which, by (4), means $W(y) = \infty$.

Thus, if (1) is true for every function $\phi(x)$ summable over (a, b), δ and y_0 *will* exist so as to satisfy (3). Since

$$W(z) \leqslant W(z+y_0) + W(-y_0) = W(z+y_0) + W(y_0),$$

this implies the existence of a finite number M such that

$$(6) \qquad W(z) < M \quad \textit{whenever} \quad \int_a^b |z(t)|\, dt \leqslant \delta.$$

Now let ξ be any point in (a, b) such that

$$(7) \qquad \frac{d}{dx}\int_a^x f_n(t)\, dt = f_n(x) \quad \text{for} \quad x = \xi \quad \text{and} \quad n = 1, 2, \ldots,$$

and let $0 < h < b - \xi$. If we take $z(t) = \delta/h$ in the interval $(\xi, \xi + h)$ and 0 elsewhere, then (6) implies

$$M > W(z) \geqslant W_n(z) = \sum_{r=1}^{n} \left| \frac{\delta}{h} \int_{\xi}^{\xi+h} f_r(t)\, dt \right| \quad (n = 1, 2, \ldots).$$

Letting $h \to 0$, and then $n \to \infty$, we obtain $M/\delta \geqslant \sum_{r=1}^{\infty} |f_r(\xi)|$. Now the set of the points ξ of (a, b) for which (7) is *false* is the union of a sequence of sets each of measure zero, and so is itself of measure zero. Hence we have shown that $M/\delta \geqslant \sum_{r=1}^{\infty} |f_r(x)|$ for almost all x in (a, b), *i.e.* $\sum_{r=1}^{\infty} |f_r(x)|$ is essentially bounded in (a, b).

COROLLARY. *If $f_n(x)$ is continuous in (a, b), for $n = 1, 2, \ldots$, then a necessary and sufficient condition for (1) to hold, for every $\phi(x)$ summable over (a, b), is that $\sum_{n=1}^{\infty} |f_n(x)|$ is bounded in (a, b).*

This may be proved directly by the same proof, since (7) now holds for *every* ξ in (a, b). Or it may be deduced from Theorem 1, since the set of points in (a, b) for which $\sum_{r=1}^{\infty} |f_r(x)| \leqslant k$, where k is any given number, is the intersection of the *closed* sets for which $\sum_{r=1}^{n} |f_r(x)| \leqslant k$, and so is itself closed.

3. We now show that summability $|C, 1|$ for a Fourier series is not a local property of the function.

THEOREM 2. *If $0 < \alpha < \beta < 2\pi$, there is a function, summable over (α, β) and zero in the remainder of $(0, 2\pi)$, whose Fourier series is not summable $|C, 1|$ at $t = 0$.*

Proof. It is known† that, if Σa_n is summable $|C, 1|$, then $\Sigma |a_n|/n < \infty$.

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† Kogbetlianz (5). Alternatively, we can show the existence of $f(t)$, such that

$$\Sigma \left| \int_a^b f(t) \{K_{n+1}(t) - K_n(t)\}\, dt \right| = \infty,$$

where $K_n(t)$ is Fejèr's kernel, by proving that, for $0 < t < 2\pi$,

$$\Sigma \left| \frac{\sin^2 (n+1) \frac{1}{2} t}{n+1} - \frac{\sin^2 \frac{1}{2} n t}{n} \right| = \infty.$$

It is therefore enough to show that there is a function $f(t)$, summable over (α, β), such that

$$\sum_{n=1}^{\infty}\left|\int_{\alpha}^{\beta} f(t)\,\frac{\cos nt}{n}\,dt\right| = \infty. \tag{8}$$

But, if $0 < t < 2\pi \quad (t \neq \pi)$,

$$\Sigma\,\frac{|\cos nt|}{n} \geqslant \Sigma\,\frac{\cos^2 nt}{n} = \tfrac{1}{2}\Sigma\,\frac{1+\cos 2nt}{n} \geqslant \tfrac{1}{2}\Sigma\,\frac{1}{n} - \tfrac{1}{2}\left|\Sigma\,\frac{\cos 2nt}{n}\right| = \infty.$$

The result therefore follows from Theorem 1.

4. We now give an explicit example of a function $f(t)$ summable over (α, β) and satisfying (8). For convenience in calculation we take $\alpha = \pi - 1$, $\beta = \pi$, and the function $f(t)$ which we define is the sum of a sequence $\{f_\nu(t)\}$ of functions each of which is constant in some interval $(\pi - \epsilon_\nu, \pi)$ and zero elsewhere. More precisely, we define a sequence $\{\epsilon_\nu\}$ satisfying $0 < \epsilon_\nu < 1$, and a sequence $\{c_\nu\}$ of positive numbers satisfying $\Sigma\, c_\nu \epsilon_\nu < \infty$; then, if

$$f_\nu(t) = \begin{cases} c_\nu & \text{in } (\pi - \epsilon_\nu, \pi), \\ 0 & \text{elsewhere,} \end{cases} \qquad \text{and} \qquad f(t) = \sum_{\nu=1}^{\infty} f_\nu(t),$$

it follows that $f(t)$ is summable over $(\pi - 1, \pi)$. Further

$$\begin{aligned} \sum_{n=1}^{\infty}\left|\int_{\pi-1}^{\pi} f(t)\,\frac{\cos nt}{n}\,dt\right| &= \sum_{n=1}^{\infty}\left|\int_{\pi-1}^{\pi} \sum_{\nu=1}^{\infty} f_\nu(t)\,\frac{\cos nt}{n}\,dt\right| \\ &= \sum_{n=1}^{\infty}\left|\sum_{\nu=1}^{\infty}\int_{\pi-\epsilon_\nu}^{\pi} c_\nu\,\frac{\cos nt}{n}\,dt\right| \\ &= \sum_{n=1}^{\infty}\left|\sum_{\nu=1}^{\infty} c_\nu\,\frac{\sin n\epsilon_\nu}{n^2}\right| = \sum_{n=1}^{\infty}\left|\sum_{\nu=1}^{\infty} I_{n,\nu}\right|, \end{aligned}$$

the term by term integration being justified by absolute convergence. Now

$$|\sin n\epsilon_\nu| \leqslant \min\,(n\epsilon_\nu, 1)$$

and, if $n\epsilon_\nu \leqslant 1$, $\qquad \sin n\epsilon_\nu > n\epsilon_\nu \cos n\epsilon_\nu > \tfrac{1}{2} n\epsilon_\nu$.

It follows that, if $n \leqslant \epsilon_k^{-1} = n_k$, say, then

$$\begin{aligned} \left|\sum_{\nu=1}^{\infty} I_{n,\nu}\right| &\geqslant |I_{n,k}| - \sum_{\nu>k} |I_{n,\nu}| - \sum_{\nu<k} |I_{n,\nu}| \\ &> \tfrac{1}{2} n^{-1} c_k\,\epsilon_k - n^{-1} \sum_{\nu>k} c_\nu\,\epsilon_\nu - n^{-2} \sum_{\nu<k} c_\nu. \end{aligned}$$

If $\Sigma c_\nu \epsilon_\nu$ converges rapidly, *e.g.* if we now narrow the choice of $\{\epsilon_\nu\}$ and $\{c_\nu\}$ so that $\epsilon_\nu c_\nu = 4^{-\nu}$ $(\nu = 1, 2, \ldots)$, then

$$\tfrac{1}{2} n^{-1} c_k \epsilon_k - n^{-1} \sum_{\nu>k} c_\nu \epsilon_\nu = n^{-1} 4^{-k} \left(\tfrac{1}{2} - \sum_{\nu=1}^{\infty} 4^{-\nu}\right) = (6n\, 4^k)^{-1}.$$

If we also agree to choose $\{\epsilon_\nu\}$ so that $\epsilon_1 > \epsilon_2 > \ldots$, then

$$n^{-2} \sum_{\nu<k} c_\nu - n^{-2} \sum_{\nu=1}^{k-1} 4^{-\nu} \epsilon_\nu^{-1} < (3n^2 \epsilon_{k-1})^{-1} = n_{k-1}(3n^2)^{-1}.$$

Thus, if $n \leqslant n_k$,

$$\left| \sum_{\nu=1}^{\infty} I_{n,\nu} \right| > (6n\, 4^k)^{-1} - n_{k-1}(3n^2)^{-1},$$

and hence, if n_k is chosen to be the square of an integer,

$$\sum_{n=1}^{\infty} \left| \sum_{\nu=1}^{\infty} I_{n,\nu} \right| \geqslant \sum_{n=\sqrt{n_k}}^{n_k} \left| \sum_{\nu=1}^{\infty} I_{n,\nu} \right| > (6\,.\,4^k)^{-1} \sum_{n=\sqrt{n_k}}^{n_k} n^{-1} - \tfrac{1}{3} n_{k-1} \sum_{n=\sqrt{n_k}}^{n_k} n^{-2}$$

$$> (12\,.\,4^k)^{-1} \log n_k - n_{k-1} n_k^{-\frac{1}{2}},$$

for $k = 1, 2, \ldots$. It only remains to choose n_k so that the last expression tends to infinity as $k \to \infty$. For example, we may take $n_k = 4^{8^k}$, *i.e.* we may take $\epsilon_\nu = 4^{-8^\nu}$, $c_\nu = 4^{8^\nu - \nu}$.

5. We end by giving a general theorem concerning functionals in an abstract space. This is a direct extension of the Banach-Steinhaus theorem*, and is a generalization of one of the main steps in the proof of Theorem 1. For an introduction to the relevant theory of operations the reader may refer to Zygmund (**5**), or Banach (**2**). We mention the following properties of a *linear normalized metric space* E: if x and y belong to E so does $x+y$ and so does αx for every finite real number α; also E includes a null element Θ satisfying $\alpha\Theta = \Theta$ and $x + \Theta = x$; to every x of E is assigned a finite number $\|x\|$ such that $0 \leqslant \|x+y\| \leqslant \|x\| + \|y\|$, $\|x\| = 0$ if and only if $x = \Theta$, and $\|\alpha x\| = |\alpha|\, \|x\|$; the *distance* between any two elements x and y of E is $\|x-y\|$. If E_1 and E_2 are linear metric spaces and to every x of E_1 is assigned $U(x)$, a member of E_2, so that

$$U(x+y) = U(x) + U(y), \quad \text{and} \quad U(x) \to U(y) \quad \text{if} \quad x \to y,$$

then $U(x)$ is a *linear operation* defined in E_1.

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* See Zygmund (**5**), pp. 97–98.

THEOREM[*] 3. *Suppose that E is a linear normalized metric space and that to every x of E is assigned a number $W(x)$ satisfying the following conditions:*

(9) $$0 \leqslant W(x) \leqslant \infty,$$

(10) $$W(x_1+x_2) \leqslant W(x_1)+W(x_2),$$

(11) *if a is a finite real number and $W(x) < \infty$, then $W(ax) = |a|\, W(x)$,*

(12) *$W(x)$ is lower semi-continuous*[†] *in E,*

(13) *the set F of the points x of E for which $W(x) < \infty$ is of the second category in E.*

Then there is a finite number M such that $W(x) \leqslant M\,||x||$ for every x of E.

Proof. If E_n is the set of those x for which $W(x) \leqslant n$, then E_n is closed by (12), and so, since $F = E_1+E_2+\ldots$ is of the second category by (13), it follows that one of the E_n contains a sphere in E. This means that there exist finite positive numbers δ and N, and a member x_0 of E, such that

$$W(x_0+x) \leqslant N \text{ for every } x \text{ of } E \text{ satisfying } ||x|| \leqslant \delta.$$

In particular, taking $||x|| = 0$, we have from (11) $W(-x_0) = W(x_0) \leqslant N$, and so, by (10),

$$\text{if} \quad ||x|| \leqslant \delta, \quad W(x) \leqslant W(x_0+x)+W(-x_0) \leqslant 2N.$$

Since $||ax|| = |a|\,||x||$, this gives

$$W\left(\frac{\delta}{||x||}\,x\right) \leqslant 2N \quad \text{whenever} \quad ||x|| > 0,$$

and so, by (11), $W(x) \leqslant \frac{2N}{\delta}\,||x||$ for all x of E with $||x|| > 0$. Finally the restriction $||x|| > 0$ may be relaxed by (11).

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[*] Theorem 3 may be regarded, geometrically, as a result concerning alternative *norms* in the same space, viz.: Suppose that E is a linear space which is complete when made metric by a norm $||x||$; if $\delta(x)$ is an alternative norm in E such that $||x-x_n|| \to 0$ implies $\delta(x) \leqslant \underline{\lim}\, \delta(x_n)$, then the ratio $\delta(x)/||x||$ is bounded for x in E $(x \neq \Theta)$.

[†] *I.e.*, for every real number λ, the set of the points x of E for which $W(x) \leqslant \lambda$ is closed.

COROLLARY. *Suppose that* $\{U_n(x)\}$ *is a sequence of linear operations defined in a* complete *space* E *and that*

$$\sum_{n=1}^{\infty} \| U_n(x) \| < \infty \quad \textit{for every } x \textit{ of } E.$$

Then there is a finite number M *such that*

$$\sum_{n=1}^{\infty} \| U_n(x) \| \leqslant M \| x \| \quad \textit{for every } x \textit{ of } E.$$

Proof. Since $\| U_n(x) \|$ is continuous in E and $U_n(x)$ is additive, it follows easily that $W(x) = \sum_{n=1}^{\infty} \| U_n(x) \|$ satisfies conditions (9) to (12) of Theorem 3. Moreover E, being complete, is of the second category, and so the result follows from Theorem 3.

Finally we remark that, by taking $W(x) = \text{upper bound}_{m \geqslant 1} \| U_m(x) \|$, where $\{U_m(x)\}$ is a sequence of linear operations, we obtain, as another corollary of Theorem 3, the Banach-Steinhaus theorem, that* *if the set for which* $\overline{\lim}_{n \to \infty} \| U_n(x) \| < \infty$ *is of the second category in* E, *then there is a finite number* M *such that* $\text{upper bound}_{m \geqslant 1} \| U_m(x) \| \leqslant M \| x \|$ *for all* x *of* E.

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* This is identical with the set F.

ON THE PRODUCT OF THREE HOMOGENEOUS LINEAR FORMS (III)

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1. *Introduction.*

The previous papers of this series were concerned with the product of three real homogeneous linear forms L_1, L_2, L_3 of determinant 1. If M denotes the lower bound of $|L_1 L_2 L_3|$ for integral values of the variables, not all zero, it was proved that

$$M \leqslant \tfrac{1}{7},$$

and that this result is best possible. The present paper solves the corresponding problem for three linear forms, of which one is real and the other two are conjugate complex.

Let L be a homogeneous linear form in u, v, w with complex coefficients, let $\bar{L}$ be the conjugate complex form, and let L_1 be a real homogeneous linear form in u, v, w. Suppose[†] that the determinant of L, $\bar{L}$, L_1 has absolute value 1. Let[‡]

$$M = \min |L \bar{L} L_1|$$

when u, v, w run through all sets of integers except 0, 0, 0. The result which will be proved is that

$$M \leqslant \frac{1}{\sqrt{23}}. \tag{1}$$

This result is best possible. For let ϕ, θ, $\bar{\theta}$ be the roots of

$$t^3 - t - 1 = 0,$$

[†] Of course there is no loss of generality in this.

[‡] min means lower bound, whether attained or not.

ϕ being real and θ, $\bar{\theta}$ conjugate complex. Let λ be any complex number and μ any real number such that $|\lambda\bar{\lambda}\mu| = 1/\sqrt{23}$.

Then
$$L = \lambda(u+v\theta+w\theta^2),$$
$$\bar{L} = \bar{\lambda}(u+v\bar{\theta}+w\bar{\theta}^2),$$
$$L_1 = \mu(u+v\phi+w\phi^2)$$

are linear forms whose determinant has† absolute value 1. Since $u+v\phi+w\phi^2$ is an integer of the cubic field $K(\phi)$, its norm is a rational integer, hence
$$|L\bar{L}L_1| \geqslant |\lambda\bar{\lambda}\mu| = \frac{1}{\sqrt{23}}$$
for $u, v, w \neq 0, 0, 0$.

The method used in this paper is similar to that of (II) only in so far as the first steps in the argument are concerned. The subsequent analysis, though elementary, is more complicated and of a different character.

2. *Reduction to a two-dimensional problem.*

Let L, $\bar{L}$, L_1 be linear forms as in §1, and let M be the lower bound of $|L\bar{L}L_1|$, as already defined. We shall suppose that $M>0$. For arbitrarily small $\epsilon>0$ there exist values of L, $\bar{L}$, L_1, say L^*, $\bar{L}^*$, $L_1{}^*$ (corresponding to integral u, v, w not all zero), such that

$$M \leqslant |L^*\bar{L}^*L_1{}^*| < \frac{M}{1-\epsilon}. \tag{2}$$

We define three real homogeneous linear forms x, y, z by
$$\frac{L_1}{L_1{}^*} = z, \qquad \frac{L}{L^*} = z+x+iy.$$

The determinant Δ of x, y, z is the same as that of
$$\frac{1}{2}\left(\frac{L}{L^*}+\frac{\bar{L}}{\bar{L}^*}\right), \quad \frac{1}{2i}\left(\frac{L}{L^*}-\frac{\bar{L}}{\bar{L}^*}\right), \quad \frac{L_1}{L_1{}^*},$$

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† This follows from the fact that the discriminant of $t^3-t-1=0$ is -23.

hence, since the determinant of L, $\bar{L}$, L_1 has absolute value 1,

$$|\Delta| = \frac{1}{2|L^* \bar{L}^* L_1^*|}.$$

Therefore, by (2),

$$\frac{1-\epsilon}{2M} < |\Delta| \leqslant \frac{1}{2M}. \tag{3}$$

For all integral values of u, v, w not all zero we have

$$|L \bar{L} L_1| \geqslant M > (1-\epsilon)|L^* \bar{L}^* L_1^*|,$$

whence

$$\big((z+x)^2+y^2\big)|z| > 1-\epsilon. \tag{4}$$

Also the values of u, v, w that give L, $\bar{L}$, L_1 the particular values L^*, $\bar{L}^*$, L_1^* make $x=0$, $y=0$, $z=1$.

Hence the points in x, y, z space which correspond to integral values of u, v, w form a lattice with the properties:

(*a*) the determinant Δ of the lattice satisfies (3);

(*b*) every lattice point except (0, 0, 0) satisfies (4);

(*c*) (0, 0, 1) is a lattice point.

It is plain that (0, 0, 1) is a *primitive* lattice point; for the existence of a lattice point $(0, 0, 1/k)$, where k is an integer greater than 1, would contradict (*b*). Hence there exist two lattice points, say (x_1, y_1, z_1), (x_2, y_2, z_2), which together with (0, 0, 1) generate the lattice. Then the points of the lattice are precisely the points given by

$$\begin{cases} x = \phantom{U+{}} Vx_1 + Wx_2, \\ y = \phantom{U+{}} Vy_1 + Wy_2, \\ z = U + Vz_1 + Wz_2, \end{cases} \tag{5}$$

when U, V, W run through all integers. Plainly

$$\begin{vmatrix} x_1, & x_2 \\ y_1, & y_2 \end{vmatrix} = \Delta.$$

We now consider the two dimensional lattice defined by the first two equations of (5). It has the properties:

(a') the determinant of the lattice is Δ;

(b') corresponding to any lattice point (x, y) other than $(0, 0)$ there exists a real number z' such that, for all integers U,

$$\big((z'+U+x)^2+y^2\big)|z'+U|>1-\epsilon. \tag{6}$$

The contrary of the assertion in (b') (with a slight change of notation) is that for any real number z there exists an integer n such that

$$\big((z+x-n)^2+y^2\big)|z-n|\leqslant 1-\epsilon.$$

Let

$$F_\epsilon(x)=\min_z\max_n\left(\frac{1-\epsilon}{|z-n|}-(z+x-n)^2\right) \tag{7}$$

(z real, n integral), and let D_ϵ be the domain consisting of all points (x, y) for which

$$y^2\leqslant F_\epsilon(x). \tag{8}$$

Then the statement (b') is equivalent to the statement that $(0, 0)$ is the only lattice point in D_ϵ.

We now see that (1) is a consequence of

THEOREM 1. *Let $\mathfrak{L}$ be a lattice of determinant Δ in the x, y plane such that* $(0, 0)$ *is the only lattice point in D_ϵ. Then*

$$|\Delta|>(1-\epsilon_1)\tfrac{1}{2}\sqrt{23},$$

where ϵ_1 depends only on ϵ and tends to zero with ϵ.

For, if Theorem 1 holds, we have, by (3),

$$M\leqslant\frac{1}{2|\Delta|}<\frac{1}{(1-\epsilon_1)\sqrt{23}},$$

and this implies (1).

3. *Notation.*

As in § 1, we denote by ϕ, θ, $\bar\theta$ the roots of

$$t^3-t-1=0. \tag{9}$$

Since $\phi+\theta+\bar{\theta}=0$, we have $\theta=-\frac{1}{2}\phi+i\psi$, where ψ is real. We may suppose that $\psi>0$. Also, since $\phi\theta\bar{\theta}=1$, we have

$$\tfrac{1}{4}\phi^2+\psi^2=\frac{1}{\phi}=\phi^2-1,$$

whence

(10) $$\psi^2=\tfrac{3}{4}\phi^2-1.$$

If in (9) we write $t^2=t'$, we obtain the equation

(11) $$t'(t'-1)^2-1=0;$$

hence the roots of this are ϕ^2, θ^2, $\bar{\theta}^2$, *i.e.* are

(12) $$\phi^2,\quad 1-\tfrac{1}{2}\phi^2+i\phi\psi,\quad 1-\tfrac{1}{2}\phi^2-i\phi\psi.$$

If in (9) we write $t-t^2=t''$, so that $t''(t+1)=t-t^3=-1$, we obtain the equation

(13) $$(t''+1)^3-t''^2=0;$$

hence the roots of this are

(14) $$\phi-\phi^2,\quad -1-\tfrac{1}{2}\phi+\tfrac{1}{2}\phi^2+i(\phi+1)\psi,\quad -1-\tfrac{1}{2}\phi+\tfrac{1}{2}\phi^2-i(\phi+1)\psi.$$

For future reference, we observe that

(15) $$(3\phi^2-1)\psi=\tfrac{1}{2}\sqrt{23}.$$

For, since the discriminant of (9) is -23, we have

$$(\theta-\bar{\theta})(\phi-\theta)(\phi-\bar{\theta})=\pm i\sqrt{23};$$

and $$\theta-\bar{\theta}=2i\psi,\quad (\phi-\theta)(\phi-\bar{\theta})=(\tfrac{3}{2}\phi)^2+\psi^2=3\phi^2-1,$$

by (10).

The numerical values of ϕ, ψ, ϕ^2, ψ^2 are

(16) $$\begin{cases} \phi=1{\cdot}32471\ldots, & \psi={\cdot}56227\ldots, \\ \phi^2=1{\cdot}75487\ldots, & \psi^2={\cdot}31615\ldots. \end{cases}$$

The following points in the plane are of particular importance in our investigations:

$$C_1=\left(1+\tfrac{3}{2}\phi-\tfrac{3}{2}\phi^2,\ (\phi+1)\psi\right)=({\cdot}35476\ldots,\ 1{\cdot}30714\ldots),$$

$$C_2=(-1+\tfrac{3}{2}\phi^2,\ \phi\psi)=(1{\cdot}63231\ldots,\ {\cdot}74486\ldots),$$

$$C_3=(\tfrac{3}{2}\phi,\ \psi)=(1{\cdot}98707\ldots,\ {\cdot}56227\ \ldots).$$

We shall use c_i, d_i for the coordinates of C_i, and we shall denote by C_i' the point $(c_i, -d_i)$. It is plain that

$$c_3 - c_2 = c_1, \quad d_3 + d_2 = d_1, \tag{17}$$

so that the line $C_2' C_3$ is parallel and equal to the line OC_1. Also

$$c_3 d_1 - c_1 d_3 = \tfrac{3}{2}\phi(\phi+1)\psi - (1 + \tfrac{3}{2}\phi - \tfrac{3}{2}\phi^2)\psi = \tfrac{1}{2}\sqrt{23}, \tag{18}$$

by (15). Thus the area of the parallelogram $OC_1 C_3 C_2'$ is $\frac{1}{2}\sqrt{23}$.

4. *The domains D, E. Enunciation of Theorems 2, 3.*

Let

$$F(x) = \min_{z} \max_{n} \left(\frac{1}{|z-n|} - (z+x-n)^2 \right) \tag{19}$$

(z real, n integral) and let D denote the domain

$$y^2 \leqslant F(x). \tag{20}$$

Plainly $F(x)$ is an even function of x, so that D is symmetrical with respect to both coordinate axes.

The inequality (20) defining D may be expressed alternatively as follows. A point (x, y) belongs to D if and only if for every z there exists $z' \equiv z \pmod 1$ such that

$$\left| z' \left((z'+x)^2 + y^2 \right) \right| \leqslant 1. \tag{21}$$

This last statement may be conveniently abbreviated to: (21) holds for all $z' \pmod 1$. The definition of D_ϵ may be expressed in the same form, with $1-\epsilon$ in place of 1 on the right of (21).

LEMMA 1. *The points C_1, C_2, C_3 (and therefore also the points C_1', ...) lie on the boundary of D.*

Proof. We have to prove that each of the polynomials

$$f_1(z) = z\left((z+1+\tfrac{3}{2}\phi - \tfrac{3}{2}\phi^2)^2 + (\phi+1)^2 \psi^2 \right), \tag{22}$$

$$f_2(z) = z\left((z-1+\tfrac{3}{2}\phi^2)^2 + \phi^2 \psi^2 \right), \tag{23}$$

$$f_3(z) = z\left((z+\tfrac{3}{2}\phi)^2 + \psi^2 \right) \tag{24}$$

satisfies $|f(z)| \leqslant 1$ for all z (mod 1), and that this ceases to be true if $(\phi+1)\psi$ or $\phi\psi$ or ψ is replaced by any larger number.

Consider first $f_3(z)$. The roots of $f_3(z)=0$ are 0, $-\frac{3}{2}\phi+i\psi$, $-\frac{3}{2}\phi-i\psi$. Hence the roots of $f_3(z-\phi)=0$ are ϕ, $-\frac{1}{2}\phi+i\psi$, $-\frac{1}{2}\phi-i\psi$. Hence

$$f_3(z-\phi)=z^3-z-1.$$

It is easily seen from this that $|f_3(z-\phi)| \leqslant 1$ for $-1 \leqslant z \leqslant 0$. On the other hand, if ψ^2 is replaced in (24) by $\psi^2+\delta$, where $\delta>0$, $f_3(z-\phi)$ becomes

$$z^3-z-1+\delta(z-\phi).$$

This has absolute value greater than 1 for all $z \equiv 0$ (mod 1). For, if z is an integer greater than ϕ,

$$z^3-z-1+\delta(z-\phi) > z^3-z-1 \geqslant 1,$$

and, if z is an integer less than ϕ,

$$z^3-z-1+\delta(z-\phi) < z^3-z-1 \leqslant -1.$$

Similarly, the roots of $f_2(z)=0$ are

$$0, \quad 1-\tfrac{3}{2}\phi^2+i\phi\psi, \quad 1-\tfrac{3}{2}\phi^2-i\phi\psi,$$

hence, by (11), (12),

$$f_2(z-\phi^2)=z(z-1)^2-1.$$

Hence $|f_2(z-\phi^2)| \leqslant 1$ for $0 \leqslant z \leqslant 2$. If $\phi^2\psi^2$ is replaced by $\phi^2\psi^2+\delta$, where $\delta>0$, the new polynomial has absolute value greater than 1 for all integral z.

Similarly, the roots of $f_1(z)=0$ are

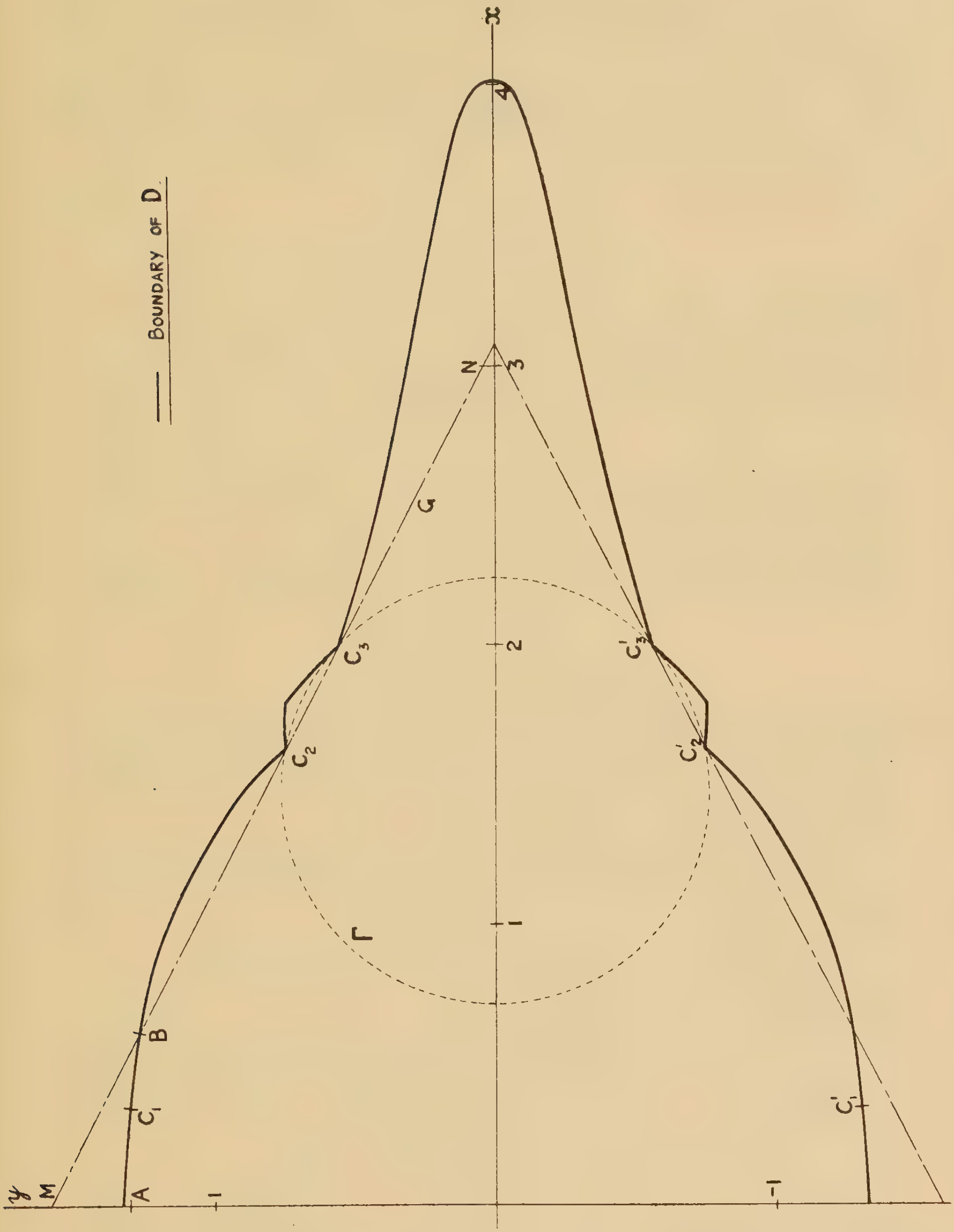
$$0, \quad -1-\tfrac{3}{2}\phi+\tfrac{3}{2}\phi^2+i(\phi+1)\psi, \quad -1-\tfrac{3}{2}\phi+\tfrac{3}{2}\phi^2-i(\phi+1)\psi,$$

hence, by (13), (14),

$$f_1(z-\phi+\phi^2)=(z+1)^3-z^2.$$

Hence $$|f_1(z-\phi+\phi^2)| \leqslant 1 \quad \text{for} \quad -1 \leqslant z \leqslant 0.$$

Again, if $(\phi+1)^2\psi^2$ is replaced by a larger number, the new polynomial has absolute value greater than 1 for all integral z.


Boundary of D.
x
4
N
3
G
2
C₃
C₃'
C₂
C₂'
Γ
1
B
C₁
C₁'
y
M
A
1
-1

Let G denote the straight line through C_2, C_3. The equation of G is easily found to be

$$(25) \qquad y=\frac{\phi^2+\phi-x}{\phi^2-\frac{1}{2}\phi}\,\psi=l(x),$$

say. Let

$$M=\left(0,\ \frac{\phi+1}{\phi-\frac{1}{2}}\,\psi\right)$$

be the point in which G meets $x=0$, and let N denote the point

$$\left(3,\ \frac{\phi^2+\phi-3}{\phi^2-\frac{1}{2}\phi}\,\psi\right)$$

on G. Let A denote the point $\bigl(0,\ (\phi+1)\psi\bigr)$. Let Γ denote the circle through C_2, C_3, C_2', C_3'.

It will be proved later (Lemma 6) that the part of the curve $y^2=F(x)$ on which $0\leqslant x<c_2$, $y>0$ meets G in a single point $B=(e, f)$, and that

$$\cdot 6<e<\cdot 62, \quad 1\cdot 27<f<1\cdot 28.$$

We define E to be the closed domain, symmetrical with respect to both coordinate axes, bounded for $x\geqslant 0$, $y\geqslant 0$ by the following curve: the straight line AC_1, the straight line C_1B, the straight line BC_2 (a part of G), the arc C_2C_3 of Γ, the straight line C_3N (a part of G), and the line $x=3$ (the ordinate at N).

The diagram shows the domains D, E for $x\geqslant 0$.

We define Q to be the interior of the quadrilateral $AMBC_1$, together with its boundary.

The proof of Theorem 1 is based on the following two results:

THEOREM 2. *E is contained in D.*

THEOREM 3. *If (x_0, y_0) is any point of Q, and $0<\lambda<\frac{1}{2}\sqrt{23}$, then the straight line*

$$xy_0-yx_0=\lambda$$

meets the boundary of E in exactly two points, say (x_1, y_1), (x_2, y_2), and these satisfy the inequality

$$|y_1-y_2|>y_0.$$

5. *Deduction of Theorem* 1 *from Theorems* 2, 3.

Let $\mathcal{L}$ be a plane lattice of determinant Δ such that $(0, 0)$ is the only point of $\mathcal{L}$ in D_ϵ. Let $\mathcal{L}'$ be the lattice formed by all points $\big(x, y/(1-10^6\epsilon)\big)$, where (x, y) is a point of $\mathcal{L}$. Plainly the determinant Δ' of $\mathcal{L}'$ is given by

$$\Delta = (1-10^6\epsilon)\,\Delta'. \tag{26}$$

LEMMA 2. $(0, 0)$ *is the only point of* $\mathcal{L}'$ *in* E.

Proof. It suffices to prove that, if (x, y) is in E, then $\big(x, (1-10^6\epsilon)\,y\big)$ is in D_ϵ. If (x, y) is in E, so is (x, y') for all y' with $|y'| \leqslant |y|$, and the same applies to D_ϵ. Hence we need consider only points (x, y) of E for which y has its maximum value for given x, *i.e.* points on the upper boundary of E. For such points we have

$$y \geqslant \frac{\phi^2+\phi-3}{\phi^2-\frac{1}{2}\phi}\,\psi > \cdot 04; \tag{27}$$

also

$$|x| \leqslant 3, \quad y \leqslant (\phi+1)\,\psi < 2. \tag{28}$$

By Theorem 2, (x, y) lies in D. Hence the inequality

$$\big|z\big((z+x)^2+y^2\big)\big| \leqslant 1 \tag{29}$$

is satisfied for all z (mod 1). To prove that $\big(x, (1-10^6\epsilon)\,y\big)$ is in D_ϵ we have to prove that

$$\big|z\big((z+x)^2+y^2(1-10^6\epsilon)^2\big)\big| \leqslant 1-\epsilon \tag{30}$$

is satisfied for all z (mod 1). Hence it suffices to prove that (29) implies (30).

If $|z| < 10^{-2}$, (30) is valid by (27) and (28). If $|z| \geqslant 10^{-2}$, the left-hand side of (29) exceeds that of (30) by

$$|z|\,y^2\{1-(1-10^6\epsilon)^2\} > 10^{-2}(\cdot 04)^2\,10^6\epsilon > \epsilon.$$

This proves the result.

LEMMA 3. *If* $|\Delta'| \leqslant \frac{1}{2}\sqrt{23}$, *there is a point of* $\mathcal{L}'$ *either in the domain* Q *or in the image* Q' *of* Q *in* $y=0$.

Proof. Let T denote the domain, symmetrical with respect to both coordinate axes, whose boundary in the positive quadrant is the straight line MN (a part of G) and the line $x=3$ (the ordinate at N). The area of T is

$$4\int_0^3 l(x)\,dx = 12\,\frac{\phi^2+\phi}{\phi^2-\frac{1}{2}\phi}\,\psi - 18\,\frac{1}{\phi^2-\frac{1}{2}\phi}\,\psi$$

$$= 19{\cdot}019\ldots - 9{\cdot}264\ldots$$

$$> 4\left(\tfrac{1}{2}\sqrt{23}\right) \geqslant 4\,|\Delta'|.$$

Hence, by the classical theorem of Minkowski, T contains a point of the lattice $\mathcal{L}'$, other than $(0, 0)$. But the only parts of T which do not belong to E consist of Q and the three domains obtained from Q by reflection in the coordinate axes. Hence either Q or Q' contains a lattice point.

By changing y into $-y$, we can suppose without loss of generality that Q contains a lattice point.

Proof of Theorem 1. By the above, we can suppose that there is a point (x_0, y_0) of the lattice $\mathcal{L}'$ in Q. Plainly (x_0, y_0) is a primitive lattice point, for otherwise there would be a lattice point in E, contrary to Lemma 2. Hence there exists a lattice point (x', y') such that

$$x'y_0 - y'x_0 = |\Delta'|.$$

We assume that $|\Delta'| < \frac{1}{2}\sqrt{23}$, and deduce a contradiction. By Theorem 3, the line

$$xy_0 - yx_0 = |\Delta'| \tag{31}$$

meets the boundary of E in exactly two points, say (x_1, y_1), (x_2, y_2), where $y_1 > y_2$, and these satisfy

$$y_1 - y_2 > y_0. \tag{32}$$

For any integer n, the point

$$(x'+nx_0,\ y'+ny_0) \tag{33}$$

is a lattice point, and lies on the line (31). By (32), we can choose n so that

$$y_2 < y'+ny_0 < y_1.$$

But then the point (33) lies in E, contrary to Lemma 2.

It follows that $|\Delta'| \geqslant \frac{1}{2}\sqrt{23}$. By (26),

$$|\Delta| \geqslant (1-10^6\epsilon)\,\tfrac{1}{2}\sqrt{23}.$$

This proves Theorem 1.

6. *Proof of Theorem* 2.

We first evaluate $F(x)$, defined by (19), for $0 \leqslant x \leqslant c_2 = \frac{3}{2}\phi^2 - 1$.

LEMMA 4. *For* $0 \leqslant x \leqslant \frac{3}{2}\phi^2 - 1$, *we have*

$$F(x) = \frac{1}{\zeta} - (\zeta + x)^2, \tag{34}$$

where ζ *is the unique solution in* $0 < \zeta < 1$ *of*

$$\frac{1}{\zeta} - (\zeta + x)^2 = \frac{1}{1-\zeta} - (\zeta - 1 + x)^2, \tag{35}$$

i.e.,

$$\frac{1}{\zeta} - \frac{1}{1-\zeta} = 2\zeta - 1 + 2x.$$

Also the interval $0 \leqslant x \leqslant \frac{3}{2}\phi^2 - 1$ *corresponds* (1, 1) *to the interval*

$$\tfrac{1}{2} \geqslant \zeta \geqslant 2 - \phi^2.$$

Proof. We first prove that (35) has a unique solution ζ in $0 < \zeta < 1$. Let

$$g(\zeta) = \frac{1}{\zeta} - (\zeta + x)^2 - \frac{1}{1-\zeta} + (\zeta - 1 + x)^2.$$

Then

$$g'(\zeta) = -\frac{1}{\zeta^2} - \frac{1}{(1-\zeta)^2} - 2 < 0.$$

Also $g(0) = \infty$, $g(1) = -\infty$. Hence $g(\zeta) = 0$ has a unique solution ζ in $0 < \zeta < 1$. Since (35) is linear in x, the correspondence between x and ζ is (1, 1). It is plain that $x = 0$ corresponds to $\zeta = \frac{1}{2}$. Also, by (35),

$$2\frac{dx}{d\zeta} = -2 - \frac{1}{\zeta^2} - \frac{1}{(1-\zeta)^2} < 0;$$

hence ζ decreases as x increases. If in (35) we take $\zeta = 2 - \phi^2$, we obtain

$$\begin{aligned} 2x + 2(2-\phi^2) - 1 &= \frac{1}{2-\phi^2} - \frac{1}{\phi^2 - 1} \\ &= (1 + \phi + \phi^2) - \phi \\ &= \phi^2 + 1, \end{aligned}$$

whence

$$x = \tfrac{3}{2}\phi^2 - 1.$$

We now come to the proof of (34). Plainly the definition (19) is unchanged if we impose on z the condition $0 < z < 1$. Hence

$$(36)\quad F(x) \geqslant \min\left\{ \min_{0<z\leqslant\zeta} \left(\frac{1}{z} - (z+x)^2\right), \quad \min_{\zeta\leqslant z<1} \left(\frac{1}{1-z} - (z-1+x)^2\right) \right\}.$$

The derivative of $(1/z) - (z+x)^2$ with respect to z is

$$-\frac{1}{z^2} - 2(z+x) < 0.$$

Hence

$$(37)\qquad \min_{0<z\leqslant\zeta} \left(\frac{1}{z} - (z+x)^2\right) = \frac{1}{\zeta} - (\zeta+x)^2.$$

The derivative of $1/(1-z) - (z-1+x)^2$ with respect to z is

$$(38)\qquad \frac{1}{(1-z)^2} - 2(z-1+x).$$

This expression is an increasing function of z for $0 < z < 1$, since its derivative is

$$\frac{2}{(1-z)^3} - 2 > 0.$$

Hence the expression (38) is non-negative for $\zeta \leqslant z < 1$ provided that

$$(39)\qquad \frac{1}{(1-\zeta)^2} - 2(\zeta - 1 + x) \geqslant 0.$$

When we substitute for x from (35), this becomes

$$(40)\qquad \frac{\zeta^3 - 4\zeta^2 + 5\zeta - 1}{\zeta(1-\zeta)^2} \geqslant 0.$$

If we write $2 - \zeta = \tau$, the numerator becomes

$$-\tau(\tau-1)^2 + 1,$$

and this is non-negative for $\tau \leqslant \phi^2$ since, by (11), (12), $\tau - \phi^2$ is its only real zero. Hence (39) holds for $\zeta \geqslant 2 - \phi^2$, *i.e.* for $x \leqslant \frac{3}{2}\phi^2 - 1$. Thus we have

$$(41)\qquad \min_{\zeta\leqslant z<1} \left(\frac{1}{1-z} - (z-1+x)^2\right) = \frac{1}{1-\zeta} - (\zeta-1+x)^2,$$

and therefore, by (35), (36), (37), (41),

$$F(x) \geqslant \frac{1}{\zeta} - (\zeta+x)^2.$$

On the other hand, we have

$$F(x) \leqslant \max_{n} \left(\frac{1}{|\zeta-n|}-(\zeta-n+x)^2\right).$$

Hence, to complete the proof of (34), it suffices to prove that, for $n \neq 0, 1$,

$$\frac{1}{|\zeta-n|}-(\zeta-n+x)^2 \tag{42}$$

is less than or equal to either expression in (35). If $n \leqslant -1$, the expression (42) is less than $1-1=0$. If $n \geqslant 3$, the expression is less than

$$\frac{1}{3-\frac{1}{2}}-(3-\tfrac{1}{2}-x)^2 < \tfrac{2}{5}-(3-\tfrac{1}{2}-1{\cdot}64)^2 < 0.$$

If $n=2$, we have to prove that

$$\frac{1}{2-\zeta}-(\zeta-2+x)^2 \leqslant \frac{1}{1-\zeta}-(\zeta-1+x)^2.$$

When we substitute for x from (35), this becomes

$$\frac{1}{\zeta}+\frac{1}{2-\zeta}-\frac{2}{1-\zeta} \leqslant 2,$$

i.e.

$$2\zeta^3-8\zeta^2+10\zeta-2 \geqslant 0.$$

This follows from (40).

Remark. It is plain from Lemma 1 that

$$F(\tfrac{3}{2}\phi-\tfrac{3}{2}\phi^2+1) = \big((\phi+1)\psi\big)^2.$$

Actually the value of ζ which corresponds to $x=1+\frac{3}{2}\phi-\frac{3}{2}\phi^2$ is

$$\zeta=\phi^2-\phi.$$

LEMMA 5. *As x increases from 0 to $\frac{3}{2}\phi^2-1$, $y=\sqrt{\{F(x)\}}$ decreases steadily from $\frac{1}{2}\sqrt{7}$ to $\phi\psi$, and $-dy/dx$ increases steadily from 0.*

Proof. With the notation of Lemma 4, we have

$$2(\zeta+x) = 1+\frac{1}{\zeta}-\frac{1}{1-\zeta}, \tag{43}$$

$$4y^2 = \frac{4}{\zeta}-\left(1+\frac{1}{\zeta}-\frac{1}{1-\zeta}\right)^2$$

$$= \frac{4}{\zeta(1-\zeta)}-\left(\frac{1}{\zeta}+\frac{1}{1-\zeta}-1\right)^2.$$

Write $t=\zeta(1-\zeta)$. As ζ decreases from $\frac{1}{2}$ to $2-\phi^2$, t decreases steadily from $\frac{1}{4}$ to $(2-\phi^2)(\phi^2-1)$. We have

$$4y^2=-1+\frac{6}{t}-\frac{1}{t^2}. \tag{44}$$

In terms of t, ζ is given by $2\zeta=1-\sqrt{(1-4t)}$. By (43),

$$2x=\left(1+\frac{1}{t}\right)\sqrt{(1-4t)}. \tag{45}$$

From (44) we have

$$4y\frac{dy}{dt}=\frac{1-3t}{t^3}. \tag{46}$$

Since $t\leqslant\frac{1}{4}$, $dy/dt>0$. Hence y decreases steadily as t decreases, *i.e.* as x increases. By Lemma 1, $x=\frac{3}{2}\phi^2-1$ corresponds to $y=\phi\psi$.

From (45) we have

$$-2\frac{dx}{dt}=\left(1+\frac{1}{t}\right)\frac{2}{\sqrt{(1-4t)}}+\frac{1}{t^2}\sqrt{(1-4t)}$$
$$=\frac{1-2t+2t^2}{t^2\sqrt{(1-4t)}}.$$

Combining this with (46), we have

$$2y\left(-\frac{dy}{dx}\right)=\frac{(1-3t)\sqrt{(1-4t)}}{t(1-2t+2t^2)}.$$

As t decreases, the numerator increases. Also the denominator decreases, since its derivative is

$$1-4t+6t^2>0.$$

We have already proved that y decreases. Hence $-dy/dx$ steadily increases as t decreases, *i.e.* as x increases. Plainly, $t=\frac{1}{4}$ (*i.e.* $x=0$) gives $dy/dx=0$.

LEMMA 6. *The equation* $F(x)=l^2(x)$ *has a unique solution* $x=e$ *for* $0\leqslant x<c_2$. *If* $f=l(e)$, *then* $\cdot 6<e<\cdot 62$, $1\cdot 27<f<1\cdot 28$.

Proof. The function $l(x)-\sqrt{\{F(x)\}}$ vanishes at $x=c_2$, by Lemma 1. Its value at $x=0$ is

$$\frac{\phi^2+\phi}{\phi^2-\frac{1}{2}\phi}\psi-\tfrac{1}{2}\sqrt{7}=1\cdot 585\ldots-1\cdot 322\ldots>0.$$

By Lemma 5, the derivative of the function steadily increases as x increases from 0 to c_2. Hence the function has at most one zero for $0 < x < c_2$. By calculation (using Lemma 4) we find that†

$$l(\cdot 6) - \surd\{F(\cdot 6)\} = 1\cdot 2761 \ldots - 1\cdot 2752 \ldots > 0,$$

$$l(\cdot 62) - \surd\{F(\cdot 62)\} = 1\cdot 2658 \ldots - 1\cdot 2717 \ldots < 0.$$

Hence there is exactly one root $x = e$, and the inequalities satisfied by e, f follow from the above.

Let $y = l_1(x)$ be the equation of the straight line BC_1.

LEMMA 7.

$$F(x) \geqslant \big((\phi+1)\psi\big)^2 \quad \textit{for} \quad 0 \leqslant x \leqslant c_1,$$

$$F(x) \geqslant \big(l_1(x)\big)^2 \quad \textit{for} \quad c_1 \leqslant x \leqslant e,$$

$$F(x) \geqslant \big(l(x)\big)^2 \quad \textit{for} \quad e \leqslant x \leqslant c_2.$$

Proof. The first inequality follows at once from the fact that $F(x)$ decreases steadily as x increases, and $\surd\{F(c_1)\} = d_1 = (\phi+1)\psi$ (Lemmas 5, 1). For the second inequality, it is sufficient to observe that

$$\surd\{F(x)\} - l_1(x)$$

vanishes for $x = c_1$ and $x = e$, and has a derivative which steadily decreases as x increases from c_1 to e (Lemma 5). The same argument applies to the third inequality.

Let R and a be the numbers defined uniquely by

$$(\tfrac{3}{2}\phi - a)^2 + \psi^2 = (\tfrac{3}{2}\phi^2 - 1 - a)^2 + \phi^2\psi^2 = R. \tag{47}$$

Their numerical values are‡

$$a = 1\cdot 4733 \ldots, \quad R = \cdot 5800 \ldots. \tag{48}$$

The equation of the circle Γ through C_2, C_3, C_2', C_3' is

$$(x-a)^2 + y^2 = R.$$

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† $x = \cdot 6$ corresponds to $\zeta = \cdot 3851$, $x = \cdot 62$ corresponds to $\zeta = \cdot 3816$.

‡ The exact value of a is $\frac{1}{11}(7\phi^2 + 9\phi - 8)$.

LEMMA 8. *$F(x) \geqslant R-(x-a)^2$ for $c_2 \leqslant x \leqslant c_3$.*

Proof. We have to prove that, for $\frac{3}{2}\phi^2-1 \leqslant x \leqslant \frac{3}{2}\phi$, the cubic polynomial

$$f(z) = z\Big((z+x)^2+R-(x-a)^2\Big) \tag{49}$$

satisfies $|f(z)| \leqslant 1$ for all z (mod 1). For $x = \frac{3}{2}\phi^2-1$, $f(z)$ is identical with $f_2(z)$ in the proof of Lemma 1, and for $x = \frac{3}{2}\phi$, $f(z)$ is identical with $f_3(z)$. From the formulae there obtained, we note that

$$f_2(-\phi^2+1) = -1. \tag{50}$$

If we write $f(x, z)$ for $f(z)$ to indicate the dependence on x, we have

$$f(x_1, z)-f(x_2, z) = 2z(z+a)(x_1-x_2). \tag{51}$$

In particular, if $x > \frac{3}{2}\phi^2-1$, we have, taking

$$x_1 = x, \quad x_2 = \tfrac{3}{2}\phi^2-1, \quad z = -\phi^2+1,$$

$$f(x, -\phi^2+1)-(-1) = 2(-\phi^2+1)(-\phi^2+1+a)(x-\tfrac{3}{2}\phi^2+1) < 0,$$

since $1 < \phi^2 < 1+a$.

Hence, for $\frac{3}{2}\phi^2-1 \leqslant x \leqslant \frac{3}{2}\phi$, $f(z)$ is a cubic polynomial in z with

$$f(-\infty) = -\infty,$$

$$f(-a) = -aR > -1,$$

$$f(-\phi^2+1) \leqslant -1,$$

$$f(0) = 0,$$

$$f(\infty) = \infty,$$

the sign of equality occurring only for $x = \frac{3}{2}\phi^2-1$. Hence the equation $f(z) = -1$ has three real roots, say $z = \alpha, \beta, \gamma$, which satisfy

$$\alpha < -a < \beta \leqslant -\phi^2+1 \leqslant \gamma < 0, \tag{52}$$

the two signs of equality occurring only when $x = \frac{3}{2}\phi^2-1$. Also $f(z) = 0$ has only one real root $z = 0$, and $f(z) = 1$ has one real root $z = \delta > 0$. The inequality $|f(z)| \leqslant 1$ is therefore satisfied for

$$\alpha \leqslant z \leqslant \beta, \quad \gamma \leqslant z \leqslant \delta. \tag{53}$$

If we denote by the suffixes 2 and 3 the values corresponding to $x=c_2$ and $x=c_3$, we have [by the formulae for $f_2(z)$, $f_3(z)$ in the proof of Lemma 1]

$$\alpha_2=-\phi^2,\quad \beta_2=\gamma_2=-\phi^2+1,\quad \delta_2=-\phi^2+2, \tag{54}$$

$$\alpha_3=-\phi-1,\quad \beta_3=-\phi,\quad \gamma_3=-\phi+1. \tag{55}$$

It follows from (51) that, as x increases, the value of $f(x, z)$ for fixed z increases if $z>0$ or $z<-\alpha$, and decreases if $-\alpha<z<0$. Hence, as x increases, α, β, δ all decrease, but γ increases. From (54) and (55), we have, therefore,

$$-\phi-1\leqslant\alpha\leqslant-\phi^2, \tag{56}$$

$$-\phi\leqslant\beta,\quad \gamma\leqslant-\phi+1, \tag{57}$$

$$0<\delta\leqslant 2-\phi^2. \tag{58}$$

Since $f(x, \alpha)=-1$, we have, on differentiating (49),

$$\begin{aligned}-\frac{d\alpha}{dx}&=\frac{-\alpha-a}{(1/2\alpha^2)-\alpha-x}\\ &\geqslant\frac{\phi^2-a}{(1/2\phi^4)-(-\phi-1)-(\frac{3}{2}\phi^2-1)}\\ &>\frac{\cdot 28}{1}=\cdot 28,\end{aligned}$$

by (16), (48), (56).

Similarly, differentiating the relation $f(x, \delta)=1$, we have

$$\begin{aligned}-\frac{d\delta}{dx}&=\frac{a+\delta}{(1/2\delta^2)+\delta+x}\\ &\leqslant\frac{a+2-\phi^2}{\{1/2(2-\phi^2)^2\}+\frac{3}{2}\phi^2-1}\\ &<\frac{1\cdot 72}{8+1\cdot 63}<\cdot 2.\end{aligned}$$

Hence $\delta-\alpha$ increases with x. Thus, by (54),

$$\delta-\alpha\geqslant 2. \tag{59}$$

By (57),

$$\gamma-\beta\leqslant 1. \tag{60}$$

From (59) and (60) it is plain that the two intervals (53) include all z (mod 1). This proves Lemma 8.

LEMMA 9. *$F(x) \geqslant l^2(x)$ for $c_3 \leqslant x \leqslant \phi^2+\phi$.*

Remark. $$\phi^2+\phi = 3{\cdot}079\ldots > 3.$$

Proof†. For $c_3 = \frac{3}{2}\phi \leqslant x \leqslant \phi^2+\phi$, we can write

$$x = \tfrac{3}{2}\phi p+(\phi^2+\phi)(1-p),$$

where $0 \leqslant p \leqslant 1$. Substituting in (25), we find

$$l(x) = p\psi.$$

Hence the result which we have to prove is that for any p $(0 \leqslant p \leqslant 1)$ the polynomial

$$g(z, p) = z\left(\left(z+\tfrac{3}{2}\phi p+(\phi^2+\phi)(1-p)\right)^2+p^2\psi^2\right) \tag{61}$$

satisfies $|g(z, p)| \leqslant 1$ for all z (mod 1).

Let $$\alpha(p) = (\phi^2-\tfrac{1}{2})p-(\phi^2+\phi+\tfrac{1}{2}).$$

We shall prove that $|g(z, p)| \leqslant 1$ for

$$\alpha(p) \leqslant z \leqslant \alpha(p)+1.$$

This suffices to establish the lemma.

Writing $z = \alpha(p)+\lambda$, where $0 \leqslant \lambda \leqslant 1$, and substituting in (61),

$$\begin{aligned} g(z, p) &= \{(\phi^2-\tfrac{1}{2})p-\phi^2-\phi-\tfrac{1}{2}+\lambda\}\left\{\left(\tfrac{1}{2}(\phi-1)p+\lambda-\tfrac{1}{2}\right)^2+p^2(\tfrac{3}{4}\phi^2-1)\right\} \\ &= \{(\phi^2-\tfrac{1}{2})p-\phi^2-\phi-\tfrac{1}{2}+\lambda\} \\ &\qquad \times\{(\phi^2-\tfrac{1}{2}\phi-\tfrac{3}{4})p^2+(\lambda-\tfrac{1}{2})(\phi-1)p+(\lambda-\tfrac{1}{2})^2\}. \end{aligned}$$

For fixed λ, this is a cubic polynomial in p with the coefficient of p^3 positive.

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† Actually, for $x \geqslant \frac{3}{2}\phi$, $F(x)$ can be determined exactly, as in Lemma 4. We have

$$F(x) = \frac{1}{\zeta}-(x-\zeta)^2, \quad \text{where} \quad \frac{1}{\zeta}-(x-\zeta)^2 = \frac{1}{\zeta+1}-(x-1-\zeta)^2.$$

ζ increases with x, and $\zeta = \phi$ corresponds to $x = \frac{3}{2}\phi$.

To prove that such a polynomial $f(p)$ satisfies $|f(p)| \leqslant 1$ for $0 \leqslant p \leqslant 1$ it is sufficient to prove:

(i) $f(p) \leqslant 0$ for $0 \leqslant p \leqslant 1$,

(ii) $f(0) \geqslant -1$,

(iii) $f(1) \geqslant -1$,

(iv) $f'(1) \leqslant 0$†.

Proof of (i):

$$(\phi^2-\tfrac{1}{2})p-\phi^2-\phi-\tfrac{1}{2}+\lambda \leqslant \phi^2-\tfrac{1}{2}-\phi^2-\phi-\tfrac{1}{2}+1=-\phi<0.$$

Proof of (ii). We have to prove that

$$(\phi^2+\phi+\tfrac{1}{2}-\lambda)(\lambda-\tfrac{1}{2})^2 \leqslant 1$$

for $0 \leqslant \lambda \leqslant 1$. This is immediate since the left-hand side does not exceed $\frac{1}{4}(\phi^2+\phi+\frac{1}{2})<1$.

Proof of (iii). We have to prove that

$$(\phi+1-\lambda)\big(\phi^2-(1-\lambda)\phi+(1-\lambda)^2-1\big) \leqslant 1$$

for $0 \leqslant \lambda \leqslant 1$. The left-hand side is

$$(1-\lambda)^3-(1-\lambda)+1=1-\lambda(1-\lambda)(2-\lambda) \leqslant 1.$$

Proof of (iv). We have to prove that

$$(\phi^2-\tfrac{1}{2})\big(\phi^2-(1-\lambda)\phi+(1-\lambda)^2-1\big) \\ +(-\phi-1+\lambda)\big(2\phi^2-\phi-\tfrac{3}{2}+(\phi-1)(\lambda-\tfrac{1}{2})\big) \leqslant 0$$

for $0 \leqslant \lambda \leqslant 1$. The left-hand side is a quadratic in λ in which the coefficient of λ^2 is $\phi^2-\frac{1}{2}+\phi-1>0$. Hence it is sufficient to verify that the inequality holds for $\lambda=0$ and $\lambda=1$. For $\lambda=0$ we have

$$(\phi^2-\tfrac{1}{2})(\phi^2-\phi)-(\phi+1)(2\phi^2-\tfrac{3}{2}\phi-1)=\phi-2<0,$$

and for $\lambda=1$ we have

$$(\phi^2-\tfrac{1}{2})(\phi^2-1)-\phi(2\phi^2-\tfrac{1}{2}\phi-2)=\phi-\tfrac{3}{2}<0.$$

Proof of Theorem 2: Lemmas 7, 8, 9.

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† If $f(p)$ had a minimum at p_0, where $0<p_0<1$, we should have $f'(p_0)=0$, $f''(p_0)>0$, whence $f'(1)>0$, contrary to (iv).

7. *Proof of Theorem* 3.

Let

$$(62)\qquad G(x)=\begin{cases}(\phi+1)\psi & \text{for}\quad 0\leqslant x\leqslant c_1,\\ l_1(x) & \text{for}\quad c_1\leqslant x\leqslant e,\\ l(x) & \text{for}\quad e\leqslant x\leqslant c_2,\\ \sqrt{\{R-(x-a)^2\}} & \text{for}\quad c_2\leqslant x\leqslant c_3,\\ l(x) & \text{for}\quad c_3\leqslant x\leqslant 3,\end{cases}$$

and let $G(-x)=G(x)$. The domain E is given by $|x|\leqslant 3$, $|y|\leqslant G(x)$. $G(x)$ is continuous, and is differentiable except for a finite number of values of x. It is convenient to collect here a few numerical results. We have

$$-l_1'(x)=\frac{d_1-f}{e-c_1},$$

whence

$$(63)\qquad \cdot 1<-l_1'(x)<\cdot 16.$$

Also

$$(64)\qquad -l'(x)=\frac{\psi}{\phi^2-\frac{1}{2}\phi}=\cdot 5146\ldots.$$

Finally $$-\frac{d}{dx}\sqrt{\{R-(x-a)^2\}}=\frac{x-a}{\sqrt{\{R-(x-a)^2\}}},$$

which lies between

$$\frac{\frac{3}{2}\phi^2-1-a}{\phi\psi}\quad\text{and}\quad\frac{\frac{3}{2}\phi-a}{\psi}$$

for $c_2\leqslant x\leqslant c_3$. Hence, for $c_2\leqslant x\leqslant c_3$,

$$(65)\qquad \cdot 2<-\frac{d}{dx}\sqrt{\{R-(x-a)^2\}}<1.$$

LEMMA 10. *If (x_0, y_0) is a point of Q, and $0<\lambda\leqslant\frac{1}{2}\sqrt{23}$, the straight line $xy_0-yx_0=\lambda$ meets the boundary of E in exactly two points. One point has $y>0$, the other $y<0$.*

Proof. The result is obvious if $x_0=0$, from the form of the definition of E. Hence we suppose that $x_0>0$. Since (x_0, y_0) is in Q, we have

$$(66)\qquad 0<x_0\leqslant e<\cdot 62,$$

$$(67)\qquad 1\cdot 27<f\leqslant y_0\leqslant\frac{\phi+1}{\phi-\frac{1}{2}}\psi<1\cdot 6.$$

Consider the function

$$G(x)-x\frac{y_0}{x_0}+\frac{\lambda}{x_0} \tag{68}$$

in the interval $-3 \leqslant x \leqslant 3$. We have

$$G(-3)+3\frac{y_0}{x_0}+\frac{\lambda}{x_0} > 0,$$

since each term is positive, and

$$G(3)-3\frac{y_0}{x_0}+\frac{\lambda}{x_0} < \frac{\phi^2+\phi-3}{\phi^2-\frac{1}{2}\phi}\psi-3\frac{1{\cdot}27}{x_0}+\frac{\frac{1}{2}\sqrt{23}}{x_0}$$

$$< {\cdot}05-\frac{1{\cdot}3}{x_0}$$

$$< {\cdot}05-\frac{1{\cdot}3}{{\cdot}62} < 0.$$

The function (68) is continuous, and is differentiable except for a finite number of values of x. By (62), (63), (64), (65), we have $|G'(x)| < 1$ wherever $G(x)$ is differentiable. Since, by (66), (67),

$$\frac{y_0}{x_0} > \frac{1{\cdot}27}{{\cdot}62} > 1,$$

the function (68) has negative derivative except at a finite number of points. Hence it has exactly one zero for $-3 \leqslant x \leqslant 3$. Thus the line $xy_0-yx_0=\lambda$ meets $y=G(x)$, the upper boundary of E, in exactly one point.

Similar calculations apply to the lower boundary $y=-G(x)$.

From now onwards, we denote by (x_1, y_1), (x_2, y_2), where $y_1 > 0 > y_2$, the points in which $xy_0-yx_0=\lambda$ meets the boundary of E. We define

$$H(x_0, y_0;\ \lambda)=y_1-y_2,$$

$$H(x_0, y_0)=H(x_0, y_0;\ \tfrac{1}{2}\sqrt{23}).$$

LEMMA 11. *If* (x_0, y_0) *is a point of* Q, *and* $0<\lambda<\frac{1}{2}\sqrt{23}$, *then*

$$H(x_0, y_0;\ \lambda) < H(x_0, y_0).$$

Proof. The result is obvious if $x_0=0$, since $G(x)$ is non-increasing for increasing $x \geqslant 0$, and is decreasing for $x > c_1$. In what follows, (x_0, y_0) is a fixed point of Q with $x_0 > 0$. It is plain from the proof of Lemma 10 that

x_1, y_1, x_2, y_2 are continuous functions of λ, and are differentiable except for a finite number of values of λ. From the formula

$$G(x_1) = x_1 \frac{y_0}{x_0} - \frac{\lambda}{x_0},$$

we have
$$\left(\frac{y_0}{x_0} - G'(x_1)\right) \frac{dx_1}{d\lambda} = \frac{1}{x_0}.$$

Since
$$y_1 = x_1 \frac{y_0}{x_0} - \frac{\lambda}{x_0},$$

$$\frac{dy_1}{d\lambda} = \frac{1}{x_0} \frac{G'(x_1)}{(y_0/x_0) - G'(x_1)}.$$

Similarly
$$\frac{dy_2}{d\lambda} = \frac{1}{x_0} \frac{-G'(x_2)}{(y_0/x_0) + G'(x_2)}.$$

Both denominators are positive, as was seen in the proof of Lemma 10. Hence the sign of $d(y_1 - y_2)/d\lambda$ is the same as that of

$$G'(x_1) + G'(x_2). \tag{69}$$

Since $x_1 \geqslant 0$, we have $G'(x_1) \leqslant 0$. Also $x_2 > -e$ since

$$-G(-e) = -f,$$

$$(-e)\frac{y_0}{x_0} - \frac{\lambda}{x_0} < -e\frac{y_0}{x_0} \leqslant -e\frac{f}{e} = -f.$$

Hence $G'(x_2) \leqslant 0$ unless

$$-e \leqslant x_2 \leqslant -c_1, \tag{70}$$

in which case $G'(x_2) = -l_1'(x)$. If x_2 satisfies (70), we have $x_1 \geqslant c_1$, since the line joining (x_1, y_1), (x_2, y_2) passes to the right of the origin. But for $x \geqslant c_1$ we have $G'(x) \leqslant l_1'(x)$, by (63), (64), (65). Hence in any case the expression (69) is non-positive. It follows that $y_1 - y_2$ is non-increasing as λ increases, hence $H(x_0, y_0;\ \lambda) \leqslant H(x_0, y_0)$.

Finally, if $\lambda > \frac{1}{2}\sqrt{23} - \delta$, where $\delta > 0$ is small, we have

$$x_1 = \frac{x_0 y_1}{y_0} + \frac{\lambda}{y_0} \geqslant \frac{\lambda}{y_0} > \frac{\frac{1}{2}\sqrt{23}-\delta}{y_0} > \frac{2{\cdot}3}{1{\cdot}6} > e,$$

hence $G'(x_1) < -{\cdot}2$ [by (64), (65)] and $G'(x_2) \leqslant -l_1'(x) < {\cdot}16$. Hence the expression (69) is negative for $\lambda > \frac{1}{2}\sqrt{23} - \delta$. This completes the proof of Lemma 11.

Let

(71) $$\alpha=\frac{\phi^2-\frac{1}{2}\phi}{\psi},$$

(72) $$\beta=\phi^2+\phi.$$

Let Q_1 denote those points of Q which satisfy

(73) $$(\alpha y-\beta)^2-x^2 \leqslant \beta^2-\alpha\sqrt{23}.$$

LEMMA 12. *If* (x_0, y_0) *is in* Q_1, *then*

$$H(x_0, y_0) \geqslant y_0.$$

Proof. We have

$$-G(e)=-f,$$

$$e\frac{y_0}{x_0}-\frac{\frac{1}{2}\sqrt{23}}{x_0}<\frac{(\cdot 62)(1\cdot 6)-2\cdot 39}{x_0}$$

$$<-\frac{1\cdot 39}{x_0}$$

$$<-\frac{1\cdot 39}{\cdot 62}$$

$$<-f.$$

Hence $x_2>e$, and *a fortiori* $x_1>e$. Therefore, if we denote by (x_3, y_3) and (x_4, y_4) the points in which the line

(74) $$xy_0-x_0y=\tfrac{1}{2}\sqrt{23}$$

meets the lines $y=l(x)$, $y=-l(x)$ respectively, we have

$$y_1-y_2 \geqslant y_3-y_4.$$

By (25), (71), (72), the equation $y=l(x)$ can be written in the form

(75) $$\alpha y=\beta-x.$$

From (74), (75), we find

$$y_3=\frac{\beta y_0-\frac{1}{2}\sqrt{23}}{\alpha y_0+x_0},$$

$$-y_4=\frac{\beta y_0-\frac{1}{2}\sqrt{23}}{\alpha y_0-x_0}.$$

Hence $$y_3-y_4=\frac{2ay_0(\beta y_0-\frac{1}{2}\sqrt{23})}{a^2y_0^2-x_0^2}.$$

We have $ay_0>y_0>1{\cdot}27>{\cdot}62>x_0$. Since (x_0, y_0) is in Q_1, (73) is satisfied by x_0, y_0, hence

$$2a\{\beta y_0-\tfrac{1}{2}\sqrt{23}\}\geqslant a^2y_0^2-x_0^2.$$

Hence $y_3-y_4\geqslant y_0$, which proves the result.

Let Q_2 denote those points of Q which satisfy

$$(ay-\beta)^2-x^2\geqslant\beta^2-a\sqrt{23}. \tag{76}$$

LEMMA 13. *The domain Q_2 is given by*

$$0\leqslant x\leqslant c_1, \tag{77}$$

$$d_1\leqslant y\leqslant\frac{1}{a}\,[\beta-\sqrt{\{x^2+\beta^2-a\sqrt{23}\}}]. \tag{78}$$

Proof. We first prove that the point $C_1=(c_1, d_1)$ satisfies

$$(ad_1-\beta)^2-c_1^2=\beta^2-a\sqrt{23}. \tag{79}$$

It follows from (17), (18) that when (x_0, y_0) is C_1, the line (74) is the line C_3C_2'. Hence, with the notation used in the proof of the previous lemma, $y_1=y_3=d_3$, $y_2=y_4=-d_2$, $y_3-y_4=d_3+d_2=d_1=y_0$. Hence, by the proof of the previous lemma, (c_1, d_1) satisfies (79).

Throughout Q (indeed throughout E),

$$y\leqslant\frac{\phi+1}{\phi-\frac{1}{2}}\,\psi=\frac{\beta}{a}.$$

Hence (76) can be written

$$y\leqslant\frac{1}{a}\,[\beta-\sqrt{\{x^2+\beta^2-a\sqrt{23}\}}]. \tag{80}$$

We now prove that, in Q_2, $x\leqslant c_1$. For this it suffices to prove that, for $c_1<x\leqslant e$,

$$\frac{1}{a}\,[\beta-\sqrt{\{x^2+\beta^2-a\sqrt{23}\}}]<l_1(x). \tag{81}$$

For then (80) proves that $y < l_1(x)$ for $x > c_1$, whereas, by definition of Q, $y \geqslant l_1(x)$ for $x \geqslant c_1$.

Since c_1, d_1 satisfy (79), the two sides of (81) are equal for $x = c_1$. Hence it suffices to prove that

$$-\frac{x}{\alpha\sqrt{\{x^2+\beta^2-\alpha\sqrt{23}\}}} < l_1'(x)$$

for $c_1 < x \leqslant e$. Both sides are negative, and the absolute value of the right-hand side is less than ·16, by (63). The absolute value of the left-hand side increases as x increases, and when $x = c_1$ it is

$$\frac{c_1}{\alpha\sqrt{\{c_1{}^2+\beta^2-\alpha\sqrt{23}\}}} = \frac{c_1}{\alpha(\beta-\alpha d_1)} = \cdot 338\ldots > \cdot 16.$$

Hence $x \leqslant c_1$. From this and (80) we have (77) and (78).

LEMMA 14. *If* (x_0, y_0) *is a point of* Q_2, *then*

(82) $$c_3 y_0 - d_3 x_0 \geqslant \tfrac{1}{2}\sqrt{23},$$

(83) $$c_2 y_0 - (-d_2)\, x_0 \leqslant \tfrac{1}{2}\sqrt{23}.$$

Proof. By (17) and (18) these are satisfied with the equality sign when $x_0 = c_1$, $y_0 = d_1$. It follows that (82) is satisfied throughout Q_2, since, by Lemma 13, $x_0 \leqslant c_1$, $y_0 \geqslant d_1$.

To prove that (83) is satisfied throughout Q_2, it suffices to prove that

$$\frac{1}{\alpha}\,[\beta-\sqrt{\{x^2+\beta^2-\alpha\sqrt{23}\}}] \leqslant \frac{\frac{1}{2}\sqrt{23}-d_2 x}{c_2}$$

for $0 \leqslant x \leqslant c_1$. We have already seen that equality occurs when $x = c_1$. Hence it suffices to prove that

$$-\frac{x}{\alpha\sqrt{\{x^2+\beta^2-\alpha\sqrt{23}\}}} \geqslant -\frac{d_2}{c_2}$$

for $0 \leqslant x \leqslant c_1$. Both sides are negative; the absolute value of the right-hand side is ·4563 ..., and that of the left-hand side is a maximum when $x = c_1$, its value then being ·338 ... (as in the proof of the previous lemma). This proves the result.

LEMMA 15. *If* (x_0, y_0) *is a point of* Q_2, *then*

$$H(x_0, y_0) \geqslant y_0.$$

Proof. Let (x_3, y_3) and (x_4, y_4), where $y_3 > 0 > y_4$, be the points in which the line (74) meets the circle Γ, whose equation is $(x-a)^2+y^2 = R$. Since Γ lies in E, it suffices to prove that $y_3-y_4 \geqslant y_0$. Now y_3, y_4 are the roots of

$$\left(\frac{\frac{1}{2}\sqrt{23}}{y_0}-a\right)^2+2\left(\frac{\frac{1}{2}\sqrt{23}}{y_0}-a\right)\frac{x_0}{y_0}y+\left(\frac{x_0^2}{y_0^2}+1\right)y^2 = R.$$

Write $$\xi = \frac{\frac{1}{2}\sqrt{23}}{y_0}, \qquad \mu = \left(\frac{x_0}{y_0}\right)^2.$$

Then the equation is

$$(1+\mu)y^2+2(\xi-a)\sqrt{\mu}\,y+(\xi-a)^2-R = 0.$$

Hence $$\tfrac{1}{4}(y_3-y_4)^2 = \frac{(\xi-a)^2\mu-(1+\mu)\big((\xi-a)^2-R\big)}{(1+\mu)^2}$$

$$= \frac{R}{1+\mu}-\frac{(\xi-a)^2}{(1+\mu)^2}.$$

We shall now prove that for fixed ξ the right-hand side decreases as μ increases ($\mu \geqslant 0$). It suffices to verify that $2(\xi-a)^2 < R$. Since (x_0, y_0) is in Q_2, we have, by Lemma 13,

$$d_1 \leqslant y_0 < \frac{\beta}{a},$$

whence $$1{\cdot}3 < y_0 < 1{\cdot}6.$$

Hence $$1{\cdot}4 < \xi < 1{\cdot}85,$$

so that $$2(\xi-a)^2 < 2({\cdot}38)^2 < R,$$

by (48).

Hence, for fixed y_0, y_3-y_4 decreases as x_0 increases. Let

$$x_0' = \sqrt{\{(\beta-ay_0)^2-\beta^2+a\sqrt{23}\}};$$

then $x_0 \leqslant x_0'$, and (x_0', y_0) belongs to both Q_1 and Q_2. Let (x_3', y_3'), (x_4', y_4'), where $y_3' > 0 > y_4'$, be the points in which the line

(84) $$xy_0-yx_0' = \tfrac{1}{2}\sqrt{23}$$

meets the circle Γ. It follows from what we have proved that

(85) $$y_3-y_4 \geqslant y_3'-y_4'.$$

Since (x_0', y_0) is a point of Q_2, it follows from (82), (83) that the line (84) passes to the left of C_3 and to the right of C_2'. Hence it passes between C_2 and C_3, and between C_2' and C_3'. Let (x_5', y_5'), (x_6', y_6') be the points in which the line (84) meets the lines $y = l(x)$, $y = -l(x)$ respectively. Then

(86) $$y_3' - y_4' \geqslant y_5' - y_6'.$$

Since (x_0', y_0) is a point of Q_1, it follows from the proof of Lemma 12 that

(87) $$y_5' - y_6' \geqslant y_0.$$

The result now follows from (85), (86), (87).

Proof of Theorem 3. Lemmas 10, 11, 12, 15.

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THE TORSION AND FLEXURE SOLUTIONS FOR THE ELLIPTIC LIMAÇON CROSS-SECTION

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1. *Introduction.*

This paper adds a further example to the slowly growing list of complete and exact solutions of the Saint-Venant flexure problem. This problem has recently been re-formulated (Stevenson, **3**) as the search for six plane harmonic "canonical" functions, one of which is the torsion function, the others satisfying boundary conditions of the same simplicity of form as the torsion function. The cross-section for which the solution is found here is bounded by the elliptic limaçon whose polar equation is $r = a - b\cos\theta$, with $a \geqslant b$. For different values of the parameters a and b we have a series of cross-sections having one axis of symmetry, passing from the circle for $b = 0$, through a series of single compact ovals for $0 < b < \frac{1}{2}a$, to a series of kidney shaped ovals for $\frac{1}{2}a < b < a$, becoming the cardioid for $b = a$.

The solution for the load at the centroid of an end-section of an elastic cylinder of this cross-section, the load having a resolute perpendicular to the axis of symmetry, involves a twist of the cross-section and the amount of this associated twist is a necessary constant of the solution, which must be determined before the stresses and displacements can be completely related to the external force system. Alternatively, other load points may be chosen. If the load point is chosen so that the mean twist over the cross-section is zero, the load point is termed the centre of flexure, and the determination of its coordinates is necessary to complete the solution. If the load-point is chosen so that the strain energy of the elastic cylinder is least, the load point is termed the centre of least strain (Stevenson, **4**), its coordinates being closely related to those of the centre of flexure.

As in the case of the cardioid cross-section (Shepherd, **2**) the solution will be found to be expressible in finite terms, although as a matter of experience in these problems, the determination of the associated twist or the coordinates of the load point for any given twist is more readily carried out by means of the series expansions of the canonical functions, at least in part.

2. *Statement of the problem.*

The z-axis is taken parallel to the generators of the elastic cylinder of cross-section S and boundary C, and the x-axis along the axis of symmetry of the cross-section. If the external forces are equivalent to loads (W, W') parallel to the x- and y-axes and acting at the centroid $x=h$, $y=0$ of the end-section $z=l$, then it has previously been shown (Stevenson, 3), that the stresses are given by

$$\widehat{pz} = \widehat{pz}_1 + \widehat{px}_2 \quad (p = x, y, z),$$

all other stresses vanishing, where

$$(EI/\mu W)\,\widehat{xz}_1 = (1+\eta)\left\{\frac{\partial \chi_1}{\partial x} - x^2 - h\left(\frac{\partial \chi_3}{\partial x} - x\right)\right\} - \eta\left(\frac{\partial \phi_2}{\partial x} - y^2\right), \qquad (2.1)$$

$$(EI/\mu W)\,\widehat{yz}_1 = (1+\eta)\left\{\frac{\partial \chi_1}{\partial y} - h\left(\frac{\partial \chi_3}{\partial y} - y\right)\right\} - \eta\,\frac{\partial \phi_2}{\partial y}, \qquad (2.2)$$

$$(I/W)\,\widehat{zz}_1 = -(l-z)\,x, \qquad (2.3)$$

$$(EI'/\mu W')\,\widehat{xz}_2$$

$$= (1+\eta)\,\frac{\partial \chi_2}{\partial x} + \eta\left\{\frac{\partial \phi_1}{\partial x} - h\left(\frac{\partial \phi_3}{\partial x} - y\right)\right\} + (EI'\tau'/W')\left(\frac{\partial \phi_3}{\partial x} - y\right), \qquad (2.4)$$

$$(EI'/\mu W')\,\widehat{yz}_2$$

$$= (1+\eta)\left(\frac{\partial \chi_2}{\partial y} - y^2\right) + \eta\left\{\frac{\partial \phi_1}{\partial y} + x^2 - h\left(\frac{\partial \phi_3}{\partial y} + x\right)\right\} + (EI'\tau'/W')\left(\frac{\partial \phi_3}{\partial y} + x\right), \qquad (2.5)$$

$$(I'/W')\,\widehat{zz}_2 = -(l-z)\,y, \qquad (2.6)$$

in which I and I' are the principal second moments of the cross-sections about the principal axes parallel to y and x respectively at the centroid $(h, 0)$, E is Young's modulus, μ is the rigidity, η is Poisson's ratio, τ' is the associated twist per unit length, and $\chi_1, \chi_2, \chi_3, \phi_1, \phi_2. \phi_3$ are the six canonical flexure functions.

These functions, which are conveniently taken as the real parts of complex canonical functions

$$\Omega_r = \chi_r + i\chi_r{}^*, \quad \omega_r = \phi_r + i\psi_r, \quad (r = 1, 2, 3) \qquad (2.7)$$

of the complex variable $x+iy$, are real plane harmonic functions, *i.e.* real solutions of $\nabla^2 U = 0$ in two dimensions, determined from the boundary conditions

$$\frac{\partial}{\partial n}(\chi_r - F_r) = 0, \quad \psi_r - F_r = \text{const.}, \tag{2.8}$$

where $$F_1 = \tfrac{1}{3}x^3, \quad F_2 = \tfrac{1}{3}y^3, \quad F_3 = \tfrac{1}{2}(x^2+y^2), \tag{2.9}$$

and n denotes the normal to the boundary.

We shall use the customary $z = x+iy$, since the symbol z occurs as the distance from the root-end of the cylinder only in equations (2.3) and (2.6), and there will be no occasion for confusion, since hereafter throughout this investigation z always means the complex variable.

Since both the displacements and the stresses must be single-valued, finite and continuous across the cross-section S, in general, this imposes certain restrictions upon the functions Ω_r, ω_r. Thus in the case of the torsion function ω_3 it is well known that ω_3 must necessarily be a non-cyclic function. In the present case, *i.e.* for uni-axial cross-sections with the axis of x as the axis of symmetry, the same holds for ω_1, ω_2, and Ω_2, but is not required of and cannot hold for Ω_1 and Ω_3. These always occur in the expressions for the displacements and stresses in the combination $\Omega_1 - h\Omega_3$, and it is this combination which has to satisfy the above requirements to give a physically admissible solution of Laplace's equation. The equation of condition that χ_r should separately be a physically admissible solution is

$$\int_C \frac{\partial \chi_r}{\partial n}\, ds = 0.$$

Using the conjugate function χ_r^* we have, on the boundary,

$$\frac{\partial \chi_r^*}{\partial s} = \frac{\partial \chi_r}{\partial n} = \frac{\partial F_r}{\partial n} = \frac{\partial F_r}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial F_r}{\partial y}\frac{\partial y}{\partial n} = \frac{\partial F_r}{\partial x}\frac{\partial y}{\partial s} - \frac{\partial F_r}{\partial y}\frac{\partial x}{\partial s},$$

from which we have an equivalent alternative boundary condition for deriving Ω_r, namely

$$\chi_r^* = \int\left(\frac{\partial F_r}{\partial x}\, dy - \frac{\partial F_r}{\partial y}\, dx\right) \tag{2.10}$$

along the boundary C. From this it is clear that the change $\Delta\chi_r^*$ in χ_r^* on passing round the boundary C is

$$\Delta\chi_r^* = \int_C \left(\frac{\partial F_r}{\partial x}\, dy - \frac{\partial F_r}{\partial y}\, dx\right),$$

which is not zero when $r=1$, 3, $\chi_1{}^*$ and $\chi_3{}^*$ being cyclic functions such that, from (2.9),

$$\left.\begin{aligned} \Delta\chi_1{}^* &= \int_C x^2\,dy = 2hS, \\ \text{and}\qquad \Delta\chi_3{}^* &= \int_C (x\,dy - y\,dx) = 2S. \end{aligned}\right\} \tag{2.11}$$

It is to be noted that $\Delta(\chi_1{}^* - h\chi_3{}^*) = 0$.

It is not difficult to work with the separate functions χ_r, ψ_r; the necessary cyclic functions usually present themselves satisfactorily in a simple manner and give, in fact, a check upon the calculated value of h for the cross-section. Any slight trouble involved in making sure that $\chi_1 - h\chi_3$ is a physically admissible solution is more than compensated for by the gain in manipulative ease in dealing with the canonical functions separately in a systematic manner.

In the case of uni-axial symmetry of cross-section, we have three moment integrals

$$\left.\begin{aligned} M_1 &= M_1' + \int x^3\,dS, \\ L_2 &= L_2' - \int xy^2\,dS, \\ M_3 &= M_3' + \int (x^2+y^2)\,dS, \end{aligned}\right\} \tag{2.12}$$

where
$$M_r' = \text{real part of } \int iz\,\frac{d\omega_r}{dz}\,dS, \tag{2.13}$$

$$L_r' = \text{real part of } \int iz\,\frac{d\Omega_r}{dz}\,dS. \tag{2.14}$$

These integrals across the cross-section can be expressed alternatively as line integrals round the boundary C of the cross-section in the forms

$$M_r' = \int_C \tfrac{1}{2}(x^2+y^2)\,\frac{\partial\phi_r}{\partial s}\,dx = \text{real part of } \int_C \tfrac{1}{2}z\bar{z}\,\frac{d\omega_r}{dz}\,dz, \tag{2.15}$$

$$L_r' = \int_C \tfrac{1}{2}(x^2+y^2)\,\frac{\partial\chi_r}{\partial s}\,dS = \text{real part of } \int_C \tfrac{1}{2}z\bar{z}\,\frac{d\Omega_r}{dz}\,dz. \tag{2.16}$$

The canonical function ϕ_3 is the well-known Saint-Venant torsion function and the torsion moment for the cylinder is $\mu\tau M_3$, for a twist τ. In terms of the moment integrals M_1, L_2, M_3, the associated twist τ' occurring in

(2.4), (2.5) is given by

$$-\tau' = (W'/EI'M_3)\{(1+\eta)(L_2-2hI')+\eta(M_1-hM_3)\}, \tag{2.17}$$

and the coordinates $(f, 0)$ of the centre of flexure are given by

$$f = \{L_2+\sigma(M_1-hM_3)\}/2I', \tag{2.18}$$

in which σ is the modified Poisson's ratio of generalized plane stress given by $(1-\sigma)(1+\eta)=1$. The coordinates of the centre of least strain are obtained from these by putting $\sigma=0$ in (2.18).

3. *Preparatory analysis.*

Consider the orthogonal curvilinear coordinates ξ, η defined by

$$z = 2c\sec^2\tfrac{1}{2}\zeta \quad (z = x+iy,\ \zeta = \xi+i\eta). \tag{3.1}$$

If $z = re^{i\theta}$, we have

$$(2c/r)^{\frac{1}{2}}\cos\tfrac{1}{2}\theta = \cos\tfrac{1}{2}\xi\cosh\tfrac{1}{2}\eta, \quad (2c/r)^{\frac{1}{2}}\sin\tfrac{1}{2}\theta = \sin\tfrac{1}{2}\xi\sinh\tfrac{1}{2}\eta. \tag{3.2}$$

Eliminating ξ, we find, for the curve $\eta=\text{const.}$,

$$r = a-b\cos\theta, \tag{3.3}$$

where
$$a = 4c\cosh\eta\,\mathrm{cosech}^2\eta, \quad b = 4c\,\mathrm{cosech}^2\eta, \tag{3.4}$$

so that
$$4bc = a^2-b^2, \quad \text{and} \quad a \geqslant b. \tag{3.5}$$

Similarly, by eliminating η, we find that the curve $\xi=\text{const.}$ is given by

$$r = b'\cos\theta-a', \tag{3.6}$$

where
$$a' = 4c\cos\xi\,\mathrm{cosec}^2\xi, \quad b' = 4c\,\mathrm{cosec}^2\xi,$$

so that $4b'c = b'^2-a'^2$, and $b' \geqslant a'$.

Basset (1, p. 242) calls these curves confocal limaçons, distinguishing the set (3.3), which are the inverses of confocal ellipses with respect to a focus, from the set (3.6), which are the inverses of confocal hyperbolas with respect to a focus, as elliptic and hyperbolic limaçons respectively. We shall here take $\eta=\beta$ to give the entire elliptic limaçon, but $\xi=\alpha$ to give the upper half of a hyperbolic limaçon, the lower half of the same limaçon corresponding to the value $\xi = -\alpha$. From (3.1) we readily find that

$$\frac{x}{4c} = \frac{1+\cos\xi\cosh\eta}{(\cosh\eta+\cos\xi)^2}, \qquad \frac{y}{4c} = \frac{\sin\xi\sinh\eta}{(\cosh\eta+\cos\xi)^2}. \tag{3.7}$$

The value $\xi = 0$ corresponds to the strip $0 < x < 2c$ of the x-axis, while $\xi = \pm\pi$ gives the negative x-axis, so that there is a discontinuity of 2π in ξ on crossing the negative x-axis. The remainder of the positive x-axis is given by $\eta = 0$, the curve $\eta = \infty$ is the point circle at the origin, and η assumes all values from 0 to ∞.

Now, from (3.1),

$$\frac{z}{4c} = \frac{d}{d\zeta}\tan\tfrac{1}{2}\zeta = -i\frac{d}{d\zeta}\left\{1-\frac{2}{1+e^{i\zeta}}\right\} = -2Z_1, \tag{3.8}$$

with the abbreviation

$$Z_r = C_r + iS_r = \sum_{n=1}^{\infty}(-1)^n n^r e^{in\zeta}, \tag{3.9}$$

where r is a positive or negative integer, or zero (and is not to be confused with the polar coordinate r). Note that C_r, S_r are plane harmonic real functions which can be exhibited in finite terms, since

$$1+2(C_0+iS_0) = -i\tan\tfrac{1}{2}\zeta = \frac{\sinh\eta - i\sin\xi}{\cosh\eta+\cos\xi}, \tag{3.10}$$

and

$$Z_{r+1} = -i\frac{\partial Z_r}{\partial\zeta}, \tag{3.11}$$

whence

$$\left.\begin{aligned} C_{r+1} &= -\frac{\partial C_r}{\partial\eta} = \frac{\partial S_r}{\partial\xi}, \\ S_{r+1} &= -\frac{\partial C_r}{\partial\xi} = -\frac{\partial S_r}{\partial\eta}. \end{aligned}\right\} \tag{3.12}$$

But

$$(r/4c) = (z\bar{z})^{\frac{1}{2}}/4c = 1/(2\cos\tfrac{1}{2}\bar{\zeta}\cos\tfrac{1}{2}\zeta) = 1/(\cosh\eta+\cos\xi), \tag{3.13}$$

(a bar indicating that i is changed to $-i$), whence, using (3.10), we have

$$R = (r/4c)\sinh\eta = 1+2C_0. \tag{3.14}$$

Now it is readily shown that

$$R^{n+1} = \coth\eta\, R^n - \frac{1}{n}\frac{\partial R^n}{\partial\eta}, \tag{3.15}$$

and so, by successive applications of (3.15), we have from (3.14)

$$R^2 = \coth\eta(1+2C_0)+2C_1, \tag{3.16}$$

$$R^3 = \tfrac{1}{2}(3\coth^2\eta-1)(1+2C_0)+3\coth\eta\, C_1+C_2, \tag{3.17}$$

$$R^4 = \tfrac{1}{2}(5\coth^3\eta-3\coth\eta)(1+2C_0)+\tfrac{1}{3}(15\coth^2\eta-4)\, C_1 + 2\coth\eta\, C_2+\tfrac{1}{3}C_3, \tag{3.18}$$

$$R^5 = \tfrac{1}{8}(35\coth^4\eta-30\coth^2\eta+3)(1+2C_0)+\tfrac{5}{12}(21\coth^3\eta-11\coth\eta)\, C_1 + \tfrac{5}{12}(9\coth^2\eta-2)\, C_2+\tfrac{5}{6}\coth\eta\, C_3+\tfrac{1}{12}C_4, \tag{3.19}$$

$$\begin{aligned} R^6 = {} & \tfrac{1}{8}(63\coth^5\eta-70\coth^3\eta+15\coth\eta)(1+2C_0) \\ & +\tfrac{1}{4}(63\coth^4\eta-49\coth^2\eta+\tfrac{64}{15})\, C_1+\tfrac{1}{4}(28\coth^3\eta-13\coth\eta)\, C_2 \\ & +\tfrac{1}{12}(21\coth^2\eta-4)\, C_3+\tfrac{1}{4}\coth\eta\, C_4+\tfrac{1}{60}C_5. \end{aligned} \tag{3.20}$$

By means of these results we can express the functions F_r, $\partial F_r/\partial\eta$ of (2.8) and (2.9) in terms of C_r and S_r.

Thus

$$F_3 = \tfrac{1}{2}r^2 = 8c^2\,\mathrm{cosech}^2\eta\, R^2 = 8c^2\,\mathrm{cosech}^2\eta\{\coth\eta(1+2C_0)+2C_1\}, \tag{3.21}$$

and, by (3.12),

$$\begin{aligned} \frac{\partial F_3}{\partial\eta} = {} & -16c^2(\mathrm{cosech}^4\eta+2\coth^2\eta\,\mathrm{cosech}^2\eta) \\ & +32c^2(\mathrm{cosech}^4\eta+2\coth^2\eta\,\mathrm{cosech}^2\eta)\frac{\partial C_{-1}}{\partial\eta} \\ & +96c^2\coth\eta\,\mathrm{cosech}^2\eta\frac{\partial C_0}{\partial\eta}+32c^2\,\mathrm{cosech}^2\eta\frac{\partial C_1}{\partial\eta}. \end{aligned} \tag{3.22}$$

Again, with a and b functions of η, it is readily seen from (3.3) and (3.4) that

$$x = r\cos\theta = ar/b-r^2/b,$$

i.e.

$$x/4c = R\coth\eta-R^2,$$

whence

$$(x/4c)^3 = \coth^3\eta\, R^3-3\coth^2\eta R^4+3\coth\eta R^5-R^6,$$

and so, from (3.17)–(3.20), we find that

$$\begin{aligned}3F_1 = x^3 = &-48c^3 \coth\eta \operatorname{cosech}^4\eta (1+2C_0)\\ &+64c^3 (\operatorname{cosech}^2\eta - \tfrac{3}{2}\coth^2\eta \operatorname{cosech}^2\eta - \tfrac{1}{15})\, C_1\\ &-48c^3 \coth\eta \operatorname{cosech}^2\eta\, C_2 + 16c^3 (\tfrac{1}{3} - \operatorname{cosech}^2\eta)\, C_3 - \tfrac{16}{15}c^3\, C_5,\end{aligned} \tag{3.23}$$

and, by (3.12),

$$\begin{aligned}3\frac{\partial F_1}{\partial\eta} = &\ 48c^3 (\operatorname{cosech}^6\eta + 4\operatorname{cosech}^4\eta \coth^2\eta)\left(1-2\frac{\partial C_{-1}}{\partial\eta}\right)\\ &-32c^3 (2\coth\eta \operatorname{cosech}^2\eta + 15\coth\eta \operatorname{cosech}^4\eta)\frac{\partial C_0}{\partial\eta}\\ &-16c^3 (12\coth^2\eta \operatorname{cosech}^2\eta - \operatorname{cosech}^2\eta + \tfrac{4}{15})\frac{\partial C_1}{\partial\eta}\\ &-64c^3 \coth\eta \operatorname{cosech}^2\eta \frac{\partial C_2}{\partial\eta} + 16c^3 (\tfrac{1}{3} - \operatorname{cosech}^2\eta)\frac{\partial C_3}{\partial\eta} - \tfrac{16}{15}c^3 \frac{\partial C_5}{\partial\eta}.\end{aligned} \tag{3.24}$$

Again, it is readily shown from (3.13) and (3.14) that

$$(r/4c)^2 \sin^2\xi = -(1-2\coth\eta R + R^2),$$

hence, since

$$y/4c = \sin\xi \operatorname{cosech}\eta\, R^2,$$

we have

$$\begin{aligned}(y/4c)^3 &= \sin\xi \operatorname{cosech}\eta\, R^4 (r/4c)^2 \sin^2\xi\\ &= -\sin\xi \operatorname{cosech}\eta (R^4 - 2\coth\eta R^5 + R^6).\end{aligned}$$

Now

$$\frac{\partial R^n}{\partial\xi} = n \sin\xi \operatorname{cosech}\eta\, R^{n+1},$$

hence

$$(y/4c)^3 = -\frac{\partial}{\partial\xi}(\tfrac{1}{3}R^3 - \tfrac{1}{2}\coth\eta\, R^4 + \tfrac{1}{5}R^5),$$

and, substituting from (3.17)–(3.19) and making use of (3.12), we get

$$\begin{aligned}3F_2 = y_3 = &-16c^3 (3\coth^2\eta \operatorname{cosech}^2\eta - \operatorname{cosech}^2\eta - \tfrac{4}{15})\, S_1\\ &-48c^3 \coth\eta \operatorname{cosech}^2\eta\, S_2\\ &-16c^3 (\operatorname{cosech}^2\eta + \tfrac{1}{3})\, S_3 + \tfrac{16}{15}c^3\, S_5,\end{aligned} \tag{3.25}$$

and

$$3\,\frac{\partial F_2}{\partial \eta}=-64c^3\,(3\coth\eta\,\operatorname{cosech}^4\eta+2\coth\eta\,\operatorname{cosech}^2\eta)\,\frac{\partial S_0}{\partial \eta}$$
$$-16c^3\,(9\coth^2\eta\,\operatorname{cosech}^2\eta+3\operatorname{cosech}^4\eta-\operatorname{cosech}^2\eta-\tfrac{4}{15})\,\frac{\partial S_1}{\partial \eta}$$
$$-80c^3\coth\eta\,\operatorname{cosech}^2\eta\,\frac{\partial S_2}{\partial \eta}-16c^3\,(\operatorname{cosech}^2\eta+\tfrac{1}{3})\,\frac{\partial S_3}{\partial \eta}$$
$$+\tfrac{16}{15}c^3\,\frac{\partial S_5}{\partial \eta}. \tag{3.26}$$

In similar fashion we also get

$$(x^2-y^2)/16c^2=(1+\coth^2\eta)\,R^2-4\coth\eta\,R^3+2R^4,$$

$$(x^3-3xy^2)/64C^3$$
$$=(3\coth\eta+\coth^3\eta)\,R^3-3(3\coth^2\eta+1)\,R^4+12\coth\eta R^5-4R^6;$$

these lead, in virtue of (3.16)–(3.20), to the results

$$(x^2-y^2)/16c^2=\tfrac{2}{3}(C_3-C_1),$$
$$(x^3-3xy^2)/64c^3=\tfrac{1}{3}(C_3-C_1)-\tfrac{1}{15}(C_5-C_1).$$

It follows that the odd functions C_{2r+1} (and hence S_{2r+1}) are expressible simply in terms of Cartesian coordinates, and in particular

$$C_1=-\frac{x}{8c},\qquad S_1=-\frac{y}{8c}, \tag{3.27}$$

$$C_3=-\frac{x}{8c}+\frac{3}{32c^2}\,(x^2-y^2),\qquad S_3=-\frac{y}{8c}+\frac{3}{16c^2}\,xy, \tag{3.28}$$

$$C_5=-\frac{x}{8c}+\frac{15}{32c^2}\,(x^2-y^2)-\frac{15}{64c^3}\,(x^3-3xy^2),$$
$$S_5=-\frac{y}{8c}+\frac{15}{16c^2}\,xy+\frac{15}{64c^3}\,(y^3-3x^2y). \tag{3.29}$$

These are useful in expressing parts of the canonical functions in terms of Cartesian coordinates. For completeness we also give the remaining functions needed in finite form. From (3.10) and (3.12) we find that

$$1+2C_0=\frac{\sinh\eta}{\cosh\eta+\cos\xi},\qquad 2S_0=-\frac{\sin\xi}{\cosh\eta+\cos\xi}, \tag{3.30}$$

and

$$C_2 = \frac{\partial^2 C_0}{\partial \eta^2} = \tfrac{1}{2}\frac{\sinh\eta\cos\xi}{(\cosh\eta+\cos\xi)^2} - \frac{\sinh\eta(1+\cosh\eta\cos\xi)}{(\cosh\eta+\cos\xi)^3}, \tag{3.31}$$

$$S_2 = \frac{\partial^2 S_0}{\partial \eta^2} = \tfrac{1}{2}\frac{\cosh\eta\sin\xi}{(\cosh\eta+\cos\xi)^2} - \frac{\sinh^2\eta\sin\xi}{(\cosh\eta+\cos\xi)^3}. \tag{3.32}$$

Again, $$Z_{-1} = C_{-1}+iS_{-1} = \sum_{n=1}^{\infty}(-1)^n\frac{e^{-n(\eta-i\xi)}}{n},$$

whence

$$\left.\begin{aligned} C_{-1} &= -\tfrac{1}{2}\log(1+2e^{-\eta}\cos\xi+e^{-2\eta}),\\ S_{-1} &= -\tan^{-1}\left(\frac{\sin\xi}{e^{\eta}+\cos\xi}\right). \end{aligned}\right\} \tag{3.33}$$

4. *The canonical flexure functions.*

We consider first the torsion function, previously obtained by Basset (**1**, p. 240) in the guise of the stream function in the corresponding hydrodynamical problem of the motion of inviscid liquid in a rotating bowl of the same cross-section S.

From (2.8), (2.9), (3.4), and (3.21), ψ_3 must satisfy

$$\psi_3 = \tfrac{1}{2}ab\sinh\beta[1+2C_0]+4bc[C_1]+\text{const.},$$

round the boundary C given by $\eta=\beta$, where the brackets [ ] are used to denote that η has been replaced by β in the functions so bracketed, and a and b are constants. This condition is satisfied by taking $\psi_3=\psi_{31}+\psi_{32}$ throughout the cross-section, where

$$\psi_{31} = \tfrac{1}{2}ab\sinh\beta(1+2C_0) = \tfrac{1}{2}ab\sinh\beta\,\frac{\sinh\eta}{\cosh\eta+\cos\xi}, \tag{4.1}$$

$$\psi_{32} = 4bc\,C_1 = -\tfrac{1}{2}bx, \tag{4.2}$$

by (3.27) and (3.30), this solution is satisfactory, since, besides satisfying the boundary condition, it gives finite stresses and displacements across the cross-section, care being needed at the origin, where η becomes infinite. These results are equivalent to Basset's result for the stream-function in his problem. The corresponding complex canonical functions are given by

$$\omega_{31} = \phi_{31}+i\psi_{31} = \tfrac{1}{2}iab\sinh\beta(1+2Z_0), \tag{4.3}$$

$$\omega_{32} = \phi_{32}+i\psi_{32} = -\tfrac{1}{2}ibz. \tag{4.4}$$

Consider next the canonical function ψ_1. From (2.8), (2.9), (3.4) and (3.23), ψ_1 must satisfy the boundary condition

$$\psi_1 = -\tfrac{1}{4}\sinh\beta\, ab^2[1+2C_0] - abc\sinh\beta[C_2] + \tfrac{4}{3}c\left(4bc - \tfrac{3}{2}a^2 - \frac{c^2}{15}\right)[C_1]$$

$$+\tfrac{4}{3}c^2\left(\tfrac{4}{3}c - b\right)[C_3] - \tfrac{16}{45}c^3[C_5] + \text{const.};$$

this is satisfied by taking $\psi_1 = \psi_{11} + \psi_{12}$ across the cross-section S, where

$$\psi_{11} = -\sinh\beta\left(\tfrac{1}{2}ab^2 C_0 + abc\, C_2\right), \tag{4.5}$$

which can be put in finite terms from (3.30) and (3.31), and

$$\psi_{12} = c\left(\tfrac{16}{3}bc - 2a^2 - \tfrac{4}{45}c^2\right)C_1 + \tfrac{4}{9}c^2(4c - 3b)\,C_3 - \tfrac{16}{45}c^3\,C_5,$$

i.e.

$$\psi_{12} = \tfrac{1}{12}(x^3 - 3xy^2) - \tfrac{1}{8}b(x^2 - y^2) + (a^2 + b^2)\tfrac{1}{8}x, \tag{4.6}$$

by using (3.27)–(3.29). The corresponding complex canonical functions are given by

$$\omega_{11} = \phi_{11} + i\psi_{11} = -i\sinh\beta\left(\tfrac{1}{2}ab^2 Z_0 + abc\, Z_2\right), \tag{4.7}$$

$$\omega_{12} = \phi_{12} + i\psi_{12} = \tfrac{1}{12}iz^3 - \tfrac{1}{8}ibz^2 + \tfrac{1}{8}i(a^2 + b^2)z. \tag{4.8}$$

Next consider the canonical function ψ_2. From (2.8), (2.9), (3.4), and (3.25), we see that, along the boundary C, ψ_2 must satisfy

$$\psi_2 = -\tfrac{1}{3}c\left(2a^2 + b^2 - \tfrac{64}{15}c^2\right)[S_1] - abc\sinh\beta\,[S_2]$$

$$-\tfrac{4}{3}c^2\left(b + \tfrac{4}{3}c\right)[S_3] + \tfrac{16}{45}c^3[S_5] + \text{const.};$$

this is satisfied by taking $\psi_2 = \psi_{21} + \psi_{22}$ across the cross-section, where

$$\psi_{21} = -abc\sinh\beta\, S_2, \tag{4.9}$$

expressible in finite terms by means of (3.32), and

$$\psi_{22} = -\tfrac{1}{3}c\left(2a^2 + b^2 - \tfrac{64}{15}c^2\right)S_1 - \tfrac{4}{3}c^2\left(b + \tfrac{4}{3}c\right)S_3 + \tfrac{16}{45}c^3\,S_5.$$

Hence, using (3.27)–(3.29), we have

$$\psi_{22} = \tfrac{1}{8}a^2 y - \tfrac{1}{4}bxy + \tfrac{1}{12}(y^3 - 3x^2 y). \tag{4.10}$$

The corresponding complex canonical functions are given by

$$\omega_{21} = \phi_{21} + i\psi_{21} = -i\, abc\sinh\beta\, Z_2, \tag{4.11}$$

$$\omega_{22} = \phi_{22} + i\psi_{22} = \tfrac{1}{8}a^2 z - \tfrac{1}{8}bz^2 - \tfrac{1}{12}z^3. \tag{4.12}$$

We now proceed to find χ_2. From (2.8), (2.9), (3.4), and (3.26), we see that, round the boundary C, χ_2 must satisfy

$$\frac{\partial\chi_2}{\partial\eta}=-\tfrac{1}{3}(3ab^2+8abc)\sinh\beta\left[\frac{\partial S_0}{\partial\eta}\right]-\tfrac{4}{3}c\left(2a^2+b^2-\frac{c^2}{15}\right)\left[\frac{\partial S_1}{\partial\eta}\right]$$
$$-\tfrac{5}{3}abc\sinh\beta\left[\frac{\partial S_2}{\partial\eta}\right]-\tfrac{4}{3}c^2(b+\tfrac{4}{3}c)\left[\frac{\partial S_3}{\partial\eta}\right]+\tfrac{16}{45}c^3\left[\frac{\partial S_5}{\partial\eta}\right];$$

this is satisfied by taking $\chi_2=\chi_{21}+\chi_{22}$, where

$$\chi_{21}=-\sinh\beta\{(ab^2+\tfrac{8}{3}abc)\,S_0+\tfrac{5}{3}abc\,S_2\},\qquad(4.13)$$

which can be put into finite terms by means of equations (3.30) and (3.32), and

$$\chi_{22}=-\tfrac{4}{3}c(2a^2+b^2-\tfrac{1}{15}c^2)\,S_1-\tfrac{4}{3}c^2(b+\tfrac{4}{3}c)\,S_3+\tfrac{16}{45}c^3\,S_5,$$

i.e.
$$\chi_{22}=\tfrac{1}{12}(y^3-3x^2y)-\tfrac{1}{4}bxy+\tfrac{1}{8}(3a^2+b^2)\,y,\qquad(4.14)$$

by using (3.27)–(3.29). The corresponding complex canonical functions are given by

$$\Omega_{21}=\chi_{21}+i\chi_{21}^{*}=i\sinh\beta\{(ab^2+\tfrac{8}{3}abc)\,Z_0+\tfrac{5}{3}abc\,Z_2\},\qquad(4.15)$$

$$\Omega_{22}=\chi_{22}+i\chi_{22}^{*}=\tfrac{1}{12}iz^3+\tfrac{1}{8}ibz^2-\tfrac{1}{8}i(3a^2+b^2)\,z.\qquad(4.16)$$

Proceeding in similar fashion for χ_1, we find from (2.8), (2.9), (3.4), and (3.24) that χ_1 must satisfy the boundary condition

$$\frac{\partial\chi_1}{\partial\eta}=\tfrac{1}{4}(b^3+4a^2b)\left[1-2\frac{\partial C_{-1}}{\partial\eta}\right]-\tfrac{1}{2}\sinh\beta(\tfrac{8}{3}abc+5ab^2)\left[\frac{\partial C_0}{\partial\eta}\right];$$
$$-\tfrac{1}{3}c(11a^2+b^2+\tfrac{4}{15}c^2)\left[\frac{\partial C_1}{\partial\eta}\right]-\tfrac{4}{3}abc\sinh\beta\left[\frac{\partial C_2}{\partial\eta}\right]$$
$$+\tfrac{4}{3}c^2(\tfrac{4}{3}c-b)\left[\frac{\partial C_3}{\partial\eta}\right]-\tfrac{16}{45}\left[\frac{\partial C_5}{\partial\eta}\right];$$

this is satisfied by taking

$$\chi_1=\tfrac{1}{4}b(b^2+4a^2)\eta+\chi_{11}+\chi_{12},$$

where

$$\chi_{11}=-\tfrac{1}{2}b(b^2+4a^2)\,C_{-1}-\tfrac{1}{2}\sinh\beta(\tfrac{8}{3}abc+5ab^2)\,C_0$$
$$-\tfrac{4}{3}abc\sinh\beta C_2,\qquad(4.17)$$

which can be expressed in finite terms from (3.30), (3.31), and (3.33), and

$$\chi_{12}=-\tfrac{1}{3}c(11a^2+b^2+\tfrac{4}{15}c^2)\,C_1+\tfrac{4}{3}c^2(\tfrac{4}{3}c-b)\,C_3-\tfrac{16}{45}C_5,$$

i.e.
$$\chi_{12}=\tfrac{1}{2}a^2x-\tfrac{1}{8}b(x^2-y^2)+\tfrac{1}{12}(x^3-3xy^2), \tag{4.18}$$

on making use of (3.27)–(3.29). The corresponding complex canonical functions are given by

$$\Omega_{11}=\chi_{11}+i\chi_{11}^*=-\tfrac{1}{2}b(b^2+4a^2)Z_{-1}-\tfrac{1}{2}\sinh\beta(\tfrac{8}{3}abc+5ab^2)Z_0$$
$$-\tfrac{4}{3}abc\sinh\beta\,Z_2, \tag{4.19}$$

$$\Omega_{12}=\chi_{12}+i\chi_{12}^*=\tfrac{1}{2}a^2z-\tfrac{1}{8}bz^2+\tfrac{1}{12}z^3. \tag{4.20}$$

It is clear that the term $\tfrac{1}{4}b(b^2+4a^2)\eta$ in χ_1 becomes infinite at the origin, where η becomes infinite, but we shall show later that this infinity disappears from the combination $\chi_1-h\chi_3$, and it is this combination which has to give a physically admissible solution across the cross-section, since χ_1, χ_3 only occur in this combination in the physical problem, and we have separated the parts χ_1 and χ_3 only for mathematical convenience.

Finally, from (2.8), (2.9), (3.4), and (3.22), the boundary condition for χ_3 is

$$\frac{\partial\chi_3}{\partial\eta}=-\tfrac{1}{2}(b^2+2a^2)+(b^2+2a^2)\left[\frac{\partial C_{-1}}{\partial\eta}\right]+3ab\sinh\beta\left[\frac{\partial C_0}{\partial\eta}\right]+4bc\left[\frac{\partial C_1}{\partial\eta}\right];$$

this is satisfied by taking

$$\chi_3=-\tfrac{1}{2}(b^2+2a^2)\eta+\chi_{31}+\chi_{32},$$

where
$$\chi_{31}=(b^2+2a^2)C_{-1}+3ab\sinh\beta C_0, \tag{4.21}$$

which can be put in finite terms from (3.30) and (3.33), and

$$\chi_{32}=4bc\,C_1=-\tfrac{1}{2}bx, \tag{4.22}$$

by (3.27). The corresponding complex canonical functions are

$$\Omega_{31}=\chi_{31}+i\chi_{31}^*=(b^2+2a^2)Z_{-1}+3ab\sinh\beta Z_0, \tag{4.23}$$

$$\Omega_{32}=\chi_{32}+i\chi_{32}^*=-\tfrac{1}{2}bz. \tag{4.24}$$

It now becomes apparent that the critical term in the combination $\chi_1-h\chi_3$ is

$$\eta\{\tfrac{1}{4}b(b^2+4a^2)+\tfrac{1}{2}h(b^2+2a^2)\},$$

which gives no trouble if the coefficient of η vanishes. This necessitates

$$h=-b(b^2+4a^2)/2(b^2+2a^2), \tag{4.25}$$

which is indeed found to be the case; hence our solutions for χ_1 and χ_3 are satisfactory in every respect.

5. *The torsion and flexure moment integrals.*

There is no difficulty in obtaining the following results for this cross-section:—

$$\int dS = \tfrac{1}{2}\pi(2a^2+b^2), \qquad \int x\,dS = -\tfrac{1}{4}\pi b(4a^2+b^2), \tag{5.1}$$

$$\int x^2\,dS = \tfrac{1}{32}\pi(8a^4+36a^2b^2+5b^4), \qquad \int y^2\,dS = \tfrac{1}{32}\pi(8a^4+12a^2b^2+b^4), \tag{5.2}$$

$$\int x^3\,dS = -\tfrac{1}{64}\pi b(48a^4+80a^2b^2+7b^4), \quad \int xy^2\,dS = -\tfrac{1}{64}\pi b(16a^4+16a^2b^2+b^4). \tag{5.3}$$

In calculating the moment integrals M_r', L_r', we shall divide them into portions corresponding to the separation of the canonical functions into two portions, *e.g.* $M_3' = M'_{31}+M'_{32}$ corresponding to $\omega_3 = \omega_{31}+\omega_{32}$. Hence we find, from (2.15),

$$M'_{31} = \int_{-\pi}^{\pi} \tfrac{1}{2}(x^2+y^2)\,\frac{\partial\phi_{31}}{\partial\xi}\,d\xi,$$

where, from (4.3) and (3.12),

$$\frac{\partial\phi_{31}}{\partial\xi} = -ab\sinh\beta\,\frac{\partial S_0}{\partial\xi} = -ab\sinh\beta\,C_1,$$

and, from (3.4) and (3.21),

$$\tfrac{1}{2}(x^2+y^2) = \tfrac{1}{2}ab\sinh\beta(1+2C_0)+4bc\,C_1,$$

whence

$$M'_{31} = -\pi a^2b^2\sinh^2\beta\sum_{n=1}^{\infty} ne^{-2n\beta} - 4\pi ab^2c\sinh\beta\sum_{n=1}^{\infty} n^2e^{-2n\beta}.$$

Now
$$\sum_{n=1}^{\infty} ne^{-2n\beta} = \tfrac{1}{4}\operatorname{cosech}^2\beta = \tfrac{1}{16}\,\frac{b}{c}, \tag{5.4}$$

and
$$\sum_{n=1}^{\infty} n^2e^{-2n\beta} = \tfrac{1}{4}\coth\beta\operatorname{cosech}^2\beta = \tfrac{1}{16}\,\frac{a}{c}\operatorname{cosech}\beta, \tag{5.5}$$

whence
$$M'_{31} = -\tfrac{1}{2}\pi a^2b^2.$$

Again, from (2.13) and (4.4),

$$M'_{32} = \text{real part of } \int \tfrac{1}{2}bz\,dS = \tfrac{1}{2}b\int x\,dS,$$

i.e.
$$M'_{32} = -\tfrac{1}{8}\pi b^2(4a^2+b^2),$$

by using (5.1). Collecting the results we find that

$$M_3' = -\tfrac{1}{8}\pi b^2(8a^2+b^2);$$

whence, finally, from (2.12) and (5.2),

$$M_3 = \tfrac{1}{16}\pi(8a^4+8a^2b^2+b^4). \tag{5.6}$$

Next, from (2.15),

$$M'_{11} = \int_{-\pi}^{\pi} \tfrac{1}{2}(x^2+y^2)\,\frac{\partial\phi_{11}}{\partial\xi}\,\partial\xi,$$

where

$$\frac{\partial\phi_{11}}{\partial\xi} = \tfrac{1}{2}\sinh\beta(ab^2C_1+2abc\,C_3),$$

whence

$$M'_{11} = 2\pi a^2b^2c\sum_{n=1}^{\infty} ne^{-2n\beta}+4\pi a^2bc^2\sum_{n=1}^{\infty} n^3e^{-2n\beta}$$
$$+2\pi ab^3c\sinh\beta\sum_{n=1}^{\infty} n^2e^{-2n\beta}+4\pi ab^2c^2\sinh\beta\sum_{n=1}^{\infty} n^4e^{-2n\beta}.$$

But, in addition to (5.4) and (5.5), we have

$$\sum_{n=1}^{\infty} n^3e^{-2n\beta} = \tfrac{1}{8}\{\operatorname{cosech}^4\beta+2\coth^2\beta\operatorname{cosech}^2\beta\} = \frac{1}{128c^2}(b^2+2a^2), \tag{5.7}$$

$$\sum_{n=1}^{\infty} n^4e^{-2n\beta} = \tfrac{1}{4}\{3\coth\beta\operatorname{cosech}^4\beta+\coth\beta\operatorname{cosech}^2\beta\},$$
$$= \frac{1}{64c^2}\operatorname{cosech}\beta(3ab+4ac), \tag{5.8}$$

which lead to

$$M'_{11} = \tfrac{1}{32}\pi a^2b(13b^2+4a^2).$$

Again, from (2.13) and (4.8), we have

$$M'_{12} = -\tfrac{1}{4}\int(x^3-3xy^2)\,dS+\tfrac{1}{4}b\int(x^2-y^2)\,dS-\tfrac{1}{8}(a^2+b^2)\int x\,dS$$
$$= \tfrac{1}{64}\pi b(8a^4+30a^2b^2+5b^4),$$

after making use of (5.1)–(5.3). Collecting the results, we have

$$M_1' = \tfrac{1}{64}\pi b(16a^4+56a^2b^2+5b^4),$$

whence, finally, from (2.12) and (5.3),

$$M_1 = -\tfrac{1}{32}\pi b\,(16a^4+12a^2b^2+b^4). \tag{5.9}$$

In similar fashion, from (2.16),

$$L'_{21} = \int_{-\pi}^{\pi} \tfrac{1}{2}(x^2+y^2)\,\frac{\partial\chi_{21}}{\partial\xi}\,d\xi,$$

where, from (4.13) and (3.12),

$$\frac{\partial \chi_{21}}{\partial \xi} = -\sinh\beta\{(ab^2+\tfrac{8}{3}abc)\,C_1+\tfrac{5}{3}abc\,C_3\},$$

whence

$$-L'_{21} = \pi c(4a^2b^2+\tfrac{32}{3}a^2bc)\sum_{n=1}^{\infty} ne^{-2n\beta}+\tfrac{20}{3}\pi a^2bc^2\sum_{n=1}^{\infty} n^3e^{-2n\beta}$$

$$+\pi\sinh\beta\{(4ab^3c+\tfrac{32}{3}ab^2c^2)\sum_{n=1}^{\infty} n^2e^{-2n\beta}+\tfrac{20}{3}ab^2c^2\sum_{n=1}^{\infty} n^4e^{-2n\beta}\},$$

and, using (5.4), (5.5), (5.7), and (5.8), we have

$$L'_{21} = -\tfrac{1}{96}\pi a^2b(52a^2+41b^2).$$

Again, from (2.14) and (4.16), we have

$$L'_{22} = -\tfrac{1}{4}\int(x^3-3xy^2)\,dS-\tfrac{1}{4}b\int(x^2-y^2)\,dS+\tfrac{1}{8}(3a^2+b^2)\int x\,dS,$$

i.e.

$$L'_{22} = -\tfrac{3}{64}\pi b(8a^4+6a^2b^2+b^4),$$

on making use of (5.1), (5.2), and (5.3). Collecting the results, we get

$$L_2' = -\tfrac{1}{192}\pi b(176a^4+136a^2b^2+9b^4),$$

whence, finally, from (2.12) and (5.3),

$$L_2 = -\tfrac{1}{96}\pi b(64a^4+44a^2b^2+3b^4). \tag{5.10}$$

6. *The associated torsion and centres of flexure and least strain.*

The value of the associated twist τ' occurring in (2.4) and (2.5) can now be obtained from (2.17), using the results (4.25), (5.2), (5.6), (5.9), and (5.10), and we find that

$$\tau' = \frac{W'}{EI'}\cdot\frac{a^2b\{(1+\eta)(4a^2-b^2)^2+3\eta b^4\}}{3(2a^2+b^2)(8a^4+8a^2b^2+b^4)}. \tag{6.1}$$

The position $(f, 0)$ of the centre of flexure is obtained in similar fashion from (2.18), and we find

$$f = -\tfrac{1}{6}b\,\frac{\{(64a^4+44a^2b^2+3b^4)(2a^2+b^2)+6\sigma a^2b^4\}}{(8a^4+12a^2b^2+b^4)(2a^2+b^2)}, \tag{6.2}$$

while the coordinates of the centre of least strain, which are readily found by putting η (and therefore σ) zero in those of the centre of flexure, are $(x_0, 0)$, where

$$x_0 = -\tfrac{1}{6}b\,\frac{64a^4+44a^2b^2+3b^4}{8a^4+12a^2b^2+b^4}. \tag{6.3}$$

As a check upon our values for the torsion moment $\mu\tau M_3$, the associated torsion τ', and the abscissa f of the centre of flexure, note that, if we put $b=a$, so that the boundary is a cardioid, these quantities become

$$\mu\tau M_3 = \tfrac{17}{16}\mu\tau\pi a^4, \tag{6.4}$$

$$\tau' = (W'a/EI')(3+4\eta)/51, \tag{6.5}$$

$$f = -a(111+2\sigma)/126. \tag{6.6}$$

These are the correct values for the cardioid (Stevenson, **3**, p. 199).

While the family of cross-sections has no great value as an engineering example, it does serve to illustrate the nature of the twist associated with the bending of an elastic cylinder under a transverse end-load when the cross-section is slit or indented. We observe that the associated twist τ' is always positive, *i.e.* the mean twist of the cross-section when a load is applied at the centroid of the end-section with a resolute transverse to the axis of symmetry of the cross-section is such that the flattened side ($0<b<\frac{1}{2}a$), see Fig. 1, or the indented side ($\frac{1}{2}a<b<a$), see Figs. 2 and 3, is turned in the direction of the load resolute as was found for the cardioid cross-section, which again behaves in this respect in a similar fashion to the circular cross-section with a radial slit. (Stevenson, **3**, p. 194.)

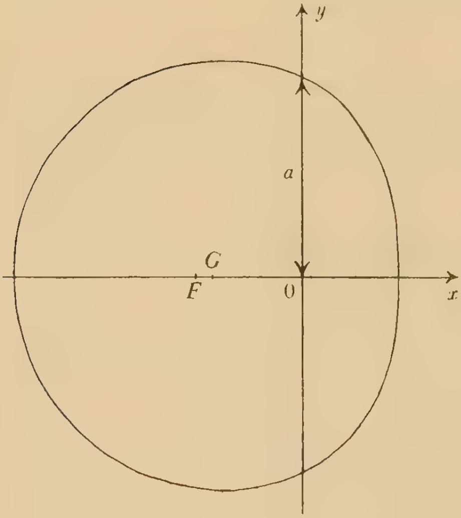

Fig. 1.
G = centroid. F = centre of flexure.
$b/a = 0.5$, $OG = 0.4722a$,
$OF = 0.5664a + \sigma 0.0126a$.

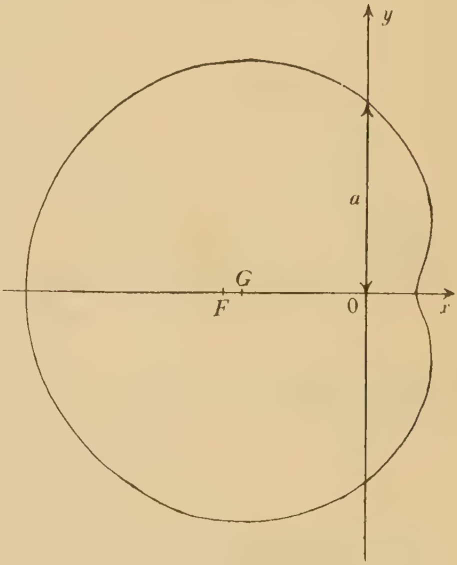

Fig. 2.
G = centroid. F = centre of flexure.
$b/a = 0.75$, $OG = 0.6677a$,
$OF = 0.7442a + \sigma 0.0061a$.

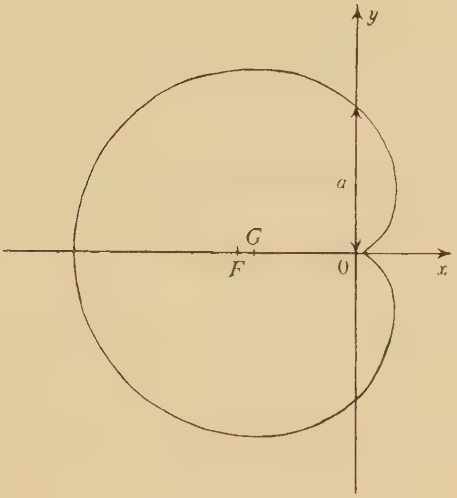

Fig. 3.
G = centroid. F = centre of flexure.
$b/a = 0.95$. $OG = 0.7179a$, $OF = 0.8556a + \sigma 0.0126a$.

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ON WARING'S PROBLEM FOR FIFTH POWERS

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Let $G(k)$ be the least integer s such that the Diophantine equation

$$N = x_1^k + \ldots + x_s^k, \quad x_\nu \geqslant 0, \quad (\nu = 1, \ldots, s)$$

is solvable for all sufficiently large integers N.

It has been proved by Hardy and Littlewood[‡] that

$$G(5) \leqslant 41.$$

By a combination of the methods of Hardy-Littlewood and of Landau[§] James[||] has proved that $G(5) \leqslant 35$. Estermann[¶] has indicated that the result $G(5) \leqslant 29$ is probably obtainable by the introduction of Vinogradow's ideas.

It seems natural to try the problem by combining these three leading methods. The object of this paper is to prove in this way that

$$G(5) \leqslant 28.$$

I wish to thank Mr. Davenport and Dr. Heilbronn for their kind suggestions and helpful remarks.

1. *Introduction.*

Let n be any sufficiently large integer, and let $P = \frac{1}{2}n^{\frac{1}{5}}$ and $Q = n^{\frac{1}{4}}$; let N be an integer satisfying

$$\tfrac{1}{2}Q < N < Q.$$

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[‡] *Math. Zeitschrift*, 23 (1925), 1–37.

[§] *Math. Annalen*, 66 (1909), 102–105.

[||] *Proc. London Math. Soc.* (2), 37 (1934), 257–291.

[¶] *Proc. London Math. Soc.* (2), 41 (1937), 126–142; also *Acta Arithmetica*, 2 (1937), 197–211.

Let
$$f(z)=10(z^4+2N^2z^2),$$

$$U(\alpha)=\sum_{z=1}^{\frac{1}{2}Q} e\bigl(\alpha f(z)\bigr), \quad e(x)=e^{2\pi i x},$$

$$V(\alpha,\ P)=\sum_{z=1}^{P} e(\alpha z^5),$$

$$V_i(\alpha)=V(\alpha,\ 2^{-i+1}P^{(\frac{4}{5})^i})-V(\alpha,\ 2^{-i}P^{(\frac{4}{5})^i}) \quad (i=0,\ 1,\ 2,\ 3,\ 4,\ 5),$$

$$Q(\alpha)=V_1(\alpha)\,V_2(\alpha)\,V_3(\alpha)\,V_4(\alpha)\,V_5{}^2(\alpha)=\sum_n r_6{}'(n)\,e(n\alpha),$$

$$R(\alpha)=V_0(\alpha)\,Q(\alpha)=\sum_n r_7{}'(n)\,e(n\alpha),$$

$$U^6(\alpha)\,V_0{}^3(\alpha)\,Q^2(\alpha)=\sum_n r(n)\,e(n\alpha).$$

Further, we write

$$S_{a,q}=\sum_{h=1}^{q} e_q\bigl(af(h)\bigr), \quad e_q(x)=e^{(2\pi i/q)x},$$

$$T_{a,q}=\sum_{h=1}^{q} e_q(ah^5),$$

$$U^*(\beta,\ a,\ q)=\frac{S_{a,q}}{q}\int_0^{\frac{1}{2}Q} e\bigl(\beta f(z)\bigr)\,dz,$$

$$V^*{}_0(\beta,\ a,\ q)=\frac{T_{a,q}}{q}\int_P^{2P} e(\beta z^5)\,dz,$$

$$\mathfrak{S}(N)=\sum_{q=1}^{\infty}\ \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-21}\,S_{a,q}^6\,T_{a,q}^{15}\,e_q(-an).$$

We use ϵ to denote an arbitrarily small positive number and $c_1,\ c_2,\ \ldots$ to denote absolute constants. The constants implied in the symbol O depend only on ϵ.

2. *Lemmas on trigonometrical sums and trigonometrical integrals.*

LEMMA 2.1. *If* $(a,\ p)=1$, *then*

$$\sum_{x=1}^{p^l} e_{p^l}\bigl(a(x^4+2x^2)\bigr)=O(p^{\frac{3}{4}l}).$$

Proof. For $l=1$, the lemma was proved by Mordell†. For $l>1$, we put $x=y+p^{\tau}z$, where $\tau=[\frac{1}{2}(l+1)]$. Then

$$\begin{aligned} S &= \sum_{y=1}^{p^{\tau}} \sum_{z=1}^{p^{l-\tau}} e_{p^l}\Big(a(y^4+2y^2)+4ap^{\tau}z(y^3+y)\Big) \\ &= p^{l-\tau} \sum_{\substack{y=1 \\ 4y(y^2+1)\equiv 0 \pmod{p^{l-\tau}}}}^{p^{\tau}} e_{p^l}\Big(a(y^4+2y^2)\Big) \\ &= O(p^{\tau}N_0), \end{aligned}$$

where N_0 denotes the number of solutions of

$$2^2y(y^2+1)\equiv 0 \pmod{p^{l-\tau}} \quad (0\leqslant y<p^{l-\tau}).$$

For $p\geqslant 3$, evidently $N_0\leqslant 3$. For $p=2$ we have either

$$y\equiv 0 \pmod{2^{l-\tau-2}}$$

or

$$y^2+1\equiv 0 \pmod{2^{l-\tau-2}}.$$

Both cases give $N_0=O(1)$. Therefore

$$S=O(p^{\tau})=O(p^{\frac{3}{4}l}).$$

LEMMA 2.2. *If* $p\nmid a$, $p\,|\,L$ *and*

$$l>\begin{cases} 1 & \textit{for} \quad p\neq 2, \\ 2 & \textit{for} \quad p=2, \end{cases}$$

then

$$\sum_{\substack{x=1 \\ p\nmid x}}^{p^l} e_{p^l}\Big(a(x^4+2Lx^2)\Big)=0.$$

Proof. Let

$$\tau=\begin{cases} [\frac{1}{2}(l+1)] & \text{for} \quad p\neq 2, \\ [\frac{1}{2}(l+3)] & \text{for} \quad p=2. \end{cases}$$

Let $x=y+p^{\tau}z$. Then the sum in question is equal to

$$\sum_{\substack{y=1 \\ p\nmid y}}^{p^{\tau}} \sum_{z=1}^{p^{l-\tau}} e_{p^l}\Big(a(y^4+2Ly_2)+4p^{\tau}za(y^3+Ly)\Big)=0$$

since $p\nmid(y^3+Ly)$.

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† Mordell, *Quart. J. of Math.*, 3 (1932), 161–167.

LEMMA 2.3. *If* $l \geqslant 1$ *and* $p \nmid a$, *then*

$$S = \sum_{x=1}^{p^l} e_{p^l}\Big(a(x^4+2N^2x^2)\Big) = O(p^{\frac{3}{4}l}).$$

Proof. (1) If $p \nmid N$, then xN runs with x through all residue classes, mod p^l. Therefore

$$S = \sum_{x=1}^{p^l} e_{p^l}\Big(aN^4(x^4+2x^2)\Big) = O(p^{\frac{3}{4}l}),$$

by Lemma 2.1.

(2) Let $p^\sigma M = N$, $p \nmid M$, $\sigma > 0$.

We now consider (i) $l \geqslant 4\sigma$, (ii) $l < 4\sigma$.

(2.1) $l \geqslant 4\sigma$. We divide S into two parts,

$$S = \sum_{\substack{x=1 \\ p^\sigma | x}}^{p^l} + \sum_{\substack{x=1 \\ p^\sigma \nmid x}}^{p^l} = S_1 + S_2, \quad \text{say.}$$

We have, by (1),

$$\begin{aligned} S_1 &= \sum_{x=1}^{p^{l-\sigma}} e_{p^{l-4\sigma}}\Big(a(x^4+2M^2x^2)\Big) \\ &= p^{3\sigma}\,O(p^{l-4\sigma})^{\frac{3}{4}} = O(p^{\frac{3}{4}l}). \end{aligned}$$

Further,

$$\begin{aligned} S_2 &= \sum_{t=0}^{\sigma-1} \sum_{\substack{x=1 \\ p \nmid x}}^{p^{l-t}} e_{p^{l-4t}}\Big(a(x^4+2M^2p^{2(\sigma-t)}x^2)\Big) \\ &= \sum_{t=0}^{\sigma-1} p^{3t} \sum_{\substack{x=1 \\ p \nmid x}}^{p^{l-4t}} e_{p^{l-4t}}\Big(a(x^4+2M^2p^{2(\sigma-t)}x^2)\Big) \\ &= \sum_{t=0}^{\sigma-1} p^{3t}\,T_t, \quad \text{say.} \end{aligned}$$

By Lemma 2.2, $T_t = 0$ for all t, since

$$l - 4t \geqslant 4.$$

Thus we have $$S_2 = 0.$$

(2.2) $l < 4\sigma$. Let σ_1 be the integer defined by

$$4(\sigma_1 - 1) < l \leqslant 4\sigma_1.$$

Then $\sigma_1 \leqslant \sigma$. We divide S into

$$\sum_{\substack{x=1 \\ p^{\sigma_1} | x}}^{p^l} + \sum_{\substack{x=1 \\ p^{\sigma_1} \nmid x}}^{p^l} = S_1 + S_2, \quad \text{say}.$$

Evidently $$|S_1| \leqslant p^{l-\sigma_1} \leqslant p^{\frac{3}{4}l}.$$

Further, as in (2.1),

$$S_2 = \sum_{t=0}^{\sigma_1 - 1} p^{3t} T_t.$$

By Lemma 2.2, $T_t = 0$ except in the following cases:

(i) $$p > 2, \quad l - 4t = 1,$$

(ii) $$p = 2, \quad l - 4t = 1 \quad \text{or} \quad 2.$$

In these exceptional cases we have

(i) $$p \neq 2, \quad l = 4(\sigma_1 - 1) + 1.$$

Then

$$\begin{aligned} S_2 &= p^{3(\sigma_1 - 1)} \sum_{\substack{x=1 \\ p \nmid x}}^{p} e_p\Big(a(x^4 + 2M^2 p^{2(\sigma - \sigma_1 + 1)} x^2)\Big) \\ &= p^{3(\sigma_1 - 1)} \sum_{x=1}^{p-1} e_p(ax^4) = O(p^{3(\sigma_1 - 1) + \frac{1}{2}}) \\ &= O(p^{\frac{3}{4}l}). \end{aligned}$$

(ii) $$p = 2, \ l = 4(\sigma_1 - 1) + 1 \text{ or } 4(\sigma_1 - 1) + 2.$$

Then evidently

$$S_2 = O(2^{3(\sigma_1 - 1)}) = O(2^{\frac{3}{4}l}).$$

Remark. $\frac{3}{4}$ is the best possible value, since, for $p > 2$,

$$\begin{aligned} \sum_{x=1}^{p^8} e_{p^8}(x^4 + 2p^4 x^2) &= p^4 \sum_{\substack{y=1 \\ y^3 \equiv 0 \ (\mathrm{mod}\ p^4)}}^{p^4} e_{p^8}(y^4 + 2p^4 y^2) \\ &= p^4 \sum_{\substack{y=1 \\ p^2 | y}}^{p^4} e_{p^8}(y^4 + 2p^4 y^2) = p^6 = (p^8)^{\frac{3}{4}}. \end{aligned}$$

LEMMA 2.4. *If* $(a, q) = 1$, *then*

$$S_{a,q} = O(q^{\frac{3}{4} + \epsilon}).$$

This follows from Lemma 2.3 by a well-known argument.

LEMMA 2.5. *If* $(a, q) = 1$, *then*

$$T_{a,q} = O(q^{\frac{4}{5}+\epsilon})$$

(Landau, Satz 315)†.

LEMMA 2.6. *If* $(a, q) = 1$, *then*

$$\sum_{h=1}^{m} e_q\Big((ah^5)\Big) = O\Big(m^{\epsilon} q^{\epsilon} (m^{\frac{15}{16}} + mq^{-\frac{1}{16}} + m^{\frac{11}{16}} q^{\frac{1}{16}})\Big)$$

and
$$\sum_{h=1}^{m} e_q\Big(af(h)\Big) = O\Big(m^{\epsilon} q^{\epsilon} (m^{\frac{7}{8}} + mq^{-\frac{1}{8}} + m^{\frac{1}{2}} q^{\frac{1}{8}})\Big)$$

(Landau, Satz 267, with a slight modification).

LEMMA 2.7. *If* $|\beta| \leqslant \frac{1}{2}$, *then*

$$U^*(\beta, a, q) = O\Big(q^{-\frac{1}{4}+\epsilon} \min(Q, |\beta|^{-\frac{1}{4}})\Big),$$

and
$$V^*(\beta, a, q) = O\Big(q^{-\frac{1}{5}+\epsilon} \min(P, |\beta|^{-\frac{1}{5}})\Big).$$

Proof. In view of Lemma 2.4 and the evident approximation

$$I(\beta) = \int_0^{\frac{1}{2}Q} e^{\pm 2\pi i f(x)|\beta|}\, dx = O(Q),$$

it is sufficient to prove that

$$I(\beta) = O(|\beta|^{-\frac{1}{4}}).$$

For $|\beta|^{-\frac{1}{4}} < x < \frac{1}{2}Q$, we have

$$f'(x)|\beta| \geqslant 4|\beta|^{\frac{1}{4}},$$
$$f''(x)|\beta| \geqslant 0.$$

Hence, by the second mean-value theorem,

$$I(\beta) = O(|\beta|^{-\frac{1}{4}}) + \int_{|\beta|^{-\frac{1}{4}}}^{\frac{1}{2}Q} e^{2\pi i f(x)\beta}\, dx = O(|\beta|^{-\frac{1}{4}}).$$

The second formula in the lemma can easily be proved by the same argument.

LEMMA 2.8.

$$J = \int_{-\infty}^{\infty} \left(\int_0^{\frac{1}{2}Q} e\Big(\beta f(x)\Big)\, dx\right)^6 \left(\int_P^{2P} e(\beta y^5)\, dy\right)^3 e^{-2\pi i\beta n}\, d\beta = KP^{5+\frac{1}{2}}$$

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† The references here and subsequently are to Landau, *Vorlesungen über Zahlentheorie* (1927).

and $$K \geqslant c_1 > 0.$$

Proof. The convergence of the integral is an easy consequence of Lemma 2.8. Let $x = \frac{1}{2}Qx'$, $y = 2Py'$ and $\beta = (2P)^{-5}\beta'$. Then

$$J = 2^{-\frac{1}{2}} P^{5+\frac{1}{2}} \int_{-\infty}^{\infty} \left(\int_0^1 e\left(\beta' f_1(x')\right) dx'\right)^6 \left(\int_{\frac{1}{2}}^1 e(\beta' y'^5)\, dy'\right)^3 e^{-2\pi i\beta'} d\beta',$$

where $$f_1(x) = 10\,.\,2^{-4}(x^4+8c'x^2) \quad (\tfrac{1}{4} < c' < 1).$$

Evidently J is equal to the limit of

$$J_X = 2^{-\frac{1}{2}} P^{5+\frac{1}{2}} \int_{-X}^{X} \left(\int_0^1 e\left(\beta' f_1(x')\right) dx'\right)^6 \left(\int_{\frac{1}{2}}^1 e(\beta' y'^5)\, dy'\right)^3 e^{-2\pi i\beta'} d\beta'$$

$$= 2^{-\frac{1}{2}} P^{5+\frac{1}{2}} \int \frac{\sin 2\pi Xt}{\pi t}\, dt \int_{\frac{1}{2}}^1 dy_1 \int_{\frac{1}{2}}^1 dy_2 \int_0^1 dx_1 \ldots \int_0^1 dx_6\, \frac{1}{5{y_3}^4}$$

$$(\tfrac{1}{2} \leqslant y_3 \leqslant 1),$$

where $$t = f_1(x_1)+\ldots+f_1(x_6)+{y_1}^5+{y_2}^5+{y_3}^5-1.$$

Since $\dfrac{\sin 2\pi Xt}{\pi t}$ is a Dirichlet kernel, we have

$$J = 2^{-\frac{1}{2}} P^{5+\frac{1}{2}} \int_{\frac{1}{2}}^1 dy_1 \int_{\frac{1}{2}}^1 dy_2 \int_0^1 dx_1 \ldots \int_0^1 dx_6\, \frac{1}{\left(1-f_1(x_1)-\ldots-f_1(x_6)-{y_1}^5-{y_2}^5\right)^{\frac{4}{5}}}$$

$$[(\tfrac{1}{2})^5 \leqslant 1-f_1(x_1)-\ldots-f_1(x_6)-{y_1}^5-{y_2}^5 \leqslant 1]$$

$$= P^{5+\frac{1}{2}} K \quad \text{(say)}.$$

For $\frac{1}{2} < y_1 \leqslant 2^{-\frac{3}{5}}$, $\frac{1}{2} < y_2 \leqslant 2^{-\frac{3}{5}}$ and $0 < x_r < \frac{1}{20}$ $(r = 1, 2, \ldots, 6)$, we have

$$f_1(x) \leqslant 2^{-4}\, 10 \left(\frac{1}{20^4} + \frac{8}{20^2}\right) < \tfrac{1}{24}$$

and $$(\tfrac{1}{2})^4 \leqslant f_1(x_1)+\ldots+f_1(x_6)+{y_1}^5+{y_2}^5 < \tfrac{1}{2}.$$

Hence we have

$$K > 2^{-\frac{1}{2}} \left(\int_{\frac{1}{2}}^{2^{-\frac{3}{5}}} dy\right)^2 \left(\int_0^{\frac{1}{20}} dx\right)^6 = c_1.$$

3. *Approximation of U and V for smaller q.*

LEMMA 3.1. (Euler's summation formula)†. *Let*

$$b_1(x) = x-[x]-\tfrac{1}{2},$$

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† For a proof see, for example, Estermann, *Proc. London Math. Soc.* (2), 41 (1936), 126–142, §2.3.

where $[x]$ *is the integer part of* x,

$$b_l(x+1) = b_l(x), \quad \int_0^x b_l(y)\,dy = b_{l+1}(x) - b_{l+1}(0)$$

for every x *and every* l. *Let* $\phi(x)$ *and its derivatives (as far as they occur below) be continuous in the interval*

$$a \leqslant x \leqslant b.$$

Then, for any t,

$$\sum_{\substack{m \\ a \leqslant m+t < b}} \phi(m+t) = \int_a^b \phi(x)\,dx + \sum_{r=0}^{l-1} \{\phi^{(r)}(b)\,b_{r+1}(t-b) - \phi^{(r)}(a)\,b_{r+1}(t-a)\}$$
$$- \int_a^b \phi^{(l)}(x)\,b_l(t-x)\,dx.$$

LEMMA 3.2. *If* $\alpha = a/q + \beta$, $q \leqslant Q^{1-\epsilon}$ *and* $|\beta| \leqslant q^{-1} Q^{-3-\epsilon}$, *then*

$$U(\alpha) - U^*(\beta, a, q) = O(q^{\frac{7}{8}+\epsilon}).$$

Proof.

$$U(\alpha) = \sum_{v=1}^{q} \sum_{\substack{0 \leqslant r \leqslant \frac{1}{2}Q \\ r \equiv v \ (\mathrm{mod}\ q)}} e_q\big(af(r)\big)\, e\big(\beta f(r)\big)$$

$$= \sum_{v=1}^{q} e_q\big(af(v)\big)\, d_v,$$

where

$$d_v = \sum_{\substack{j \\ 0 \leqslant qj+v \leqslant \frac{1}{2}Q}} e\big(\beta f(qj+v)\big) = \sum_{\substack{j \\ 0 \leqslant j+(v/q) < \frac{1}{2}Q/q}} \phi\left(j + \frac{v}{q}\right)$$

and

$$\phi(x) = \phi_1(x)\,\phi_2(x),$$

$$\phi_1(x) = \phi_3\big((20\pi i\beta)^{\frac{1}{4}}\, qx\big), \quad \phi_3(x) = e(z^4),$$

$$\phi_2(x) = \phi_4\big((40\pi i\beta)^{\frac{1}{2}}\, qxN\big), \quad \phi_4(x) = e(z^2).$$

Take $l = \left[\dfrac{1}{\epsilon}\right] + 1$. Applying Lemma 3.1, we have

$$d_v = \int_0^{\frac{1}{2}Q/q} \phi(x)\,dx + \sum_{r=0}^{l-1} \left\{\phi^{(r)}\left(\frac{Q}{2q}\right) b_{r+1}\left(\frac{v}{q} - \frac{Q}{2q}\right) - \phi^{(r)}(0)\, b_{r+1}\left(\frac{v}{q}\right)\right\}$$
$$- \int_0^{\frac{1}{2}Q/q} \phi^{(l)}(x)\, b_l\left(\frac{v}{q} - x\right) dx.$$

Since
$$\int_0^{\frac{1}{2}Q/q} \phi(x)\,dx = \frac{1}{q}\int_0^{\frac{1}{2}Q} e^{2\pi i f(y)\beta}\,dy,$$
we have
$$U(a) - U^*(\beta, a, q) = \sum_{r=0}^{l-1}\left\{\phi^{(r)}\left(\frac{Q}{2q}\right) a_{r+1}\left(\frac{v}{q} - \frac{Q}{2q}\right) - \phi^{(r)}(0)\, a_{r+1}\left(\frac{v}{q}\right)\right\} - R,$$
where
$$a_r = \sum_{v=1}^{q} e_q\bigl(af(v)\bigr)\, b_{r+1}\left(\frac{v}{q} - t\right)$$
and
$$R = \sum_{v=1}^{q} e_q\bigl(af(x)\bigr)\int_0^{\frac{1}{2}Q/q} \phi^{(l)}(x)\, b_l\left(\frac{v}{q} - x\right) dx.$$

It is easy to see that
$$\phi_3^{(r)}(z) = P_r(z)\, e(z^4),$$
and
$$\phi_4^{(r)}(z) = Q_r(z)\, e(z^2),$$
where $P_r(z)$ and $Q_r(z)$ are two polynomials of degrees $3r$ and r respectively. Therefore
$$P_r(z) = O(1 + |z|^{3r})$$
and
$$Q_r(z) = O(1 + |z|^r).$$

Consequently
$$\phi_1^{(r)}(z) = (20\pi i\beta)^{\frac{1}{4}r} q^r \phi_1^{(r)}\bigl((10\pi i\beta)^{\frac{1}{4}} qx\bigr)$$
$$= O\bigl(|\beta|^{\frac{1}{4}r} q^r (1 + |\beta|^{\frac{3}{4}r} q^{3r} x^{3r})\bigr)$$
and
$$\phi_2^{(r)}(x) = O\bigl(|\beta|^{\frac{1}{2}r} q^r N^r (1 + |\beta|^{\frac{1}{2}r} q^r x^r N^r)\bigr)$$
for $x \geqslant 0$, $0 \leqslant r \leqslant l$. Since $|\beta| \leqslant q^{-1} Q^{-3-\epsilon}$, $q \leqslant Q^{1-\epsilon}$, $N < Q$ and $x \leqslant \frac{1}{2}Q/q$, we have then, for $0 \leqslant x \leqslant \frac{1}{2}Q/q$ and $0 \leqslant r \leqslant l$,
$$\phi_1^{(r)}(x) = O(Q^{-r\epsilon}), \quad \phi_2^{(r)}(x) = O(Q^{-r\epsilon}).$$

Therefore
$$\phi^{(r)}(x) = O\left(\sum_{t=1}^{r} |\phi_1^{(r-t)}(x)\,\phi_2^{(t)}(x)|\right) = O(Q^{-r\epsilon}),$$
for $0 \leqslant x \leqslant \frac{1}{2}Q/q$ and $0 \leqslant r \leqslant l$. In particular we have
$$\phi^{(l)}(x) = O(Q^{-1}) \quad (0 \leqslant x \leqslant \tfrac{1}{2}Q/q).$$

Since $b_r(x)$ is a bounded function of x, we have
$$R = O\left(q\int_0^{\frac{1}{2}Q/q} Q^{-1}\,dx\right) = O(1).$$

Now let us consider $a_r(t)$. Let

$$s_v = \sum_{m=1}^{v} e_q\big(af(m)\big).$$

Then

$$a_r(t) = s_1 b_{r+1}\left(\frac{1}{q}-t\right) + \sum_{v=2}^{q} (s_v - s_{v-1})\, b_{r+1}\left(\frac{v}{q}-t\right)$$

$$= \sum_{m=1}^{q-1} s_m \left(b_{r+1}\left(\frac{m}{q}-t\right) - b_{r+1}\left(\frac{m+1}{q}-t\right)\right) + s_q\, b_{r+1}(1-t).$$

By Lemma 2.6, $$s_v = O(q^{\frac{7}{8}+\epsilon}) \quad (v \leqslant q).$$

Hence

$$a_r(t) = O\left(q^{\frac{7}{8}+\epsilon} \left\{ \sum_{m=1}^{q-1} \left| b_{r+1}\left(\frac{m}{q}-t\right) - b_{r+1}\left(\frac{m+1}{q}-t\right)\right| + b_l(x)\right\}\right)$$

$$= O(q^{\frac{7}{8}+\epsilon}),$$

since $b_{r+1}(x)$ is of bounded variation. Thus

$$\phi^{(r)}\left(\frac{Q}{2q}\right) a_r\left(\frac{Q}{2q}\right) - \phi^{(r)}(0)\, a_r(0) = O(q^{\frac{7}{8}+\epsilon})$$

for $0 \leqslant r \leqslant l-1$. Hence the lemma follows.

LEMMA 3.3. *If* $q \leqslant P^{1-\epsilon}$, $|\beta| \leqslant q^{-1} P^{-4-\epsilon}$, *then*

$$V(\alpha) - V^*(\beta, a, q) = O(q^{\frac{15}{16}+\epsilon}).$$

The proof of this lemma is easier than that of Lemma 3.2.

4. *The main part of the proof.*

Divide the interval $0 \leqslant \alpha \leqslant 1$ in the usual way into Farey arcs belonging to all rational points a/q with $1 \leqslant q \leqslant Q^{3-\epsilon}$, $0 \leqslant a \leqslant q$, $(a, q) = 1$. Each arc has the form

$$\alpha = \frac{a}{q} + \beta, \quad -\vartheta_1 q^{-1} Q^{-3+\epsilon} \leqslant \beta \leqslant q^{-1} Q^{-3+\epsilon},$$

where $\frac{1}{2} \leqslant \vartheta_i \leqslant 1$.

Let $\mathfrak{M}_{a,q}$ denote the interval

$$|\beta| \leqslant q^{-1} Q^{-\frac{7}{2}}, \quad q \leqslant Q^{\frac{1}{2}}$$

and let E be the set of those points in $(0, 1)$ which do not belong to any $\mathfrak{M}_{a,q}$.

LEMMA 4.1.

$$\sum_m r_7'^2(m) = O\Big(R(0)\,P^{\epsilon}\Big) = O\Big(Q(0)\,P^{1+\epsilon}\Big).$$

Proof. $\sum\limits_m r_7'^2(m)$ is equal to the number of solutions of

$$x_0^5+x_1^5+x_2^5+x_3^5+x_4^5+x_5^5+x_5'^5 = y_0^5+y_1^5+y_2^5+y_3^5+y_4^5+y_5^5+y_5'^5,$$

where

$$2^{-v}\,P^{(\frac{4}{5})^v} < -x_v \leqslant 2^{1-v}\,P^{(\frac{4}{5})^v}, \quad -2^v\,P^{(\frac{4}{5})^v} < y_v \leqslant 2^{1-v}\,P^{(\frac{4}{5})^v}.$$

Then $x_i = y_i$ $(i = 0, 1, 2, 3, 4)$. For, if v is the first of the suffixes 0, 1, 2, 3, 4 for which $x_v \neq y_v$ $(x_v > y_v$, say), then

$$x_v^5 - y_v^5 \geqslant (y_v+1)^5 - y_v^5 \geqslant 5(2^{-v}\,P^{(\frac{4}{5})^v})^4$$

$$> \sum_{u=v+1}^{5} (2^{1-n}\,P^{(\frac{4}{5})^4})^5 + (2^{-4}\,P^{(\frac{4}{5})^5})^5 \geqslant y_{v+1}^5 + \dots + y_5'^5.$$

Also it is well known that the number of solutions of

$$x_5^5 + x_5'^5 = y_5^5 + y_5'^5$$

is $O(P^{2(\frac{4}{5})^5+\epsilon})$. Therefore

$$\sum_m r_7'^2(m) = O(P^{1+(\frac{4}{5})+(\frac{4}{5})^2+(\frac{4}{5})^3+(\frac{4}{5})^4+2(\frac{4}{5})^5+\epsilon}).$$

LEMMA 4.2.

$$\int_E |U^6(a)\,V_0(a)\,R^2(a)|\,da = O\Big(P^{5+\frac{1}{2}-\frac{1}{100}}\,Q^2(0)\Big).$$

Proof. If the point a belongs to E, then at least one of the three following conditions is satisfied:

(i) $Q^{1-\epsilon} < q \leqslant Q^{3-\epsilon}$,

(ii) $Q^{\frac{1}{2}} < q \leqslant Q^{1-\epsilon}$,

(iii) $q \leqslant Q^{\frac{1}{2}}$, $|\beta| \geqslant q^{-1}\,P^{-\frac{7}{2}}$.

Suppose that (i) holds. Then, by ordinary partial summation and Lemma 2.6, we have

$$U(a) = O(Q^{\frac{7}{8}+\epsilon}).$$

Next suppose that (ii) holds. Then, by Lemmas 2.7 and 3.2,

$$U(a) = O(q^{-\frac{1}{4}+\epsilon}\,Q) + O(q^{\frac{7}{8}+\epsilon}) = O(Q^{\frac{7}{8}+\epsilon}).$$

Finally suppose that (iii) holds. Then, by Lemmas 2.7 and 3.2,

$$U(\alpha)=O(q^{-\frac{1}{4}+\epsilon}|\beta|^{-\frac{1}{4}})+O(q^{\frac{7}{8}+\epsilon})=O(Q^{\frac{7}{8}+\epsilon}).$$

Next we introduce a second Farey section of order $P^{4-\epsilon}$. We assume that α satisfies one of the following three conditions with respect to the new intervals

$$\text{(i}'\text{)}\quad P^{1-\epsilon}<q\leqslant P^{4-\epsilon},$$

$$\text{(ii}'\text{)}\quad P^{\frac{5}{16}}<q\leqslant P^{1-\epsilon},$$

$$\text{(iii}'\text{)}\quad q\leqslant P^{\frac{5}{16}},\quad |\beta|\geqslant q^{-1}P^{-\frac{75}{16}}.$$

Then we can prove in the same way as above that

$$V_0(\alpha)=O(P^{\frac{15}{16}+\epsilon}).$$

Next we show that each point of E satisfies one of the three conditions (i′), (ii′), or (iii′).

Since $Q^{3-\epsilon}<P^{4-\epsilon}$, each old Farey arc contains a new Farey arc with the same centre. Hence a point α, belonging to the Farey arc with centre a/q in the old division, belongs either to the arc with the same centre in the new division, or to an arc with centre a'/q', where $q'>Q^{3-\epsilon}>P^{1-\epsilon}$. This means that every α belonging to (i) and (ii) satisfies also (i′) or (ii′), as $Q^{\frac{1}{2}}>P^{\frac{5}{16}}$.

If α satisfies the condition (iii) and does not belong to the same arc in the new Farey division, then it satisfies condition (i′). If it does belong to the same arc in the new division, it satisfies either

$$P^{\frac{5}{16}}<q\leqslant Q^{\frac{1}{2}},$$

which implies (ii′), or

$$q\leqslant P^{\frac{5}{16}},\quad |\beta|\geqslant q^{-1}P^{-\frac{7}{2}}.$$

Since $P^{-\frac{7}{2}}>P^{-\frac{75}{16}}$, this implies (iii′). Hence we have shown that

$$V_0(\alpha)=O(P^{\frac{15}{16}+\epsilon})$$

for α on E.

Therefore

$$\begin{aligned}\int_E|U^6(\alpha)\,V_0(\alpha)\,R^2(\alpha)|\,d\alpha&=O\Big(Q^{6\cdot\frac{7}{8}+\epsilon}\,P^{\frac{15}{16}+\epsilon}\int_0^1|R^2(\alpha)|\,d\alpha\Big)\\&=O\Big(P^{6\cdot\frac{7}{8}\cdot\frac{5}{4}+\frac{15}{16}+\epsilon}\sum_{n=1}^{P}r_7'^2(n)\Big)\\&=O\Big(P^{8+\frac{1}{2}+\epsilon}\,Q(0)\Big)=O\Big(P^{5+\frac{1}{2}-\frac{1}{100}}\,Q^2(0)\Big),\end{aligned}$$

by Lemma 4.1 and because

$$Q(0) \geqslant c_2 P^{(\frac{4}{5})+(\frac{4}{5})^2+(\frac{4}{5})^3+(\frac{4}{5})^4+2(\frac{4}{5})^5}$$

$$= c_2 P^{3.01696} \geqslant c_2 P^{3+\frac{1}{100}+\epsilon}.$$

LEMMA 4.3.

$$\sum_{q \leqslant Q^{\frac{1}{2}}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_0^{q^{-1}Q^{-\frac{7}{2}}} |U^6(\alpha) V_0^3(\alpha) - U^{*6}(\beta, a, q) V^{*3}(\beta, a, q)| \, d\beta$$

$$= O(P^{5+\frac{1}{2}-\frac{45}{128}+\epsilon}).$$

Proof. We divide the interval $(0, q^{-1}Q^{\frac{7}{2}})$ into two parts $I_1(0, q^{-1}Q^{-\frac{15}{4}})$ and $I_2(q^{-1}Q^{-\frac{15}{4}}, q^{-1}Q^{\frac{7}{2}})$. By Lemmas 3.2, 3.3, and 2.7 we have

$$U - U^* = O(q^{\frac{7}{8}+\epsilon}), \quad V_0 - V_0^* = O(\beta^{\frac{15}{16}+\epsilon}),$$

$$U^* = O(q^{-\frac{1}{4}+\epsilon} Q), \quad V_0^* = O\left(q^{-\frac{1}{5}+\epsilon} \min(P, |\beta|^{-\frac{1}{5}})\right)$$

and consequently

$$U = O(q^{-\frac{1}{4}+\epsilon} Q), \quad V_0 = O(q^{-\frac{1}{5}+\epsilon} P).$$

If α belongs to I_1, then

$$\begin{aligned}
&U^6 V_0^3 - U^{*6} V_0^{*3} \\
&\quad = (U^6 - U^{*6}) V_0^{*3} + U^6 (V_0^3 - V_0^{*3}) \\
&\quad = O\left(|U - U^*| |V_0^*|^3 (|U|^5 + |U^*|^5) + |U|^6 |V_0 - V_0^*| (|V_0|^2 + |V_0^*|^2)\right) \\
&\quad = O(q^{\frac{7}{8}+\epsilon} q^{-\frac{3}{5}+\epsilon} P^3 q^{-\frac{5}{4}+\epsilon} Q^5) + O(q^{-\frac{6}{4}+\epsilon} Q^6 q^{\frac{15}{16}+\epsilon} q^{-\frac{2}{5}+\epsilon} P^2) \\
&\quad = O(P^{9+\frac{1}{2}+\epsilon} q^{-\frac{77}{80}+\epsilon}).
\end{aligned}$$

Next, if α belongs to I_2, then $V_0 = O(P^{\frac{15}{16}+\epsilon})$ and

$$\begin{aligned}
&U^6 V_0^3 - U^{*6} V_0^{*3} \\
&\quad = V_0^3 (U^6 - U^{*6}) + (V_0^3 - V_0^{*3}) U^{*6} \\
&\quad = O\left(|V_0|^3 |U - U^*| (|U|^5 + |U^*|^5) + |V_0 - V_0^*| (|V_0|^2 + |V_0^*|^2) |U^*|^6\right) \\
&\quad = O\left(P^{3 \cdot \frac{15}{16}+\epsilon} q^{\frac{7}{8}+\epsilon} (q^{-\frac{1}{4}+\epsilon} Q)^5\right) + O\left(q^{\frac{15}{16}+\epsilon} (q^{-\frac{1}{5}+\epsilon} P)^2 q^{-\frac{6}{4}+\epsilon} Q^6\right) \\
&\quad = O(P^{9+\frac{1}{2}+\epsilon} q^{-\frac{77}{80}+\epsilon}).
\end{aligned}$$

Therefore the left-hand side of the equation in the enunciation is

$$O\left(\sum_{q \leqslant Q^{\frac{1}{2}}} q P^{9+\frac{1}{2}+\epsilon} q^{-\frac{77}{80}+\epsilon} q^{-1} Q^{-\frac{7}{2}}\right) = O(P^{5+\frac{1}{2}-\frac{45}{128}+\epsilon}).$$

LEMMA 4.4.

$$\sum_{q \leqslant Q^{\frac12}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{q^{-1}Q^{-\frac72}}^{\infty} |U^{*6}(\beta, a, q)\, V_0^{*3}(\beta, a, q)|\, d\beta = O(p^{5+\frac12-\frac1{16}+\epsilon}).$$

Proof. By Lemma 2.7, the left-hand side does not exceed

$$O\left(\sum_{q \leqslant Q^{\frac12}} q \int_{q^{-1}Q^{-\frac72}}^{\infty} q^{-\frac64+\epsilon} |\beta|^{-\frac64} q^{-\frac35+\epsilon} |\beta|^{-\frac35} d\beta\right)$$

$$= O\left(\sum_{q \leqslant Q^{\frac12}} q^{1-\frac32-\frac35+\epsilon} (qQ^{\frac72})^{\frac32+\frac35-1}\right) = O(Q^{\frac72(\frac32+\frac35-1)+\frac12+\epsilon})$$

$$= O(P^{5+\frac12-\frac1{16}+\epsilon}).$$

LEMMA 4.5.

$$\sum_{q \leqslant Q^{\frac12}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{0}^{q^{-1}Q^{-\frac72}} |U^{*6}(\beta, a, q)\, V_0^{*3}(\beta, a, q)| \left| Q^2(\alpha) - Q^2\left(\frac{a}{q}\right) \right| d\beta$$

$$= O\left(P^{5+\frac12-\frac38+\epsilon} Q^2(0) \right).$$

Proof.

$$\left| Q(\alpha) - Q\left(\frac{a}{q}\right) \right| \leqslant \sum_{n \leqslant 6P^4} r_6'(n)\, |e(n\alpha) - e_q(na)|$$

$$\leqslant 2\pi \sum_{n} r_6'(n)\, |\beta|\, n = O\left(P^4 |\beta|\, Q(0)\right).$$

Hence $$Q^2(\alpha) - Q^2\left(\frac{a}{q}\right) = O\left(P^4 |\beta|\, Q^2(0)\right).$$

By Lemma 2.7, the left-hand side of the equation in the enunciation is

$$O\left(\sum_{q \leqslant Q^{\frac12}} q \int_{0}^{q^{-1}Q^{-\frac72}} Q^2 (q^{-\frac14+\epsilon} \beta^{-\frac14})^4 P^3 P^4 |\beta|\, Q^2(0)\, d\beta\right)$$

$$= O\left(\sum_{q \leqslant Q^{\frac12}} q^{-1+\epsilon} Q^{2-\frac72} P^7 Q^2(0)\right)$$

$$= O\left(Q^{-\frac32+\epsilon} P^7 Q^2(0) \right) = O\left(P^{\frac{41}{8}+\epsilon} Q^2(0) \right)$$

$$= O\left(P^{5+\frac12-\frac38+\epsilon} Q^2(0) \right).$$

LEMMA 4.6.

$$r(n) = K\, P^{5+\frac12} \sum_{q \leqslant Q^{\frac12}} \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-9} Q^2\left(\frac{a}{q}\right) S_{a,q}^6\, T_{a,q}^3\, e_q(-an) + O\left(P^{5+\frac12-\frac1{100}} Q^2(0) \right).$$

Proof. We have

$$r(n)=\int_0^1 U^6(\alpha)\,V_0{}^3(\alpha)\,Q^2(\alpha)\,e(-n\alpha)\,d\alpha$$

$$=\sum_{q\leqslant Q^{\frac{1}{4}}}\ \sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{-\infty}^{\infty}U^{*6}(\beta,a,q)\,V_0^{*3}(\beta,a,q)\,Q^2\left(\frac{a}{q}\right)e(-n\alpha)\,d\alpha$$

$$+O\left(P^{5+\frac{1}{2}-\frac{1}{100}}\,Q^2(0)\right)$$

(by Lemmas 4.2, 4.3, 4.4, and 4.5)

$$=\sum_{q\leqslant Q^{\frac{1}{4}}}\ \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-9}\,S_{a,q}^6\,T_{a,q}^3\,Q^2\left(\frac{a}{q}\right)e_q(-an)\,J+O\left(P^{5+\frac{1}{2}-\frac{1}{100}}\,Q^2(0)\right),$$

where J is defined in Lemma 2.9. Hence the lemma is proved by Lemma 2.9.

LEMMA 4.7.

$$\sum_{q\leqslant P^{(\frac{4}{5})^6}}\left|\sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-9}\,Q^2\left(\frac{a}{q}\right)S_{a,q}^6\,T_{a,q}^3\,e_q(-an)\right|=O\left(P^{-\frac{1}{10}(\frac{4}{5})^6+\epsilon}\,Q^2(0)\right).$$

Proof. By Lemmas 2.4 and 2.5 the sum in question does not exceed

$$O\left(\sum_{q\leqslant P^{(\frac{4}{5})^6}} q\,Q^2(0)\,q^{-\frac{6}{4}-\frac{3}{5}+\epsilon}\right)=O\left(Q^2(0)\,P^{-\frac{1}{10}(\frac{4}{5})^6+\epsilon}\right).$$

LEMMA 4.8. *If* $q\leqslant P^{(\frac{4}{5})^6}$, *then*

$$Q^2\left(\frac{a}{q}\right)-2^{-20}\left(\frac{T_{a,q}}{q}\right)^{12}Q^2(0)=O\left(Q^2(0)\,P^{-(\frac{4}{5})^5+(\frac{4}{5})^6}\right).$$

Proof. It is evident that

$$V_i\left(\frac{a}{q}\right)-q^{-1}2^{-i}P^{(\frac{4}{5})^i}\,T_{a,q}=O(q)=O(P^{(\frac{4}{5})^6})\quad(i=1,2,\ldots,5).$$

Consequently

$$Q^2\left(\frac{a}{q}\right)-2^{-20}\left(\frac{T_{a,q}}{q}\right)^{12}Q^2(0)=O\left(Q^2(0)\,P^{-(\frac{4}{5})^5+(\frac{4}{5})^6}\right).$$

THEOREM 1.

$$r(n)=2^{-20}K P^{5+\frac{1}{2}}\,Q^2(0)\,\mathfrak{S}(N)+O\left(Q^2(0)\,P^{5+\frac{1}{2}-\frac{1}{100}}\right).$$

Proof. By Lemmas 4.6, 4.7, and 4.8, we have

$$r(n) = 2^{-20} K P^{5+\frac{1}{2}} Q^2(0) \sum_{q \leqslant P^{(\frac{4}{5})^6}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-21} S_{a,q}^{6} T_{a,q}^{15} e_q(-an)$$
$$+ O\left(Q^2(0) P^{5+\frac{1}{2}-\frac{1}{100}}\right).$$

This lemma follows, because

$$\sum_{q \leqslant P^{(\frac{4}{5})^6}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-21} S_{a,q}^{6} T_{a,q}^{15} e_q(-an) = O\left(\sum_{q \leqslant P^{(\frac{4}{5})^6}} q\, q^{-\frac{6}{4}-\frac{15}{5}+\epsilon}\right)$$
$$= O\left(P^{-\frac{5}{2}(\frac{4}{5})^6+\epsilon}\right) = O\left(P^{-\frac{1}{100}}\right).$$

LEMMA 4.9.

$$\mathfrak{S}(n) \geqslant c_3 > 0.$$

This lemma can be proved easily by the usual argument†.

THEOREM 2. *Every integer $n > c$ is representable in the form*

$$n = \sum_{i=1}^{15} x_i^5 + 10 \sum_{j=1}^{6} (y_j^4 + 2N^2 y_j^2) \quad (0 < y_j \leqslant \tfrac{1}{2} n^{\frac{1}{4}},\ x_i \geqslant 0)$$

for every N satisfying

$$\tfrac{1}{2} n^{\frac{1}{4}} < N < n^{\frac{1}{4}}.$$

This is a consequence of Theorem 1 and Lemma 4.9.

THEOREM 3. *Every sufficiently large integer is a sum of twenty-eight fifth powers.*

Proof. For every sufficiently large integer n, there is a prime p satisfying

$$p \equiv 2 \pmod{5} \quad (p + n),$$
$$(28)^{-\frac{1}{25}} n^{\frac{1}{25}} < p < (14)^{\frac{1}{25}} n^{\frac{1}{25}}.$$

The congruence $$n \equiv n_1^5 \pmod{p^5}$$

has a solution with $0 < n_1 < p^5$. Let

$$n = n_1^5 + p^5 n_2,$$

then $$14p^{25} < n < 28p^{25},$$

hence $$13p^{20} < n_2 < 28p^{20}.$$

---

† The singular series given by eleven fifth powers is positive, and in our case we have fifteen fifth powers and other additional terms at our disposal.

Let $$n_2 = 12p^{20} + n_3,$$
then $$p^{20} < n_3 < 16p^{20},$$
i.e. $$\tfrac{1}{2}n_3^{\frac{1}{4}} < p^5 < n_3^{\frac{1}{4}}.$$

Hence, by Theorem 2,

$$n_3 = \sum_{i=1}^{15} x_i^5 + 10 \sum_{j=1}^{6} (y_j^4 + 2p^{10}y_j^2)$$

$$= \sum_{i=1}^{15} x_i^5 + \sum_{j=1}^{6} p^{-5}\Big((p^5+y_j)^5 + (p^5-y_j)^5 - 2p^{25}\Big) \quad (0 < y_j \leqslant \tfrac{1}{2}n_3^{\frac{1}{4}} < p^5),$$

and so

$$n = n_1^5 + \sum_{i=1}^{15} (x_i p)^5 + \sum_{j=1}^{6} \Big((p^5+y_j)^5 + (p^5-y_j)^5\Big).$$

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SYSTEMS OF LOGIC BASED ON ORDINALS†

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Introduction ... 161
1. The conversion calculus. Gödel representations ... 162
2. Effective calculability. Abbreviation of treatment ... 166
3. Number-theoretic theorems ... 168
4. A type of problem which is not number-theoretic ... 172
5. Syntactical theorems as number-theoretic theorems ... 174
6. Logic formulae ... 174
7. Ordinals ... 178
8. Ordinal logics ... 189
9. Completeness questions ... 198
10. The continuum hypothesis. A digression ... 213
11. The purpose of ordinal logics ... 214
12. Gentzen type ordinal logics ... 217
Index of definitions ... 225
Bibliography ... 228

The well-known theorem of Gödel (Gödel [**1**], [**2**]) shows that every system of logic is in a certain sense incomplete, but at the same time it indicates means whereby from a system L of logic a more complete system L' may be obtained. By repeating the process we get a sequence $L, L_1 = L', L_2 = L_1', \ldots$ each more complete than the preceding. A logic L_ω may then be constructed in which the provable theorems are the totality of theorems provable with the help of the logics $L, L_1, L_2, \ldots$. We may then form $L_{2\omega}$ related to L_ω in the same way as L_ω was related to L. Proceeding in this way we can associate a system of logic with any constructive ordinal‡. It may be asked whether a sequence of logics of this kind is complete in the sense that to any problem A there corresponds an ordinal α such that A is solvable by means of the logic L_α. I propose to investigate this question in a rather more general case, and to give some other examples of ways in which systems of logic may be associated with constructive ordinals.

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† This paper represents work done while a Jane Eliza Procter Visiting Fellow at Princeton University, where the author received most valuable advice and assistance from Prof. Alonzo Church.

‡ The situation is not quite so simple as is suggested by this crude argument. See pages 189–193, 202, 203.

1. *The calculus of conversion. Gödel representations.*

It will be convenient to be able to use the "conversion calculus" of Church for the description of functions and for some other purposes. This will make greater clarity and simplicity of expression possible. I give a short account of this calculus. For detailed descriptions see Church [**3**], [**2**], Kleene [**1**], Church and Rosser [**1**].

The formulae of the calculus are formed from the symbols $\{$, $\}$, $($, $)$, $[$, $]$, λ, δ, and an infinite list of others called variables; we shall take for our infinite list $a, b, \ldots, z, x', x'', \ldots$. Certain finite sequences of such symbols are called *well-formed formulae* (abbreviated to W.F.F.); we define this class inductively, and define simultaneously the free and the bound variables of a W.F.F. Any variable is a W.F.F.; it is its only free variable, and it has no bound variables. δ is a W.F.F. and has no free or bound variables. If $\mathbf{M}$ and $\mathbf{N}$ are W.F.F. then $\{\mathbf{M}\}(\mathbf{N})$ is a W.F.F., whose free variables are the free variables of $\mathbf{M}$ together with the free variables of $\mathbf{N}$, and whose bound variables are the bound variables of $\mathbf{M}$ together with those of $\mathbf{N}$. If $\mathbf{M}$ is a W.F.F. and $\mathbf{V}$ is one of its free variables, then $\lambda\mathbf{V}[\mathbf{M}]$ is a W.F.F. whose free variables are those of $\mathbf{M}$ with the exception of $\mathbf{V}$, and whose bound variables are those of $\mathbf{M}$ together with $\mathbf{V}$. No sequence of symbols is a W.F.F. except in consequence of these three statements.

In metamathematical statements we use heavy type letters to stand for variable or undetermined formulae, as was done in the last paragraph, and in future such letters will stand for well-formed formulae unless otherwise stated. Small letters in heavy type will stand for formulae representing undetermined positive integers (see below).

A W.F.F. is said to be in normal form if it has no parts of the form $\{\lambda\mathbf{V}[\mathbf{M}]\}(\mathbf{N})$ and none of the form $\{\{\delta\}(\mathbf{M})\}(\mathbf{N})$, where $\mathbf{M}$ and $\mathbf{N}$ have no free variables.

We say that one W.F.F. is *immediately convertible* into another if it is obtained from it either by:

(i) Replacing one occurrence of a well-formed part $\lambda\mathbf{V}[\mathbf{M}]$ by $\lambda\mathbf{U}[\mathbf{N}]$, where the variable $\mathbf{U}$ does not occur in $\mathbf{M}$, and $\mathbf{N}$ is obtained from $\mathbf{M}$ by replacing the variable $\mathbf{V}$ by $\mathbf{U}$ throughout.

(ii) Replacing a well-formed part $\{\lambda \mathbf{V}[\mathbf{M}]\}(\mathbf{N})$ by the formula which is obtained from **M** by replacing **V** by **N** throughout, provided that the bound variables of **M** are distinct both from **V** and from the free variables of **N**.

(iii) The process inverse to (ii).

(iv) Replacing a well-formed part $\Big\{\{\delta\}(\mathbf{M})\Big\}(\mathbf{M})$ by

$$\lambda f\Big[\lambda x\Big[\{f\}\Big(\{f\}(x)\Big)\Big]\Big]$$

if **M** is in normal form and has no free variables.

(v) Replacing a well-formed part $\Big\{\{\delta\}(\mathbf{M})\Big\}(\mathbf{N})$ by

$$\lambda f\Big[\lambda x[\{f\}(x)]\Big]$$

if **M** and **N** are in normal form, are not transformable into one another by repeated application of (i), and have no free variables.

(vi) The process inverse to (iv).

(vii) The process inverse to (v).

These rules could have been expressed in such a way that in no case could there be any doubt about the admissibility or the result of the transformation [in particular this can be done in the case of process (v)].

A formula **A** is said to be *convertible* into another **B** (abbreviated to "**A** conv **B**") if there is a finite chain of immediate conversions leading from one formula to the other. It is easily seen that the relation of convertibility is an equivalence relation, *i.e.* it is symmetric, transitive, and reflexive.

Since the formulae are liable to be very lengthy, we need means for abbreviating them. If we wish to introduce a particular letter as an abbreviation for a particular lengthy formula we write the letter followed by "→" and then by the formula, thus

$$I \rightarrow \lambda x[x]$$

indicates that I is an abbreviation for $\lambda x[x]$. We also use the arrow in less sharply defined senses, but never so as to cause any real confusion. In these cases the meaning of the arrow may be rendered by the words "stands for".

If a formula $\mathbf{F}$ is, or is represented by, a single symbol we abbreviate $\{\mathbf{F}\}(\mathbf{X})$ to $\mathbf{F}(\mathbf{X})$. A formula $\big\{\{\mathbf{F}\}(\mathbf{X})\big\}(\mathbf{Y})$ may be abbreviated to

$$\{\mathbf{F}\}(\mathbf{X}, \mathbf{Y}),$$

or to $\mathbf{F}(\mathbf{X}, \mathbf{Y})$ if $\mathbf{F}$ is, or is represented by, a single symbol. Similarly for $\Big\{\big\{\{\mathbf{F}\}(\mathbf{X})\big\}(\mathbf{Y})\Big\}(\mathbf{Z})$, etc. A formula $\lambda\mathbf{V}_1\Big[\lambda\mathbf{V}_2\ldots\big[\lambda\mathbf{V}_r[\mathbf{M}]\big]\ldots\Big]$ may be abbreviated to $\lambda\mathbf{V}_1\mathbf{V}_2\ldots\mathbf{V}_r.\mathbf{M}$.

We have not as yet assigned any meanings to our formulae, and we do not intend to do so in general. An exception may be made for the case of the positive integers, which are very conveniently represented by the formulae $\lambda fx.f(x)$, $\lambda fx.f\big(f(x)\big)$, ... In fact we introduce the abbreviations

$$1 \rightarrow \lambda fx.f(x)$$

$$2 \rightarrow \lambda fx.f\big(f(x)\big)$$

$$3 \rightarrow \lambda fx.f\Big(f\big(f(x)\big)\Big), \quad \text{etc.,}$$

and we also say, for example, that $\lambda fx.f\big(f(x)\big)$, or in full

$$\lambda f\Big[\lambda x\Big[\{f\}\big(\{f\}(x)\big)\Big]\Big],$$

represents the positive integer 2. Later we shall allow certain formulae to represent ordinals, but otherwise we leave them without explicit meaning; an implicit meaning may be suggested by the abbreviations used. In any case where any meaning is assigned to formulae it is desirable that the meaning should be invariant under conversion. Our definitions of the positive integers do not violate this requirement, since it may be proved that no two formulae representing different positive integers are convertible the one into the other.

In connection with the positive integers we introduce the abbreviation

$$S \rightarrow \lambda ufx.f\big(u(f, x)\big).$$

This formula has the property that, if $\mathbf{n}$ represents a positive integer, $S(\mathbf{n})$ is convertible to a formula representing its successor†.

Formulae representing undetermined positive integers will be represented by small letters in heavy type, and we adopt once for all the convention that, if a small letter, n say, stands for a positive integer, then the same letter in heavy type, $\mathbf{n}$, stands for the formula representing the positive integer. When no confusion arises from so doing, we shall not trouble to distinguish between an integer and the formula which represents it.

---

† This follows from (A) below.

Suppose that $f(n)$ is a function of positive integers taking positive integers as values, and that there is a W.F.F. $\mathbf{F}$ not containing δ such that, for each positive integer n, $\mathbf{F}(\mathbf{n})$ is convertible to the formula representing $f(n)$. We shall then say that $f(n)$ is *λ-definable* or *formally definable*, and that $\mathbf{F}$ *formally defines* $f(n)$. Similar conventions are used for functions of more than one variable. The sum function is, for instance, formally defined by $\lambda abfx\,.\,a\big(f, b(f, x)\big)$; in fact, for any positive integers m, n, p for which $m+n=p$, we have

$$\big\{\lambda abfx\,.\,a\big(f, b(f, x)\big)\big\}(\mathbf{m}, \mathbf{n}) \text{ conv } \mathbf{p}.$$

In order to emphasize this relation we introduce the abbreviation

$$\mathbf{X}+\mathbf{Y} \rightarrow \big\{\lambda abfx\,.\,a\big(f, b(f, x)\big)\big\}(\mathbf{X}, \mathbf{Y})$$

and we shall use similar notations for sums of three or more terms, products, etc.

For any W.F.F. $\mathbf{G}$ we shall say that $\mathbf{G}$ *enumerates* the sequence $\mathbf{G}(1)$, $\mathbf{G}(2)$, ... and any other sequence whose terms are convertible to those of this sequence.

When a formula is convertible to another which is in normal form, the second is described as a *normal form* of the first, which is then said to *have a normal form*. I quote here some of the more important theorems concerning normal forms.

(A) *If a formula has two normal forms they are convertible into one another by the use of* (i) *alone.* (Church and Rosser [**1**], 479, 481.)

(B) *If a formula has a normal form then every well-formed part of it has a normal form.* (Church and Rosser [**1**], 480–481.)

(C) *There is (demonstrably) no process whereby it can be said of a formula whether it has a normal form.* (Church [**3**], 360, Theorem XVIII.)

We often need to be able to describe formulae by means of positive integers. The method used here is due to Gödel (Gödel [**1**]). To each single symbol s of the calculus we assign an integer $r[s]$ as in the table below.

s	{, (, or [	},), or]	λ	δ	a	...	z	x'	x''	x'''	...
$r[s]$	1	2	3	4	5	...	30	31	32	33	...

If $s_1, s_2, \dots, s_k$ is a sequence of symbols, then $2^{r[s_1]}3^{r[s_2]}\dots p_k^{r[s_k]}$ (where p_k is the k-th prime number) is called the *Gödel representation* (G.R.) of that sequence of symbols. No two W.F.F. have the same G.R.

Two theorems on G.R. of W.F.F. are quoted here.

(D) *There is a* W.F.F. "form" *such that if a is the* G.R. *of a* W.F.F. **A** *without free variables, then* form (a) conv **A**. (This follows from a similar theorem to be found in Church [**3**], 53–66. Metads are used there in place of G.R.)

(E) *There is a* W.F.F. Gr *such that, if* **A** *is a* W.F.F. *with a normal form without free variables, then* Gr(**A**) conv **a**, *where a is the* G.R. *of a normal form of* **A**. [Church [**3**], 53, 66, as (D).]

2. *Effective calculability. Abbreviation of treatment.*

A function is said to be "effectively calculable" if its values can be found by some purely mechanical process. Although it is fairly easy to get an intuitive grasp of this idea, it is nevertheless desirable to have some more definite, mathematically expressible definition. Such a definition was first given by Gödel at Princeton in 1934 (Gödel [**2**], 26), following in part an unpublished suggestion of Herbrand, and has since been developed by Kleene [2]). These functions were described as "general recursive" by Gödel. We shall not be much concerned here with this particular definition. Another definition of effective calculability has been given by Church (Church [**3**], 356-358), who identifies it with λ-definability. The author has recently suggested a definition corresponding more closely to the intuitive idea (Turing [**1**], see also Post [**1**]). It was stated above that "a function is effectively calculable if its values can be found by some purely mechanical process". We may take this statement literally, understanding by a purely mechanical process one which could be carried out by a machine. It is possible to give a mathematical description, in a certain normal form, of the structures of these machines. The development of these ideas leads to the author's definition of a computable function, and to an identification of computability[†] with effective calculability. It is not difficult, though somewhat laborious, to prove that these three definitions are equivalent (Kleene [**3**], Turing [**2**]).

[†] We shall use the expression "computable function" to mean a function calculable by a machine, and we let "effectively calculable" refer to the intuitive idea without particular identification with any one of these definitions. We do not restrict the values taken by a computable function to be natural numbers; we may for instance have computable propositional functions.

In the present paper we shall make considerable use of Church's identification of effective calculability with λ-definability, or, what comes to the same thing, of the identification with computability and one of the equivalence theorems. In most cases where we have to deal with an effectively calculable function, we shall introduce the corresponding W.F.F. with some such phrase as "the function f is effectively calculable, let F be a formula λ defining it", or "let F be a formula such that $F(\mathbf{n})$ is convertible to . . . whenever $\mathbf{n}$ represents a positive integer". In such cases there is no difficulty in seeing how a machine could in principle be designed to calculate the values of the function concerned; and, assuming this done, the equivalence theorem can be applied. A statement of what the formula F actually is may be omitted. We may immediately introduce on this basis a W.F.F. ϖ with the property that

$$\varpi(\mathbf{m}, \mathbf{n}) \text{ conv } \mathbf{r},$$

if r is the greatest positive integer, if any, for which m^r divides n and r is 1 if there is none. We also introduce Dt with the properties

$$\begin{aligned} \mathrm{Dt}(\mathbf{n}, \mathbf{n}) &\text{ conv } 3, \\ \mathrm{Dt}(\mathbf{n}+\mathbf{m}, \mathbf{n}) &\text{ conv } 2, \\ \mathrm{Dt}(\mathbf{n}, \mathbf{n}+\mathbf{m}) &\text{ conv } 1. \end{aligned}$$

There is another point to be made clear in connection with the point of view that we are adopting. It is intended that all proofs that are given should be regarded no more critically than proofs in classical analysis. The subject matter, roughly speaking, is constructive systems of logic, but since the purpose is directed towards choosing a particular constructive system of logic for practical use, an attempt at this stage to put our theorems into constructive form would be putting the cart before the horse.

Those computable functions which take only the values 0 and 1 are of particular importance, since they determine and are determined by computable properties, as may be seen by replacing "0" and "1" by "true" and "false". But, besides this type of property, we may have to consider a different type, which is, roughly speaking, less constructive than the computable properties, but more so than the general predicates of classical mathematics. Suppose that we have a computable function of the natural numbers taking natural numbers as values, then corresponding to this function there is the property of being a value of the function. Such a property we shall describe as "axiomatic"; the reason for using this term is that it is possible to define such a property by giving a set of axioms, the property to hold for a given argument if and only if it is possible to deduce that it holds from the axioms.

Axiomatic properties may also be characterized in this way. A property ψ of positive integers is axiomatic if and only if there is a computable property ϕ of two positive integers, such that $\psi(x)$ is true if and only if there is a positive integer y such that $\phi(x, y)$ is true. Or again ψ is axiomatic if and only if there is a W.F.F. **F** such that $\psi(n)$ is true if and only if **F**(**n**) conv 2.

3. *Number-theoretic theorems.*

By a *number-theoretic theorem*[†] we shall mean a theorem of the form "$\theta(x)$ vanishes for infinitely many natural numbers x", where $\theta(x)$ is a primitive recursive[‡] function.

We shall say that a problem is number-theoretic if it has been shown that any solution of the problem may be put in the form of a proof of one or more number-theoretic theorems. More accurately we may say that a class of problems is number-theoretic if the solution of any one of them can be transformed (by a uniform process) into the form of proofs of number-theoretic theorems.

I shall now draw a few consequences from the definition of "number theoretic theorems", and in section 5 I shall try to justify confining our consideration to this type of problem.

[†] I believe that there is no generally accepted meaning for this term, but it should be noticed that we are using it in a rather restricted sense. The most generally accepted meaning is probably this: suppose that we take an arbitrary formula of the functional calculus of the first order and replace the function variables by primitive recursive relations. The resulting formula represents a typical number-theoretic theorem in this (more general) sense.

[‡] Primitive recursive functions of natural numbers are defined inductively as follows. Suppose that $f(x_1, \ldots, x_{n-1})$, $g(x_1, \ldots, x_n)$, $h(x_1, \ldots, x_{n+1})$ are primitive recursive, then $\phi(x_1, \ldots, x_n)$ is primitive recursive if it is defined by one of the sets of equations (a) to (e).

$$(a)\quad \phi(x_1, \ldots, x_n) = h\Big(x_1, \ldots, x_{m-1}, g(x_1, \ldots, x_n), x_{m+1}, \ldots, x_{n-1}, x_m\Big) \quad (1 \leqslant m \leqslant n);$$

$$(b)\quad \phi(x_1, \ldots, x_n) = f(x_2, \ldots, x_n);$$

$$(c)\quad \phi(x_1) = a, \text{ where } n = 1 \text{ and } a \text{ is some particular natural number;}$$

$$(d)\quad \phi(x_1) = x_1 + 1 \quad (n = 1);$$

$$(e)\quad \phi(x_1, \ldots, x_{n-1}, 0) = f(x_1, \ldots, x_{n-1});$$

$$\phi(x_1, \ldots, x_{n-1}, x_n + 1) = h\Big(x_1, \ldots, x_n, \phi(x_1, \ldots, x_n)\Big).$$

The class of primitive recursive functions is more restricted than the class of computable functions, but it has the advantage that there is a process whereby it can be said of a set of equations whether it defines a primitive recursive function in the manner described above.

If $\phi(x_1, \ldots, x_n)$ is primitive recursive, then $\phi(x_1, \ldots, x_n) = 0$ is described as a primitive recursive relation between $x_1, \ldots, x_n$.

An alternative form for number-theoretic theorems is "for each natural number x there exists a natural number y such that $\phi(x, y)$ vanishes", where $\phi(x, y)$ is primitive recursive. In other words, there is a rule whereby, given the function $\theta(x)$, we can find a function $\phi(x, y)$, or given $\phi(x, y)$, we can find a function $\theta(x)$, such that "$\theta(x)$ vanishes infinitely often" is a necessary and sufficient condition for "for each x there is a y such that $\phi(x, y) = 0$". In fact, given $\theta(x)$, we define

$$\phi(x, y) = \theta(x) + \alpha(x, y),$$

where $\alpha(x, y)$ is the (primitive recursive) function with the properties

$$\begin{aligned} \alpha(x, y) &= 1 \quad (y \leqslant x), \\ &= 0 \quad (y > x). \end{aligned}$$

If on the other hand we are given $\phi(x, y)$ we define $\theta(x)$ by the equations

$$\theta_1(0) = 3,$$

$$\theta_1(x+1) = 2^{(1+\varpi_2(\theta_1(x)))\,\sigma(\phi(\varpi_3(\theta_1(x))-1,\ \varpi_2(\theta_1(x))))}\, 3^{\varpi_3(\theta_1(x))+1-\sigma(\phi(\varpi_3(\theta_1(x))-1,\ \varpi_2(\theta_1(x))))},$$

$$\theta(x) = \phi\Big(\varpi_3\big(\theta_1(x)\big) - 1,\ \varpi_2\big(\theta_1(x)\big)\Big),$$

where $\varpi_r(x)$ is defined so as to mean "the largest s for which r^s divides x". The function $\sigma(x)$ is defined by the equations $\sigma(0) = 0$, $\sigma(x+1) = 1$. It is easily verified that the functions so defined have the desired properties.

We shall now show that questions about the truth of the statements of the form "does $f(x)$ vanish identically", where $f(x)$ is a computable function, can be reduced to questions about the truth of number-theoretic theorems. It is understood that in each case the rule for the calculation of $f(x)$ is given and that we are satisfied that this rule is valid, *i.e.* that the machine which should calculate $f(x)$ is circle free (Turing [**1**], 233). The function $f(x)$, being computable, is general recursive in the Herbrand-Gödel sense, and therefore, by a general theorem due to Kleene†, is expressible in the form

$$\psi\Big(\epsilon y[\phi(x, y) = 0]\Big), \tag{3.2}$$

where $\epsilon y[\mathfrak{A}(y)]$ means "the least y for which $\mathfrak{A}(y)$ is true" and $\psi(y)$ and $\phi(x, y)$ are primitive recursive functions. Without loss of generality, we may suppose that the functions ϕ, ψ take only the values 0, 1. Then, if we define $\rho(x)$ by the equations (3.1) and

---

† Kleene [3], 727. This result is really superfluous for our purpose, since the proof that every computable function is general recursive proceeds by showing that these functions are of the form (3 . 2). (Turing [2], 161).

$$\rho(0) = \psi(0)\big(1-\theta(0)\big),$$

$$\rho(x+1) = 1-\big(1-\rho(x)\big)\,\sigma\Big[1+\theta(x)-\psi\Big\{\varpi_2\big(\theta_1(x)\big)\Big\}\Big]$$

it will be seen that $f(x)$ vanishes identically if and only if $\rho(x)$ vanishes for infinitely many values of x.

The converse of this result is not quite true. We cannot say that the question about the truth of any number-theoretic theorem is reducible to a question about whether a corresponding computable function vanishes identically; we should have rather to say that it is reducible to the problem of whether a certain machine is circle free and calculates an identically vanishing function. But more is true: every number-theoretic theorem is equivalent to the statement that a corresponding machine is circle free. The behaviour of the machine may be described roughly as follows: the machine is one for the calculation of the primitive recursive function $\theta(x)$ of the number-theoretic problem, except that the results of the calculation are first arranged in a form in which the figures 0 and 1 do not occur, and the machine is then modified so that, whenever it has been found that the function vanishes for some value of the argument, then 0 is printed. The machine is circle free if and only if an infinity of these figures are printed, *i.e.* if and only if $\theta(x)$ vanishes for infinitely many values of the argument. That, on the other hand, questions of circle freedom may be reduced to questions of the truth of number-theoretic theorems follows from the fact that $\theta(x)$ is primitive recursive when it is defined to have the value 0 if a certain machine $\mathcal{M}$ prints 0 or 1 in its $(x+1)$-th complete configuration, and to have the value 1 otherwise.

The conversion calculus provides another normal form for the number-theoretic theorems, and the one which we shall find the most convenient to use. Every number-theoretic theorem is equivalent to a statement of the form "$\mathbf{A}(\mathbf{n})$ is convertible to 2 for every W.F.F. $\mathbf{n}$ representing a positive integer", $\mathbf{A}$ being a W.F.F. determined by the theorem; the property of $\mathbf{A}$ here asserted will be described briefly as "$\mathbf{A}$ is dual". Conversely such statements are reducible to number theoretic theorems. The first half of this assertion follows from our results for computable functions, or directly in this way. Since $\theta(x-1)+2$ is primitive recursive, it is formally definable, say, by means of a formula $\mathbf{G}$. Now there is (Kleene [1], 232) a W.F.F. $\mathcal{P}$ with the property that, if $\mathbf{T}(\mathbf{r})$ is convertible to a formula representing a positive integer for each positive integer r, then $\mathcal{P}(\mathbf{T}, \mathbf{n})$ is convertible to s, where s is the n-th positive integer t (if there is one) for which

$\mathbf{T}(\mathbf{t})$ conv 2; if $\mathbf{T}(\mathbf{t})$ conv 2 for less than n values of t then $\mathscr{P}(\mathbf{T}, \mathbf{n})$ has no normal form. The formula $\mathbf{G}\big(\mathscr{P}(\mathbf{G}, \mathbf{n})\big)$ is therefore convertible to 2 if and only if $\theta(x)$ vanishes for at least n values of x, and is convertible to 2 for every positive integer x if and only if $\theta(x)$ vanishes infinitely often. To prove the second half of the assertion, we take Gödel representations for the formulae of the conversion calculus. Let $c(x)$ be 0 if x is the G.R. of 2 (*i.e.* if x is $2^3 . 3^{10} . 5 . 7^3 . 11^{28} . 13 . 17 . 19^{10} . 23^2 . 29 . 31 . 37^{10} . 41^2 . 43 . 47^{28} . 53^2 . 59^2 . 61^2 . 67^2$) and let $c(x)$ be 1 otherwise. Take an enumeration of the G.R. of the formulae into which $\mathbf{A}(\mathbf{m})$ is convertible: let $a(m; n)$ be the n-th number in the enumeration. We can arrange the enumeration so that $a(m, n)$ is primitive recursive. Now the statement that $\mathbf{A}(\mathbf{m})$ is convertible to 2 for every positive integer m is equivalent to the statement that, corresponding to each positive integer m, there is a positive integer n such that $c\big(a(m, n)\big) = 0$; and this is number-theoretic.

It is easy to show that a number of unsolved problems, such as the problem of the truth of Fermat's last theorem, are number-theoretic. There are, however, also problems of analysis which are number-theoretic. The Riemann hypothesis gives us an example of this. We denote by $\zeta(s)$ the function defined for $\mathfrak{R}s = \sigma > 1$ by the series $\sum_{n=1}^{\infty} n^{-s}$ and over the rest of the complex plane with the exception of the point $s = 1$ by analytic continuation. The Riemann hypothesis asserts that this function does not vanish in the domain $\sigma > \frac{1}{2}$. It is easily shown that this is equivalent to saying that it does not vanish for $2 > \sigma > \frac{1}{2}$, $\mathfrak{I}s = t > 2$, *i.e.* that it does not vanish inside any rectangle $2 > \sigma > \frac{1}{2} + 1/T$, $T > t > 2$, where T is an integer greater than 2. Now the function satisfies the inequalities

$$\left.\begin{array}{r} \left| \zeta(s) - \sum_{1}^{N} n^{-s} - \dfrac{N^{1-s}}{s-1} \right| < 2t(N-2)^{-\frac{1}{2}}, \quad 2 < \sigma < \frac{1}{2}, \quad t \geqslant 2, \\ |\zeta(s) - \zeta(s')| < 60t\,|s - s'|, \quad 2 < \sigma' < \frac{1}{2}, \quad t' \geqslant 2, \end{array}\right\}$$

and we can define a primitive recursive function $\xi(l, l', m, m', N, M)$ such that

$$\left| \xi(l, l', m, m', N, M) - M \left| \sum_{1}^{N} n^{-s} + \frac{N^{1-s}}{s-1} \right| \right| < 2, \quad \left(s = \frac{l}{l'} + i\frac{m}{m'} \right),$$

and therefore, if we put

$$\xi(l, M, m, M, M^2+2, M) = X(l, m, M),$$

we have

$$\left|\zeta\left(\frac{l+\vartheta}{M}+i\frac{m+\vartheta'}{M}\right)\right| \geqslant \frac{X(l,\, m,\, M)-122T}{M},$$

provided that

$$\tfrac{1}{2}+\frac{1}{T} \leqslant \frac{l-1}{M} < \frac{l+1}{M} < 2-\frac{1}{M}, \quad 2<\frac{m-1}{M}<\frac{m+1}{M}<T$$

$$(-1<\vartheta<1, \quad -1<\vartheta'<1).$$

If we define $B(M,\, T)$ to be the smallest value of $X(l,\, m,\, M)$ for which

$$\tfrac{1}{2}+\frac{1}{T}+\frac{1}{M} \leqslant \frac{l}{M} < 2-\frac{1}{M}, \quad 2+\frac{1}{M}<\frac{m}{M}<T-\frac{1}{M},$$

then the Riemann hypothesis is true if for each T there is an M satisfying

$$B(M,\, T) > 122T.$$

If on the other hand there is a T such that, for all M, $B(M,\, T) \leqslant 122T$, the Riemann hypothesis is false; for let l_M, m_M be such that

$$X(l_M,\, m_M,\, M) \leqslant 122T,$$

then

$$\left|\zeta\left(\frac{l_M+im_M}{M}\right)\right| \leqslant \frac{244T}{M}.$$

Now if a is a condensation point of the sequence $(l_M+im_M)/M$ then since $\zeta(s)$ is continuous except at $s=1$ we must have $\zeta(a)=0$ implying the falsity of the Riemann hypothesis. Thus we have reduced the problem to the question whether for each T there is an M for which

$$B(M,\, T) > 122T.$$

$B(M,\, T)$ is primitive recursive, and the problem is therefore number-theoretic.

4. *A type of problem which is not number-theoretic*[†].

Let us suppose that we are supplied with some unspecified means of solving number-theoretic problems; a kind of oracle as it were. We shall not go any further into the nature of this oracle apart from saying that it cannot be a machine. With the help of the oracle we could form a new kind of machine (call them *o*-machines), having as one of its fundamental processes that of solving a given number-theoretic problem. More definitely these machines are to behave in this way. The moves of the machine are determined as usual by a table except in the case of moves from a certain internal configuration $\mathfrak{o}$. If the machine is in the internal configuration $\mathfrak{o}$ and if the sequence of symbols marked with l is then the well-formed[†] formula **A**, then the machine goes into the internal configuration $\mathfrak{p}$ or $\mathfrak{t}$ according as it is or is not true that **A** is dual. The decision as to which is the case is referred to the oracle.

[†] Compare Rosser [1].

These machines may be described by tables of the same kind as those used for the description of *a*-machines, there being no entries, however, for the internal configuration $\mathfrak{o}$. We obtain description numbers from these tables in the same way as before. If we make the convention that, in assigning numbers to internal configurations, $\mathfrak{o}$, $\mathfrak{p}$, $\mathfrak{t}$ are always to be q_2, q_3, q_4, then the description numbers determine the behaviour of the machines uniquely.

Given any one of these machines we may ask ourselves the question whether or not it prints an infinity of figures 0 or 1; I assert that this class of problem is not number-theoretic. In view of the definition of "number theoretic problem" this means that it is not possible to construct an *o*-machine which, when supplied[‡] with the description of any other *o*-machine, will determine whether that machine is *o*-circle free. The argument may be taken over directly from Turing [**1**], § 8. We say that a number is *o*-satisfactory if it is the description number of an *o*-circle free machine. Then, if there is an *o*-machine which will determine of any integer whether it is *o*-satisfactory, there is also an *o*-machine to calculate the values of the function $1-\phi_n(n)$. Let $r(n)$ be the n-th *o*-satisfactory number and let $\phi_n(m)$ be the m-th figure printed by the *o*-machine whose description number is $r(n)$. This *o*-machine is circle free and there is therefore an *o*-satisfactory number K such that $\phi_K(n) = 1-\phi_n(n)$ for all n. Putting $n = K$ yields a contradiction. This completes the proof that problems of circle freedom of *o*-machines are not number-theoretic.

Propositions of the form that an *o*-machine is *o*-circle free can always be put in the form of propositions obtained from formulae of the functional calculus of the first order by replacing *some* of the functional variables by primitive recursive relations. Compare foot-note † on page 168.

[†] Without real loss of generality we may suppose that **A** is always well formed.

[‡] Compare Turing [**1**], §6, 7.

5. *Syntactical theorems as number-theoretic theorems.*

I now mention a property of number-theoretic theorems which suggests that there is reason for regarding them as of particular importance.

Suppose that we have some axiomatic system of a purely formal nature. We do not concern ourselves at all in interpretations for the formulae of this system; they are to be regarded as of interest for themselves. An example of what is in mind is afforded by the conversion calculus (§ 1). Every sequence of symbols " **A** conv **B** ", where **A** and **B** are well formed formulae, is a formula of the axiomatic system and is provable if the W.F.F. **A** is convertible to **B**. The rules of conversion give us the rules of procedure in this axiomatic system.

Now consider a new rule of procedure which is reputed to yield only formulae provable in the original sense. We may ask ourselves whether such a rule is valid. The statement that such a rule is valid would be number-theoretic. To prove this, let us take Gödel representations for the formulae, and an enumeration of the provable formulae; let $\phi(r)$ be the G.R. of the r-th formula in the enumeration. We may suppose $\phi(r)$ to be primitive recursive if we are prepared to allow repetitions in the enumeration. Let $\psi(r)$ be the G.R. of the r-th formula obtained by the new rule, then the statement that this new rule is valid is equivalent to the assertion of

$$(r)(\exists s)[\psi(r)=\phi(s)]$$

(the domain of individuals being the natural numbers). It has been shown in § 3 that such statements are number-theoretic.

It might plausibly be argued that all those theorems of mathematics which have any significance when taken alone are in effect syntactical theorems of this kind, stating the validity of certain "derived rules" of procedure. Without going so far as this, I should assert that theorems of this kind have an importance which makes it worth while to give them special consideration.

6. *Logic formulae.*

We shall call a formula **L** a *logic formula* (or, if it is clear that we are speaking of a W.F.F., simply a *logic*) if it has the property that, if **A** is a formula such that **L**(**A**) conv 2, then **A** is dual.

A logic formula gives us a means of satisfying ourselves of the truth of number-theoretic theorems. For to each number-theoretic proposition there corresponds a W.F.F. **A** which is dual if and only if the proposition is true. Now, if **L** is a logic and **L**(**A**) conv 2, then **A** is dual and we know that the corresponding number-theoretic proposition is true. It does not follow that, if **L** is a logic, we can use **L** to satisfy ourselves of the truth of *any* number-theoretic theorem.

If **L** is a logic, the set of formulae **A** for which **L**(**A**) conv 2 will be called the *extent* of **L**.

It may be proved by the use of (D), (E), p. 166, that there is a formula X such that, if **M** has a normal form, has no free variables and is not convertible to 2, then $X(\mathbf{M})$ conv 1, but, if **M** conv 2, then $X(\mathbf{M})$ conv 2. If **L** is a logic, then $\lambda x \,.\, X\big(\mathbf{L}(x)\big)$ is also a logic whose extent is the same as that of **L**, and which has the property that, if **A** has no free variables, then

$$\big\{\lambda x \,.\, X\big(\mathbf{L}(x)\big)\big\}(\mathbf{A})$$

either is always convertible to 1 or to 2 or else has no normal form. A logic with this property will be said to be *standardized*.

We shall say that a logic **L**′ is *at least as complete as* a logic **L** if the extent of **L** is a subset of the extent of **L**′. The logic **L**′ is *more complete than* **L** if the extent of **L** is a proper subset of the extent of **L**′.

Suppose that we have an effective set of rules by which we can prove formulae to be dual; *i.e.* we have a system of symbolic logic in which the propositions proved are of the form that certain formulae are dual. Then we can find a logic formula whose extent consists of just those formulae which can be proved to be dual by the rules; that is to say, there is a rule for obtaining the logic formula from the system of symbolic logic. In fact the system of symbolic logic enables us to obtain[†] a computable function of positive integers whose values run through the Gödel representations of the formulae provable by means of the given rules. By the theorem of equivalence of computable and λ-definable functions, there is a formula **J** such that **J**(1), **J**(2), ... are the G.R. of these formulae. Now let

$$W \rightarrow \lambda jv \,.\, \mathcal{P}\Big(\lambda u \,.\, \delta\big(j(u), v\big), 1, I, 2\Big).$$

Then I assert that $W(\mathbf{J})$ is a logic with the required properties. The properties of $\mathcal{P}$ imply that $\mathcal{P}(\mathbf{C}, 1)$ is convertible to the least positive integer **n** for which **C**(**n**) conv 2, and has no normal form if there is no such integer. Consequently $\mathcal{P}(\mathbf{C}, 1, I, 2)$ is convertible to 2 if **C**(**n**) conv 2 for some positive integer n, and it has no normal form otherwise. That is to say that $W(\mathbf{J}, \mathbf{A})$ conv 2 if and only if $\delta\big(\mathbf{J}(\mathbf{n}), \mathbf{A}\big)$ conv 2, some n, *i.e.* if **J**(**n**) conv **A** some n.

[†] Compare Turing [**1**], 252, second footnote, [**2**], 156.

There is conversely a formula W' such that, if $\mathbf{L}$ is a logic, then $W'(\mathbf{L})$ enumerates the extent of $\mathbf{L}$. For there is a formula Q such that $Q(\mathbf{L}, \mathbf{A}, \mathbf{n})$ conv 2 if and only if $\mathbf{L}(\mathbf{A})$ is convertible to 2 in less than n steps. We then put

$$W' \rightarrow \lambda ln \,.\, \mathrm{form}\Big(\varpi\Big(2,\ \wp\big(\lambda x \,.\, Q\big(l,\ \mathrm{form}\,\big(\varpi(2, x)\big),\ \varpi(3, x)\big),\ n\big)\Big)\Big).$$

Of course, $W'\big(W(\mathbf{J})\big)$ normally entirely different from $\mathbf{J}$ and $W\big(W'(\mathbf{L})\big)$ from $\mathbf{L}$.

In the case where we have a symbolic logic whose propositions can be interpreted as number-theoretic theorems, but are not expressed in the form of the duality of formulae, we shall again have a corresponding logic formula, but its relation to the symbolic logic is not so simple. As an example let us take the case where the symbolic logic proves that certain primitive recursive functions vanish infinitely often. As was shown in § 3, we can associate with each such proposition a W.F.F. which is dual if and only if the proposition is true. When we replace the propositions of the symbolic logic by theorems on the duality of formulae in this way, our previous argument applies and we obtain a certain logic formula $\mathbf{L}$. However, $\mathbf{L}$ does not determine uniquely which are the propositions provable in the symbolic logic; for it is possible that "$\theta_1(x)$ vanishes infinitely often" and "$\theta_2(x)$ vanishes infinitely often" are both associated with "$\mathbf{A}$ is dual", and that the first of these propositions is provable in the system, but the second not. However, if we suppose that the system of symbolic logic is sufficiently powerful to be able to carry out the argument on pp. 170–171 then this difficulty cannot arise. There is also the possibility that there may be formulae in the extent of $\mathbf{L}$ with no propositions of the form "$\theta(x)$ vanishes infinitely often" corresponding to them. But to each such formula we can assign (by a different argument) a proposition p of the symbolic logic which is a necessary and sufficient condition for $\mathbf{A}$ to be dual. With p is associated (in the first way) a formula $\mathbf{A}'$. Now $\mathbf{L}$ can always be modified so that its extent contains $\mathbf{A}'$ whenever it contains $\mathbf{A}$.

We shall be interested principally in questions of completeness. Let us suppose that we have a class of systems of symbolic logic, the propositions of these systems being expressed in a uniform notation and interpretable as number-theoretic theorems; suppose also that there is a rule by which we can assign to each proposition p of the notation a W.F.F. $\mathbf{A}_p$ which is dual if and only if p is true, and that to each W.F.F. $\mathbf{A}$ we can assign a proposition $p_{\mathbf{A}}$ which is a necessary and sufficient condition for $\mathbf{A}$ to be dual. $p_{\mathbf{A}_p}$ is to be expected to differ from p. To each symbolic logic C we can assign two logic formulae $\mathbf{L}_C$ and $\mathbf{L}_C'$. A formula $\mathbf{A}$ belongs to the extent of $\mathbf{L}_C$ if $p_{\mathbf{A}}$ is provable in C, while the extent of $\mathbf{L}_C'$ consists of all $\mathbf{A}_p$, where p is provable in C. Let us say that the class of symbolic logics is complete if each true proposition is provable in one of them: let us also say that a class of logic formulae is complete if the set-theoretic sum of the extents of these logics includes all dual formulae. I assert that a necessary condition for a class of symbolic logics C to be complete is that the class of logics $\mathbf{L}_C$ is complete, while a sufficient condition is that the class of logics $\mathbf{L}_C'$ is complete. Let us suppose that the class of symbolic logics is complete; consider $p_{\mathbf{A}}$, where $\mathbf{A}$ is arbitrary but dual. It must be provable in one of the systems, C say. $\mathbf{A}$ therefore belongs to the extent of $\mathbf{L}_C$, *i.e.* the class of logics $\mathbf{L}_C$ is complete. Now suppose the class of logics $\mathbf{L}_C'$ to be complete. Let p be an arbitrary true proposition of the notation; $\mathbf{A}_p$ must belong to the extent of some $\mathbf{L}_C'$, and this means that p is provable in C.

We shall say that a single logic formula $\mathbf{L}$ is complete if its extent includes all dual formulae; that is to say, it is complete if it enables us to prove every true number-theoretic theorem. It is a consequence of the theorem of Gödel (if suitably extended) that no logic formula is complete, and this also follows from (C), p. 165, or from the results of Turing [**1**], § 8, when taken in conjunction with § 3 of the present paper. The idea of completeness of a logic formula is not therefore very important, although it is useful to have a term for it.

Suppose $\mathbf{Y}$ to be a W.F.F. such that $\mathbf{Y}(\mathbf{n})$ is a logic for each positive integer n. The formulae of the extent of $\mathbf{Y}(\mathbf{n})$ are enumerated by $W\big(\mathbf{Y}(\mathbf{n})\big)$, and the combined extents of these logics by

$$\lambda r \,.\, W\Big(\mathbf{Y}\big(\varpi(2, r), \varpi(3, r)\big)\Big).$$

If we put

$$\Gamma \to \lambda y \,.\, W'\bigg(\lambda r \,.\, W\Big(y\big(\varpi(2, r), \; \varpi(3, r)\big)\Big)\bigg),$$

then $\Gamma(\mathbf{Y})$ is a logic whose extent is the combined extent of

$$\mathbf{Y}(1), \quad \mathbf{Y}(2), \quad \mathbf{Y}(3), \quad \ldots.$$

To each W.F.F. $\mathbf{L}$ we can assign a W.F.F. $V(\mathbf{L})$ such that a necessary and sufficient condition for $\mathbf{L}$ to be a logic formula is that $V(\mathbf{L})$ is dual. Let Nm be a W.F.F. which enumerates all formulae with normal forms and no free variables. Then the condition for $\mathbf{L}$ to be a logic is that $\mathbf{L}\big(\mathrm{Nm}(\mathbf{r}), \mathbf{s}\big)$ conv 2 for all positive integers r, s, *i.e.* that

$$\lambda a\,.\,\mathbf{L}\Big(\mathrm{Nm}\big(\varpi(2, a)\big),\ \varpi(3, a)\Big)$$

is dual. We may therefore put

$$V \to \lambda l a\,.\,l\Big(\mathrm{Nm}\big(\varpi(2, a)\big),\ \varpi(3, a)\Big).$$

7. *Ordinals.*

We begin our treatment of ordinals with some brief definitions from the Cantor theory of ordinals, but for the understanding of some of the proofs a greater amount of the Cantor theory is necessary than is set out here.

Suppose that we have a class determined by the propositional function $D(x)$ and a relation $G(x, y)$ ordering its members, *i.e.* satisfying

$$\left.\begin{array}{ll} G(x, y) \,\&\, G(y, z) \supset G(x, z), & \text{(i)} \\ D(x) \,\&\, D(y) \supset G(x, y) \vee G(y, x) \vee x = y, & \text{(ii)} \\ G(x, y) \supset D(x) \,\&\, D(y), & \text{(iii)} \\ \sim G(x, x). & \text{(iv)} \end{array}\right\} \qquad (7.1)$$

The class defined by $D(x)$ is then called a *series* with the ordering relation $G(x, y)$. The series is said to be *well ordered* and the ordering relation is called an *ordinal* if every sub-series which is not void has a first term, *i.e.* if

$$(D')\Big\{(\exists x)\big(D'(x)\big) \,\&\, (x)\big(D'(x) \supset D(x)\big)$$
$$\supset (\exists z)(y)\Big[D'(z) \,\&\, \big(D'(y) \supset G(z, y) \vee z = y\big)\Big]\Big\}. \qquad (7.2)$$

The condition (7.2) is equivalent to another, more suitable for our purposes, namely the condition that every descending subsequence must terminate; formally

$$(x)\Big\{D'(x) \supset D(x) \,\&\, (\exists y)\big(D'(y) \,\&\, G(y, x)\big)\Big\} \supset (x)\big(\sim D'(x)\big). \qquad (7.3)$$

The ordering relation $G(x, y)$ is said to be similar to $G'(x, y)$ if there is a one-one correspondence between the series transforming the one relation into the other. This is best expressed formally, thus

$$(\exists M)\Big[(x)\{D(x)\supset(\exists x')\,M(x,x')\}\,\&\,(x')\{D'(x')\supset(\exists x)\,M(x,x')\}$$
$$\&\Big\{\big(M(x,x')\,\&\,M(x,x'')\big)\vee\big(M(x',x)\,\&\,M(x'',x)\big)\supset x'=x''\Big\}$$
$$\&\Big\{M(x,x')\,\&\,M(y,y')\supset\big(G(x,y)\equiv G(x',y')\big)\Big\}\Big]. \qquad (7.4)$$

Ordering relations are regarded as belonging to the same ordinal if and only if they are similar.

We wish to give names to all the ordinals, but this will not be possible until they have been restricted in some way; the class of ordinals, as at present defined, is more than enumerable. The restrictions that we actually impose are these: $D(x)$ is to imply that x is a positive integer; $D(x)$ and $G(x, y)$ are to be computable properties. Both of the propositional functions $D(x)$, $G(x, y)$ can then be described by means of a single W.F.F. Ω with the properties:

$\Omega(\mathbf{m}, \mathbf{n})$ conv 4 unless both $D(m)$ and $D(n)$ are true,

$\Omega(\mathbf{m}, \mathbf{m})$ conv 3 if $D(m)$ is true,

$\Omega(\mathbf{m}, \mathbf{n})$ conv 2 if $D(m)$, $D(n)$, $G(m, n)$, $\sim(m=n)$ are true,

$\Omega(\mathbf{m}, \mathbf{n})$ conv 1 if $D(m)$, $D(n)$, $\sim G(m, n)$, $\sim(m=n)$ are true.

In consequence of the conditions to which $D(x)$, $G(x, y)$ are subjected, Ω must further satisfy:

(*a*) if $\Omega(\mathbf{m}, \mathbf{n})$ is convertible to 1 or 2, then $\Omega(\mathbf{m}, \mathbf{m})$ and $\Omega(\mathbf{n}, \mathbf{n})$ are convertible to 3,

(*b*) if $\Omega(\mathbf{m}, \mathbf{m})$ and $\Omega(\mathbf{n}, \mathbf{n})$ are convertible to 3, then $\Omega(\mathbf{m}, \mathbf{n})$ is convertible to 1, 2, or 3,

(*c*) if $\Omega(\mathbf{m}, \mathbf{n})$ is convertible to 1, then $\Omega(\mathbf{n}, \mathbf{m})$ is convertible to 2 and conversely,

(*d*) if $\Omega(\mathbf{m}, \mathbf{n})$ and $\Omega(\mathbf{n}, \mathbf{p})$ are convertible to 1, then $\Omega(\mathbf{m}, \mathbf{p})$ is also,

(*e*) there is no sequence m_1, m_2, ... such that $\Omega(\mathbf{m}_{i+1}, \mathbf{m}_i)$ conv 2 for each positive integer i,

(*f*) $\Omega(\mathbf{m}, \mathbf{n})$ is always convertible to 1, 2, 3, or 4.

If a formula Ω satisfies these conditions then there are corresponding propositional functions $D(x)$, $G(x, y)$. We shall therefore say that Ω is an *ordinal formula* if it satisfies the conditions (a)–(f). It will be seen that a consequence of this definition is that Dt is an ordinal formula; it represents the ordinal ω. The definition that we have given does not pretend to have virtues such as elegance or convenience. It has been introduced rather to fix our ideas and to show how it is possible in principle to describe ordinals by means of well formed formulae. The definitions could be modified in a number of ways. Some such modifications are quite trivial; they are typified by modifications such as changing the numbers 1, 2, 3, 4, used in the definition, to others. Two such definitions will be said to be equivalent; in general, we shall say that two definitions are equivalent if there are W.F.F. **T**, **T**$'$ such that, if **A** is an ordinal formula under one definition and represents the ordinal α, then **T**$'$(**A**) is an ordinal formula under the second definition and represents the same ordinal; and, conversely, if **A**$'$ is an ordinal formula under the second definition representing α, then **T**(**A**$'$) represents α under the first definition. Besides definitions equivalent in this sense to our original definition, there are a number of other possibilities open. Suppose for instance that we do not require $D(x)$ and $G(x, y)$ to be computable, but that we require only that $D(x)$ and $G(x, y) \,\&\, x < y$ are axiomatic†. This leads to a definition of an ordinal formula which is (presumably) not equivalent to the definition that we are using‡. There are numerous possibilities, and little to guide us in choosing one definition rather than another. No one of them could well be described as "wrong"; some of them may be found more valuable in applications than others, and the particular choice that we have made has been determined partly by the applications that we have in view. In the case of theorems of a negative character, one would wish to prove them for each one of the possible definitions of "ordinal formula". This programme could, I think, be carried through for the negative results of § 9, 10.

Before leaving the subject of possible ways of defining ordinal formulae, I must mention another definition due to Church and Kleene (Church and Kleene [**1**]). We can make use of this definition in constructing ordinal logics, but it is more convenient to use a slightly different definition which is equivalent (in the sense just described) to the Church-Kleene definition as modified in Church [**4**].

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† To require $G(x, y)$ to be axiomatic amounts to requiring $G(x, y)$ to be computable on account of (7. 1) (ii).

‡ On the other hand, if $D(x)$ is axiomatic and $G(x, y)$ is computable in the modified sense that there is a rule for determining whether $G(x, y)$ is true which leads to a definite result in all cases where $D(x)$ and $D(y)$ are true, the corresponding definition of ordinal formula is equivalent to our definition. To give the proof would be too much of a digression. Probably other equivalences of this kind hold.

Introduce the abbreviations

$$U \to \lambda ufx\,.\,u\Big(\lambda y\,.\,f\big(y(I,\,x)\big)\Big),$$

$$\mathrm{Suc} \to \lambda aufx\,.\,f\big(a(u,\,f,\,x)\big).$$

We define first a partial ordering relation "$<$" which holds between certain pairs of W.F.F. [conditions (1)–(5)].

(1) If $\mathbf{A}$ conv $\mathbf{B}$, then $\mathbf{A} < \mathbf{C}$ implies $\mathbf{B} < \mathbf{C}$ and $\mathbf{C} < \mathbf{A}$ implies $\mathbf{C} < \mathbf{B}$.

(2) $\mathbf{A} < \mathrm{Suc}\,(\mathbf{A})$.

(3) For any positive integers m and n, $\lambda ufx\,.\,\mathbf{R}(\mathbf{n}) < \lambda ufx\,.\,\mathbf{R}(\mathbf{m})$ implies $\lambda ufx\,.\,\mathbf{R}(\mathbf{n}) < \lambda ufx\,.\,u(\mathbf{R})$.

(4) If $\mathbf{A} < \mathbf{B}$ and $\mathbf{B} < \mathbf{C}$, then $\mathbf{A} < \mathbf{C}$. (1)–(4) are required for any W.F.F. $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\lambda ufx\,.\,\mathbf{R}$.

(5) The relation $\mathbf{A} < \mathbf{B}$ holds only when compelled to do so by (1)–(4).

We define C-K ordinal formulae by the conditions (6)–(10).

(6) If $\mathbf{A}$ conv $\mathbf{B}$ and $\mathbf{A}$ is a C-K ordinal formula, then $\mathbf{B}$ is a C-K ordinal formula.

(7) U is a C-K ordinal formula.

(8) If $\mathbf{A}$ is a C-K ordinal formula, then $\mathrm{Suc}\,(\mathbf{A})$ is a C-K ordinal formula.

(9) If $\lambda ufx\,.\,\mathbf{R}(\mathbf{n})$ is a C-K ordinal formula and

$$\lambda ufx\,.\,\mathbf{R}(\mathbf{n}) < \lambda ufx\,.\,\mathbf{R}\big(S(\mathbf{n})\big)$$

for each positive integer n, then $\lambda ufx\,.\,u(\mathbf{R})$ is a C-K ordinal formula†.

(10) A formula is a C-K ordinal formula only if compelled to be so by (6)–(9).

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† If we also allow $\lambda ufx\,.\,u(\mathbf{R})$ to be a C-K ordinal formula when

$$\lambda ufx\,.\,\mathbf{n}(\mathbf{R}) \text{ conv } \lambda ufx\,.\,S(\mathbf{n},\,\mathbf{R})$$

for all n, then the formulae for sum, product and exponentiation of C-K ordinal formulae can be much simplified. For instance, if $\mathbf{A}$ and $\mathbf{B}$ represent α and β, then

$$\lambda ufx\,.\,\mathbf{B}(u,\,f,\,\mathbf{A}(u,\,f,\,x))$$

represents $\alpha+\beta$. Property (6) remains true.

The representation of ordinals by formulae is described by (11)–(15).

(11) If $\mathbf{A}$ conv $\mathbf{B}$ and $\mathbf{A}$ represents α, then $\mathbf{B}$ represents α.

(12) U represents 1.

(13) If $\mathbf{A}$ represents α, then Suc $(\mathbf{A})$ represents $\alpha+1$.

(14) If $\lambda ufx\,.\,\mathbf{R}(\mathbf{n})$ represents α_n for each positive integer n, then $\lambda ufx\,.\,u(\mathbf{R})$ represents the upper bound of the sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$.

(15) A formula represents an ordinal only when compelled to do so by (11)–(14).

We denote any ordinal represented by $\mathbf{A}$ by $\Xi_{\mathbf{A}}$ without prejudice to the possibility that more than one ordinal may be represented by $\mathbf{A}$. We shall write $\mathbf{A} \leqslant \mathbf{B}$ to mean $\mathbf{A} < \mathbf{B}$ or $\mathbf{A}$ conv $\mathbf{B}$.

In proving properties of C-K ordinal formulae we shall often use a kind of analogue of the principle of transfinite induction. If ϕ is some property and we have:

(*a*) If $\mathbf{A}$ conv $\mathbf{B}$ and $\phi(\mathbf{A})$, then $\phi(\mathbf{B})$,

(*b*) $\phi(U)$,

(*c*) If $\phi(\mathbf{A})$, then $\phi\big(\mathrm{Suc}\,(\mathbf{A})\big)$,

(*d*) If $\phi\big(\lambda ufx\,.\,\mathbf{R}(\mathbf{n})\big)$ and $\lambda ufx\,.\,\mathbf{R}(\mathbf{n}) < \lambda ufx\,.\,\mathbf{R}\big(S(\mathbf{n})\big)$ for each positive integer n, then

$$\phi\big(\lambda ufx\,.\,u(\mathbf{R})\big);$$

(7.5)

then $\phi(\mathbf{A})$ for each C-K ordinal formula $\mathbf{A}$. To prove the validity of this principle we have only to observe that the class of formulae $\mathbf{A}$ satisfying $\phi(\mathbf{A})$ is one of those of which the class of C-K ordinal formulae was defined to be the smallest. We can use this principle to help us to prove:—

(i) Every C-K ordinal formula is convertible to the form $\lambda ufx\,.\,\mathbf{B}$, where $\mathbf{B}$ is in normal form.

(ii) There is a method by which for any C-K ordinal formula, we can determine into which of the forms U, Suc $(\lambda ufx\,.\,\mathbf{B})$, $\lambda ufx\,.\,u(\mathbf{R})$ (where u is free in $\mathbf{R}$) it is convertible, and by which we can determine $\mathbf{B}$, $\mathbf{R}$. In each case $\mathbf{B}$, $\mathbf{R}$ are unique apart from conversions.

(iii) If $\mathbf{A}$ represents any ordinal, $\Xi_{\mathbf{A}}$ is unique. If $\Xi_{\mathbf{A}}$, $\Xi_{\mathbf{B}}$ exist and $\mathbf{A} < \mathbf{B}$, then $\Xi_{\mathbf{A}} < \Xi_{\mathbf{B}}$.

(iv) If **A**, **B**, **C** are C-K ordinal formulae and $\mathbf{B} < \mathbf{A}$, $\mathbf{C} < \mathbf{A}$, then either $\mathbf{B} < \mathbf{C}$, $\mathbf{C} < \mathbf{B}$, or **B** conv **C**.

(v) A formula **A** is a C-K ordinal formula if:

(A) $U \leqslant \mathbf{A}$,

(B) If $\lambda ufx \,.\, u(\mathrm{R}) \leqslant \mathbf{A}$ and n is a positive integer, then

$$\lambda ufx \,.\, \mathrm{R}(\mathbf{n}) < \lambda ufx \,.\, \mathrm{R}\big(S(\mathbf{n})\big),$$

(C) For any two W.F.F. **B**, **C** with $\mathbf{B} < \mathbf{A}$, $\mathbf{C} < \mathbf{A}$ we have $\mathbf{B} < \mathbf{C}$, $\mathbf{C} < \mathbf{B}$, or **B** conv **C**, but never $\mathbf{B} < \mathbf{B}$,

(D) There is no infinite sequence $\mathbf{B}_1$, $\mathbf{B}_2$, ... for which

$$\mathbf{B}_{\mathrm{r}} < \mathbf{B}_{\mathrm{r}-1} < \mathbf{A}$$

for each r.

(vi) There is a formula H such that, if **A** is a C-K ordinal formula, then $H(\mathbf{A})$ is an ordinal formula representing the same ordinal. $H(\mathbf{A})$ is not an ordinal formula unless **A** is a C-K ordinal formula.

Proof of (i). Take $\phi(\mathbf{A})$ to be "**A** is convertible to the form $\lambda ufx \,.\, \mathbf{B}$, where **B** is in normal form". The conditions (*a*) and (*b*) are trivial. For (*c*), suppose that **A** conv $\lambda ufx \,.\, \mathbf{B}$, where **B** is in normal form; then

$$\mathrm{Suc}\,(\mathbf{A}) \text{ conv } \lambda ufx \,.\, f(\mathbf{B})$$

and $f(\mathbf{B})$ is in normal form. For (*d*) we have only to show that $u(\mathrm{R})$ has a normal form, *i.e.* that **R** has a normal form; and this is true since $\mathrm{R}(1)$ has a normal form.

Proof of (ii). Since, by hypothesis, the formula is a C-K ordinal formula we have only to perform conversions on it until it is in one of the forms described. It is not possible to convert it into two of these three forms. For suppose that $\lambda ufx \,.\, f\big(\mathbf{A}(u, f, x)\big)$ conv $\lambda ufx \,.\, u(\mathrm{R})$ and is a C-K ordinal formula; it is then convertible to the form $\lambda ufx \,.\, \mathbf{B}$, where **B** is in normal form. But the normal form of $\lambda ufx \,.\, u(\mathbf{R})$ can be obtained by conversions on **R**, and that of $\lambda ufx \,.\, f\big(\mathbf{A}(u, f, x)\big)$ by conversions on $\mathbf{A}(u, f, x)$ (as follows from Church and Rosser [1], Theorem 2); this, however, would imply that the formula in question had two normal forms, one of form $\lambda ufx \,.\, u(\mathrm{S})$ and one of form $\lambda ufx \,.\, f(\mathbf{C})$, which is impossible. Or let U conv $\lambda ufx \,.\, u(\mathrm{R})$, where **R** is a well formed formula with u as a free variable. We may suppose **R** to be in normal form. Now U is $\lambda ufx \,.\, u\Big(\lambda y \,.\, f\big(y(I, x)\big)\Big)$. By

(A), p. 165, **R** is identical with $\lambda y\,.f\big(y(I,x)\big)$, which does not have u as a free variable. It now remains to show only that if

$$\text{Suc}\,(\lambda ufx\,.\,\mathbf{B})\ \text{conv}\ \text{Suc}\,(\lambda ufx\,.\,\mathbf{B}') \quad \text{and} \quad \lambda ufx\,.\,u(\mathbf{R})\ \text{conv}\ \lambda ufx\,.\,u(\mathbf{R}'),$$

then **B** conv **B**′ and **R** conv **R**′.

If $$\text{Suc}\,(\lambda ufx\,.\,\mathbf{B})\ \text{conv}\ \text{Suc}\,(\lambda ufx\,.\,\mathbf{B}'),$$

then $$\lambda ufx\,.f(\mathbf{B})\ \text{conv}\ \lambda ufx\,.f(\mathbf{B}');$$

but both of these formulae can be brought to normal form by conversions on **B**, **B**′ and therefore **B** conv **B**′. The same argument applies in the case in which $\lambda ufx\,.\,u(\mathbf{R})$ conv $\lambda ufx\,.\,u(\mathbf{R}')$.

Proof of (iii). To prove the first half, take $\phi(\mathbf{A})$ to be "$\Xi_{\mathbf{A}}$ is unique". Then (7.5) (*a*) is trivial, and (*b*) follows from the fact that U is not convertible either to the form Suc (**A**) or to $\lambda ufx\,.u(\mathbf{R})$, where **R** has u as a free variable. For (*c*): Suc (**A**) is not convertible to the form $\lambda ufx\,.\,u(\mathbf{R})$; the possibility that Suc (**A**) represents an ordinal on account of (12) or (14) is therefore eliminated. By (13), Suc (**A**) represents $\alpha'+1$ if **A**′ represents α' and Suc (**A**) conv Suc (**A**′). If we suppose that **A** represents α, then **A**, **A**′, being C-K ordinal formulae, are convertible to the forms $\lambda ufx\,.\,\mathbf{B}$, $\lambda ufx\,.\,\mathbf{B}'$; but then, by (ii), **B** conv **B**′, *i.e.* **A** conv **A**′, and therefore $\alpha=\alpha'$ by the hypothesis $\phi(\mathbf{A})$. Then $\Xi_{\text{Suc}\,(\mathbf{A})}=\alpha'+1$ is unique. For (*d*): $\lambda ufx\,.u(\mathbf{R})$ is not convertible to the form Suc (**A**) or to U if **R** has u as a free variable. If $\lambda ufx\,.u(\mathbf{R})$ represents an ordinal, it is so therefore in virtue of (14), possibly together with (11). Now, if $\lambda ufx\,.u(\mathbf{R})$ conv $\lambda ufx\,.u(\mathbf{R}')$, then **R** conv **R**′, so that the sequence $\lambda ufx\,.\,\mathbf{R}(1)$, $\lambda ufx,\ \mathbf{R}(2)$, ... in (14) is unique apart from conversions. Then, by the induction hypothesis, the sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ is unique. The only ordinal that is represented by $\lambda ufx\,.\,u(\mathbf{R})$ is the upper bound of this sequence; and this is unique.

For the second half we use a type of argument rather different from our transfinite induction principle. The formulae **B** for which $\mathbf{A}<\mathbf{B}$ form the smallest class for which:

Suc (**A**) belongs to the class.
If **C** belongs to the class, then Suc (**C**) belongs to it.
If $\lambda ufx\,.\,\mathbf{R}(\mathbf{n})$ belongs to the class and

$$\lambda ufx\,.\,\mathbf{R}(\mathbf{n})<\lambda ufx\,.\,\mathbf{R}(\mathbf{m}), \tag{7.6}$$

where m, n are some positive integers, then $\lambda ufx\,.u(\mathbf{R})$ belongs to it.
If **C** belongs to the class and **C** conv **C**′, then **C**′ belongs to it.

It will be sufficient to prove that the class of formulae **B** for which either $\Xi_{\mathbf{B}}$ does not exist or $\Xi_{\mathbf{A}} < \Xi_{\mathbf{B}}$ satisfies the conditions (7.6). Now

$$\Xi_{\mathrm{Suc}\,(\mathbf{A})} = \Xi_{\mathbf{A}} + 1 > \Xi_{\mathbf{A}},$$

$$\Xi_{\mathrm{Suc}\,(\mathbf{C})} > \Xi_{\mathbf{C}} > \Xi_{\mathbf{A}} \text{ if } \mathbf{C} \text{ is in the class.}$$

If $\Xi_{\lambda ufx\,.\,\mathbf{R}(\mathbf{n})}$ does not exist, then $\Xi_{\lambda ufx\,.\,u(\mathbf{R})}$ does not exist, and therefore $\lambda ufx\,.\,u(\mathbf{R})$ is in the class. If $\Xi_{\lambda ufx\,.\,\mathbf{R}(\mathbf{n})}$ exists and is greater than $\Xi_{\mathbf{A}}$, and $\lambda ufx.\, \mathbf{R}(\mathbf{n}) < \lambda ufx\,.\,\mathbf{R}(\mathbf{m})$, then

$$\Xi_{\lambda ufx\,.\,u(\mathbf{R})} \geqslant \Xi_{\lambda ufx\,.\,\mathbf{R}(\mathbf{n})} > \Xi_{\mathbf{A}},$$

so that $\lambda ufx\,.\,u(\mathbf{R})$ belongs to the class.

Proof of (iv). We prove this by induction with respect to **A**. Take $\phi(\mathbf{A})$ to be "whenever $\mathbf{B} < \mathbf{A}$ and $\mathbf{C} < \mathbf{A}$ then $\mathbf{B} < \mathbf{C}$ or $\mathbf{C} < \mathbf{B}$ or **B** conv **C**". $\phi(U)$ follows from the fact that we never have $\mathbf{B} < U$. If we have $\phi(\mathbf{A})$ and $\mathbf{B} < \mathrm{Suc}\,(\mathbf{A})$, then either $\mathbf{B} < \mathbf{A}$ or **B** conv **A**; for we can find **D** such that $\mathbf{B} \leqslant \mathbf{D}$, and then $\mathbf{D} < \mathrm{Suc}\,(\mathbf{A})$ can be proved without appealing either to (1) or (5); (4) does not apply, so we must have **D** conv **A**. Then, if $\mathbf{B} < \mathrm{Suc}\,(\mathbf{A})$ and $\mathbf{C} < \mathrm{Suc}\,(\mathbf{A})$, we have four possibilities,

B conv **A**, **C** conv **A**,

B conv **A**, $\mathbf{C} < \mathbf{A}$,

$\mathbf{B} < \mathbf{A}$, **C** conv **A**,

$\mathbf{B} < \mathbf{A}$, $\mathbf{C} < \mathbf{A}$.

In the first case **B** conv **C**, in the second $\mathbf{C} < \mathbf{B}$, in the third $\mathbf{B} < \mathbf{C}$, and in the fourth the induction hypothesis applies.

Now suppose that $\lambda ufx\,.\,\mathbf{R}(\mathbf{n})$ is a C-K ordinal formula, that

$$\lambda ufx\,.\,\mathbf{R}(\mathbf{n}) < \lambda ufx\,.\,\mathbf{R}\big(S(\mathbf{n})\big) \quad \text{and} \quad \phi\big(\mathbf{R}(\mathbf{n})\big),$$

for each positive integer n, and that **A** conv $\lambda ufx\,.\,u(\mathbf{R})$. Then, if $\mathbf{B} < \mathbf{A}$, this means that $\mathbf{B} < \lambda ufx\,.\,\mathbf{R}(\mathbf{n})$ for some n; if we have also $\mathbf{C} < \mathbf{A}$, then $\mathbf{B} < \lambda ufx\,.\,\mathbf{R}(\mathbf{q})$, $\mathbf{C} < \lambda ufx\,.\,\mathbf{R}(\mathbf{q})$ for some q. Thus, for these **B** and **C**, the required result follows from $\phi\big(\lambda ufx\,.\,\mathbf{R}(\mathbf{q})\big)$.

Proof of (v). The conditions (C), (D) imply that the classes of interconvertible formulae **B**, $\mathbf{B} < \mathbf{A}$ are well-ordered by the relation "$<$". We prove (v) by (ordinary) transfinite induction with respect to the order type α of the series formed by these classes; (α is, in fact, the solution of the equation $1+\alpha=\Xi_{\mathbf{A}}$, but we do not need this). We suppose then that (v) is true for all order types less than α. If $\mathbf{E}<\mathbf{A}$, then $\mathbf{E}$ satisfies the conditions of (v) and the corresponding order type is smaller: $\mathbf{E}$ is therefore a C-K ordinal formula. This expresses all consequences of the induction hypothesis that we need. There are three cases to consider:

(x) $\alpha=0$.

(y) $\alpha=\beta+1$.

(z) α is of neither of the forms (x), (y).

In case (x) we must have $\mathbf{A}$ conv U on account of (A). In case (y) there is a formula $\mathbf{D}$ such that $\mathbf{D}<\mathbf{A}$, and $\mathbf{B}\leqslant\mathbf{D}$ whenever $\mathbf{B}<\mathbf{A}$. The relation $\mathbf{D}<\mathbf{A}$ must hold in virtue of either (1), (2), (3), or (4). It cannot be in virtue of (4); for then there would be $\mathbf{B}$, $\mathbf{B}<\mathbf{A}$, $\mathbf{D}<\mathbf{B}$ contrary to (C), taken in conjunction with the definition of $\mathbf{D}$. If it is in virtue of (3), then α is the upper bound of a sequence α_1, α_2, ... of ordinals, which are increasing by reason of (iii) and the conditions $\lambda ufx\,.\,\mathbf{R}(\mathbf{n})<\lambda ufx\,.\,\mathbf{R}\big(S(\mathbf{n})\big)$ in (B). This is inconsistent with $\alpha=\beta+1$. This means that (2) applies [after we have eliminated (1) by suitable conversions on $\mathbf{A}$, $\mathbf{D}$] and we see that $\mathbf{A}$ conv Suc ($\mathbf{D}$); but, since $\mathbf{D}<\mathbf{A}$, $\mathbf{D}$ is a C-K ordinal formula, and $\mathbf{A}$ must therefore be a C-K ordinal formula by (8). Now take case (z). It is impossible for $\mathbf{A}$ to be of the form Suc ($\mathbf{D}$), for then we should have $\mathbf{B}<\mathbf{D}$ whenever $\mathbf{B}<\mathbf{A}$, and this would mean that we had case (y). Since $U<\mathbf{A}$, there must be an $\mathbf{F}$ such that $\mathbf{F}<\mathbf{A}$ is demonstrable either by (2) or by (3) (after a possible conversion on $\mathbf{A}$); it must of course be demonstrable by (3). Then $\mathbf{A}$ is of the form $\lambda ufx\,.\,u(\mathbf{R})$. By (3), (B) we see that $\lambda ufx\,.\,\mathbf{R}(\mathbf{n})<\mathbf{A}$ for each positive integer n; each $\lambda ufx\,.\,\mathbf{R}(\mathbf{n})$ is therefore a C-K ordinal formula. Applying (9), (B) we see that $\mathbf{A}$ is a C-K ordinal formula.

Proof of (vi). To prove the first half, it is sufficient to find a method whereby from a C-K ordinal formula $\mathbf{A}$ we can find the corresponding ordinal formula Ω. For then there is a formula H_1 such that $H_1(\mathbf{a})$ conv $\mathbf{p}$ if a is the G.R. of $\mathbf{A}$ and p is that of Ω. H is then to be defined by

$$H\rightarrow\lambda a\,.\,\mathrm{form}\Big(H_1\big(\mathrm{Gr}(a)\big)\Big).$$

The method of finding Ω may be replaced by a method of finding $\Omega(\mathbf{m},\mathbf{n})$, given $\mathbf{A}$ and any two positive integers m, n. We shall arrange the method so that, whenever $\mathbf{A}$ is not an ordinal formula, either the calculation of the values does not terminate or else the values are not consistent with

Ω being an ordinal formula. In this way we can prove the second half of (vi).

Let Ls be a formula such that Ls($\mathbf{A}$) enumerates the classes of formulae $\mathbf{B}$, $\mathbf{B} < \mathbf{A}$ [*i.e.* if $\mathbf{B} < \mathbf{A}$ there is one and only one positive integer n for which Ls($\mathbf{A}$, $\mathbf{n}$) conv $\mathbf{B}$]. Then the rule for finding the value of $\Omega(\mathbf{m}, \mathbf{n})$ is as follows:—

First determine whether $U \leqslant \mathbf{A}$ and whether $\mathbf{A}$ is convertible to the form $\mathbf{r}(\mathrm{Suc}, U)$. This terminates if $\mathbf{A}$ is a C-K ordinal formula.

If $\mathbf{A}$ conv $\mathbf{r}(\mathrm{Suc}, U)$ and either $m > r+1$ or $n > r+1$, then the value is 4. If $m < n \leqslant r+1$, the value is 2. If $n < m \leqslant r+1$, the value is 1. If $m = n \leqslant r+1$, the value is 3.

If $\mathbf{A}$ is not convertible to this form, we determine whether either $\mathbf{A}$ or Ls($\mathbf{A}$, $\mathbf{m}$) is convertible to the form $\lambda ufx\,.\,u(\mathbf{R})$; and if either of them is, we verify that $\lambda ufx\,.\,\mathbf{R}(\mathbf{n}) < \lambda ufx\,.\,\mathbf{R}\big(S(\mathbf{n})\big)$. We shall eventually come to an affirmative answer if $\mathbf{A}$ is a C-K ordinal formula.

Having checked this, we determine concerning m and n whether $\mathrm{Ls}(\mathbf{A}, \mathbf{m}) < \mathrm{Ls}(\mathbf{A}, \mathbf{n})$, $\mathrm{Ls}(\mathbf{A}, \mathbf{n}) < \mathrm{Ls}(\mathbf{A}, \mathbf{m})$, or $m = n$, and the value is to be accordingly 1, 2, or 3.

If $\mathbf{A}$ is a C-K ordinal formula, this process certainly terminates. To see that the values so calculated correspond to an ordinal formula, and one representing $\Xi_{\mathbf{A}}$, first observe that this is so when $\Xi_{\mathbf{A}}$ is finite. In the other case (iii) and (iv) show that $\Xi_{\mathbf{B}}$ determines a one-one correspondence between the ordinals β, $1 \leqslant \beta \leqslant \Xi_{\mathbf{A}}$, and the classes of interconvertible formulae $\mathbf{B}$, $\mathbf{B} < \mathbf{A}$. If we take $G(m, n)$ to be $\mathrm{Ls}(\mathbf{A}, \mathbf{m}) < \mathrm{Ls}(\mathbf{A}, \mathbf{n})$, we see that $G(m, n)$ is the ordering relation of a series of order type† $\Xi_{\mathbf{A}}$ and on the other hand that the values of $\Omega(\mathbf{m}, \mathbf{n})$ are related to $G(m, n)$ as on p. 179.

To prove the second half suppose that $\mathbf{A}$ is not a C-K ordinal formula. Then one of the conditions (A)–(D) in (v) must not be satisfied. If (A) is not satisfied we shall not obtain a result even in the calculation of $\Omega(1, 1)$. If (B) is not satisfied, we shall have for some positive integers p and q,

$$\mathrm{Ls}(\mathbf{A}, \mathbf{p}) \text{ conv } \lambda ufx\,.\,u(\mathbf{R})$$

but not $\lambda ufx\,.\,\mathbf{R}(\mathbf{q}) < \lambda ufx\,.\,\mathbf{R}\big(S(\mathbf{q})\big)$. Then the process of calculating $\Omega(\mathbf{p}, \mathbf{q})$ will not terminate. In case of failure of (C) or (D) the values of $\Omega(\mathbf{m}, \mathbf{n})$ may all be calculable, but if so conditions (*a*)–(*f*), p. 179, will be violated. Thus, if $\mathbf{A}$ is not a C-K ordinal formula, then $H(\mathbf{A})$ is not an ordinal formula.

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† The order type is β, where $1+\beta = \Xi_{\mathbf{A}}$; but $\beta = \Xi_{\mathbf{A}}$, since $\Xi_{\mathbf{A}}$ is infinite.

I propose now to define three formulae Sum, Lim, Inf of importance in connection with ordinal formulae. Since they are comparatively simple, they will for once be given almost in full. The formula Ug is one with the property that Ug(**m**) is convertible to the formula representing the largest odd integer dividing m: it is not given in full. P is the predecessor function; $P\big(S(\mathbf{m})\big)$ conv **m**, $P(1)$ conv 1.

$$\text{Al} \rightarrow \lambda pxy\,.\,p\big(\lambda guv\,.\,g(v, u),\ \lambda uv\,.\,u(I, v),\ x,\ y\big),$$

$$\text{Hf} \rightarrow \lambda m\,.\,P\Big(m\Big(\lambda guv\,.\,g\big(v,\ S(u)\big),\ \lambda uv\,.\,v(I, u),\ 1,\ 2\Big)\Big),$$

$$\text{Bd} \rightarrow \lambda ww'aa'x\,.\,\text{Al}\Big(\lambda f\,.\,w\big(a, a, w'(a', a', f)\big), x, 4\Big),$$

$$\text{Sum} \rightarrow \lambda ww'pq\,.\,\text{Bd}\Big(w, w', \text{Hf}(p), \text{Hf}(q),$$

$$\text{Al}\Big(p, \text{Al}\big(q, w'\big(\text{Hf}(p), \text{Hf}(q)\big), 1\big), \text{Al}\big(S(q), w\big(\text{Hf}(p), \text{Hf}(q)\big), 2\big)\Big)\Big),$$

$$\text{Lim} \rightarrow \lambda zpq\,.\,\Big\{\lambda ab\,.\,\text{Bd}\Big(z(a), z(b), \text{Ug}(p), \text{Ug}(q), \text{Al}\Big(\text{Dt}(a, b)+\text{Dt}(b, a),$$

$$\text{Dt}(a, b), z\big(a, \text{Ug}(p), \text{Ug}(q)\big)\Big)\Big)\Big\}\big(\varpi(2, p), \varpi(2, q)\big),$$

$$\text{Inf} \rightarrow \lambda wapq\,.\,\text{Al}\Big(\lambda f\,.\,w\big(a, p, w(a, q, f)\big), w(p, q), 4\Big).$$

The essential properties of these formulae are described by:

Al(2**r**$-$1, **m**, **n**) conv **m**, Al(2**r**, **m**, **n**) conv **n**,

Hf(2**m**) conv **m**, Hf(2**m**$-$1) conv **m**,

Bd(Ω, Ω', **a**, **a**$'$, **x**) conv 4, unless both

Ω(**a**, **a**) conv 3 and Ω'(**a**$'$, **a**$'$) conv 3,

it is then convertible to **x**.

If Ω, Ω' are ordinal formulae representing α, β respectively, then Sum(Ω, Ω') is an ordinal formula representing $\alpha+\beta$. If **Z** is a W.F.F. enumerating a sequence of ordinal formulae representing $\alpha_1, \alpha_2, \ldots$, then Lim (**Z**) is an ordinal formula representing the infinite sum $\alpha_1+\alpha_2+\alpha_3 \ldots$.

If $\mathbf{\Omega}$ is an ordinal formula representing α, then $\mathrm{Inf}(\mathbf{\Omega})$ enumerates a sequence of ordinal formulae representing all the ordinals less than α without repetitions other than repetitions of the ordinal 0.

To prove that there is no general method for determining about a formula whether it is an ordinal formula, we use an argument akin to that leading to the Burali-Forti paradox; but the emphasis and the conclusion are different. Let us suppose that such an algorithm is available. This enables us to obtain a recursive enumeration $\mathbf{\Omega}_1, \mathbf{\Omega}_2, \ldots$ of the ordinal formulae in normal form. There is a formula $\mathbf{Z}$ such that $\mathbf{Z}(\mathbf{n})$ conv $\mathbf{\Omega}_{\mathbf{n}}$. Now $\mathrm{Lim}(\mathbf{Z})$ represents an ordinal greater than any represented by an $\mathbf{\Omega}_{\mathbf{n}}$, and it has therefore been omitted from the enumeration.

This argument proves more than was originally asserted. In fact, it proves that, if we take any class E of ordinal formulae in normal form, such that, if $\mathbf{A}$ is any ordinal formula, then there is a formula in E representing the same ordinal as $\mathbf{A}$, then there is no method whereby one can determine whether a W.F.F. in normal form belongs to E.

8. *Ordinal logics.*

An ordinal logic is a W.F.F. $\mathbf{\Lambda}$ such that $\mathbf{\Lambda}(\mathbf{\Omega})$ is a logic formula whenever $\mathbf{\Omega}$ is an ordinal formula.

This definition is intended to bring under one heading a number of ways of constructing logics which have recently been proposed or which are suggested by recent advances. In this section I propose to show how to obtain some of these ordinal logics.

Suppose that we have a class W of logical systems. The symbols used in each of these systems are the same, and a class of sequences of symbols called "formulae" is defined, independently of the particular system in W. The rules of procedure of a system C define an axiomatic subset of the formulae, which are to be described as the "provable formulae of C". Suppose further that we have a method whereby, from any system C of W, we can obtain a new system C', also in W, and such that the set of provable formulae of C' includes the provable formulae of C (we shall be most interested in the case in which they are included as a proper subset). It is to be understood that this "method" is an effective procedure for obtaining the rules of procedure of C' from those of C.

Suppose that to certain of the formulae of W we make number-theoretic theorems correspond: by modifying the definition of formula, we may suppose that this is donef or all formulae. We shall say that one of the systems C is *valid* if the provability of a formula in C implies the truth of the corresponding number-theoretic theorem. Now let the relation of

C' to C be such that the validity of C implies the validity of C', and let there be a valid system C_0 in W. Finally, suppose that, given any computable sequence $C_1, C_2, \ldots$ of systems in W, the "limit system", in which a formula is provable if and only if it is provable in one of the systems C_j, also belongs to W. These limit systems are to be regarded, not as functions of the sequence given in extension, but as functions of the rules of formation of their terms. A sequence given in extension may be described by various rules of formation, and there will be several corresponding limit systems. Each of these may be described as *a* limit system of the sequence.

In these circumstances we may construct an ordinal logic. Let us associate positive integers with the systems in such a way that to each C there corresponds a positive integer m_C, and that m_C completely describes the rules of procedure of C. Then there is a W.F.F. **K**, such that

$$\mathbf{K}(\mathbf{m}_C) \text{ conv } \mathbf{m}_{C'}$$

for each C in W, and there is a W.F.F. $\mathbf{\Theta}$ such that, if $\mathbf{D}(\mathbf{r}) \text{ conv } \mathbf{m}_{C_r}$ for each positive integer r, then $\mathbf{\Theta}(\mathbf{D}) \text{ conv } \mathbf{m}_C$, where C is a limit system of $C_1, C_2, \ldots$. With each system C of W it is possible to associate a logic formula $\mathbf{L}_C$: the relation between them is that, if G is a formula of W and the number-theoretic theorem corresponding to G (assumed expressed in the conversion calculus form) asserts that **B** is dual, then $\mathbf{L}_C(\mathbf{B}) \text{ conv } 2$ if and only if G is provable in C. There is a W.F.F. **G** such that

$$\mathbf{G}(\mathbf{m}_C) \text{ conv } \mathbf{L}_C$$

for each C of W. Put

$$\mathbf{N} \to \lambda a \,.\, \mathbf{G}\big(a(\Theta, \mathbf{K}, \mathbf{m}_{C_0})\big).$$

I assert that $\mathbf{N}(\mathbf{A})$ is a logic formula for each C-K ordinal formula **A**, and that, if $\mathbf{A} < \mathbf{B}$, then $\mathbf{N}(\mathbf{B})$ is more complete than $\mathbf{N}(\mathbf{A})$, provided that there are formulae provable in C' but not in C for each valid C of W.

To prove this we shall show that to each C-K ordinal formula **A** there corresponds a unique system $C[\mathbf{A}]$ such that:

(i) $\mathbf{A}(\Theta, \mathbf{K}, \mathbf{m}_{C_0}) \text{ conv } \mathbf{m}_{C[\mathbf{A}]}$,

and that it further satisfies:

(ii) $C[U]$ is a limit system of $C_0', C_0', \ldots$,

(iii) $C[\text{Suc}(\mathbf{A})]$ is $(C[\mathbf{A}])'$,

(iv) $C[\lambda ufx \,.\, u(\mathbf{R})]$ is a limit system of $C[\lambda ufx \,.\, \mathbf{R}(1)]$, $C[\lambda ufx \,.\, \mathbf{R}(2)]$, $\ldots$,

A and $\lambda ufx\,.\,u(\mathbf{R})$ being assumed to be C-K ordinal formulae. The uniqueness of the system follows from the fact that m_C determines C completely. Let us try to prove the existence of $C[\mathbf{A}]$ for each C-K ordinal formula **A**. As we have seen (p. 182) it is sufficient to prove

(*a*) $C[U]$ exists,

(*b*) if $C[\mathbf{A}]$ exists, then $C[\mathrm{Suc}\,(\mathbf{A})]$ exists,

(*c*) if $C[\lambda ufx\,.\,\mathbf{R}(1)]$, $C[\lambda ufx\,.\,\mathbf{R}(2)]$, ... exist, then $C[\lambda ufx\,.\,u(\mathbf{R})]$ exists.

Proof of (*a*).

$$\left\{\lambda y\,.\,\mathbf{K}\big(y(I,\ \mathbf{m}_{C_0})\big)\right\}(\mathbf{n})\ \mathrm{conv}\ \mathbf{K}(\mathbf{m}_{C_0})\ \mathrm{conv}\ \mathbf{m}_{C_0'}$$

for all positive integers n, and therefore, by the definition of Θ, there is a system, which we call $C[U]$ and which is a limit system of C_0', C_0', ..., satisfying

$$\Theta\Big(\lambda y\,.\,\mathbf{K}\big(y(I,\ \mathbf{m}_{C_0})\big)\Big)\ \mathrm{conv}\ \mathbf{m}_{C[U]}.$$

But, on the other hand,

$$U\big(\Theta,\ \mathbf{K},\ \mathbf{m}_{C_0}\big)\ \mathrm{conv}\ \Theta\Big(\lambda y\,.\,\mathbf{K}\big(y(I,\ \mathbf{m}_{C_0})\big)\Big).$$

This proves (*a*) and incidentally (ii).

Proof of (*b*).

$$\begin{aligned}\mathrm{Suc}\,(\mathbf{A},\ \Theta,\ \mathbf{K},\ \mathbf{m}_{C_0})\ &\mathrm{conv}\ \mathbf{K}\big(\mathbf{A}(\Theta,\ \mathbf{K},\ \mathbf{m}_{C_0})\big)\\ &\mathrm{conv}\ \mathbf{K}(\mathbf{m}_{C[\mathbf{A}]})\\ &\mathrm{conv}\ \mathbf{m}_{(C[\mathbf{A}])'}.\end{aligned}$$

Hence $C[\mathrm{Suc}\,(\mathbf{A})]$ exists and is given by (iii).

Proof of (*c*).

$$\begin{aligned}\big\{\{\lambda ufx\,.\,\mathbf{R}\}(\Theta,\ \mathbf{K},\ \mathbf{m}_{C_0})\big\}(\mathbf{n})\ &\mathrm{conv}\ \{\lambda ufx\,.\,\mathbf{R}(\mathbf{n})\}(\Theta,\ \mathbf{K},\ \mathbf{m}_{C_0})\\ &\mathrm{conv}\ \mathbf{m}_{C[\lambda ufx\,.\,\mathbf{R}(\mathbf{n})]}\end{aligned}$$

by hypothesis. Consequently, by the definition of Θ, there exists a C which is a limit system of

$$C[\lambda ufx\,.\,\mathbf{R}(1)],\quad C[\lambda ufx\,.\,\mathbf{R}(2)],\ \ldots,$$

and satisfies

$$\Theta\Big(\{\lambda ufx\,.\,u(\mathrm{R})\}(\Theta,\ \mathbf{K},\ \mathbf{m}_{C_0})\Big)\ \mathrm{conv}\ \mathbf{m}_C.$$

We define $C[\lambda ufx\,.\,u(\mathbf{R})]$ to be this C. We then have (iv) and

$$\{\lambda ufx\,.\,u(\mathrm{R})\}(\Theta,\ \mathbf{K},\ \mathbf{m}_{C_0})\ \mathrm{conv}\ \Theta\Big(\{\lambda ufx\,.\,\mathbf{R}\}(\Theta,\ \mathbf{K},\ \mathbf{m}_{C_0})\Big)$$

$$\mathrm{conv}\ \mathbf{m}_{C[\lambda ufx\,.\,u(\mathbf{R})]}.$$

This completes the proof of the properties (i)–(iv). From (ii), (iii), (iv), the fact that C_0 is valid, and that C' is valid when C is valid, we infer that $C[\mathbf{A}]$ is valid for each C-K ordinal formula $\mathbf{A}$: also that there are more formulae provable in $C[\mathbf{B}]$ than in $C[\mathbf{A}]$ when $\mathbf{A} < \mathbf{B}$. The truth of our assertions regarding **N** now follows in view of (i) and the definitions of **N** and **G**.

We cannot conclude that **N** is an ordinal logic, since the formulae **A** are C-K ordinal formulae; but the formula H enables us to obtain an ordinal logic from **N**. By the use of the formula Gr we obtain a formula Tn such that, if **A** has a normal form, then Tn(**A**) enumerates the G.R.'s of the formulae into which **A** is convertible. Also there is a formula Ck such that, if h is the G.R. of a formula $H(\mathbf{B})$, then Ck(**h**) conv **B**, but otherwise Ck(**h**) conv U. Since $H(\mathbf{B})$ is an ordinal formula only if **B** is a C-K ordinal formula, $\mathrm{Ck}\big(\mathrm{Tn}(\Omega,\ \mathbf{n})\big)$ is a C-K ordinal formula for each ordinal formula Ω and each integer n. For many ordinal formulae it will be convertible to U, but, for suitable Ω, it will be convertible to any given C-K ordinal formula. If we put

$$\Lambda \to \lambda wa\,.\,\Gamma\Big(\lambda n\,.\,\mathbf{N}\Big(\mathrm{Ck}\Big(\mathrm{Tn}(w,\ n)\Big)\Big),\ a\Big),$$

Λ is the required ordinal logic. In fact, on account of the properties of Γ, $\Lambda(\Omega,\ \mathbf{A})$ will be convertible to 2 if and only if there is a positive integer n such that

$$\mathbf{N}\Big(\mathrm{Ck}\Big(\mathrm{Tn}(\Omega,\ \mathbf{n})\Big),\ \mathbf{A}\Big)\ \mathrm{conv}\ 2.$$

If Ω conv $H(\mathbf{B})$, there will be an integer n such that $\mathrm{Ck}\big(\mathrm{Tn}(\Omega,\ \mathbf{n})\big)$ conv **B**, and then

$$\mathbf{N}\Big(\mathrm{Ck}\Big(\mathrm{Tn}(\Omega,\ \mathbf{n})\Big),\ \mathbf{A}\Big)\ \mathrm{conv}\ \mathbf{N}(\mathbf{B},\ \mathbf{A}).$$

For any n, $\mathrm{Ck}\Big(\mathrm{Tn}(\Omega, \mathbf{n})\Big)$ is convertible to U or to some $\mathbf{B}$, where $\Omega \operatorname{conv} H(\mathbf{B})$. Thus $\Lambda(\Omega, \mathbf{A}) \operatorname{conv} 2$ if $\Omega \operatorname{conv} H(\mathbf{B})$ and $\mathbf{N}(\mathbf{B}, \mathbf{A}) \operatorname{conv} 2$ or if $\mathbf{N}(U, \mathbf{A}) \operatorname{conv} 2$, but not in any other case.

We may now specialize and consider particular classes W of systems. First let us try to construct the ordinal logic described roughly in the introduction. For W we take the class of systems arising from the system of *Principia Mathematica*[†] by adjoining to it axiomatic (in the sense described on p. 167) sets of axioms[‡]. Gödel has shown that primitive recursive relations[§] can be expressed by means of formulae in P. In fact, there is a rule whereby, given the recursion equations defining a primitive recursive relation, we can find a formula[||] $\mathfrak{A}[x_0, \ldots, z_0]$ such that

$$\mathfrak{A}[f^{(m_1)}0, \ \ldots, \ f^{(m_r)}0]$$

is provable in P if $F(m_1, \ldots, m_r)$ is true, and its negation is provable otherwise. Further, there is a method by which we can determine about a formula $\mathfrak{A}[x_0, \ldots, z_0]$ whether it arises from a primitive recursive relation in this way, and by which we can find the equations which defined the relation. Formulae of this kind will be called *recursion formulae.* We shall make use of a property that they possess, which we cannot prove formally here without giving their definition in full, but which is essentially trivial. $\mathrm{Db}[x_0, y_0]$ is to stand for a certain recursion formula such that $\mathrm{Db}[f^{(m)}0, f^{(n)}0]$ is provable in P if $m = 2n$ and its negation is provable otherwise. Suppose that $\mathfrak{A}[x_0]$, $\mathfrak{B}[x_0]$ are two recursion formulae. Then the theorem which I am assuming is that there is a recursion relation $\mathfrak{C}_{\mathfrak{A}, \mathfrak{B}}[x_0]$ such that we can prove

$$\mathfrak{C}_{\mathfrak{A}, \mathfrak{B}}[x_0] = (\exists y_0)\Big(\big(\mathrm{Db}[x_0, y_0] . \mathfrak{A}[y_0]\big) \vee \big(\mathrm{Db}[fx_0, fy_0] . \mathfrak{B}[y_0]\big)\Big) \qquad (8.1)$$

in P.

[†] Whitehead and Russell [1]. The axioms and rules of procedure of a similar system P will be found in a convenient form in Gödel [1], and I follow Gödel. The symbols for the natural numbers in P are $0, f0, ff0, \ldots, f^{(n)}0 \ldots$. Variables with the suffix "0" stand for natural numbers.

[‡] It is sometimes regarded as necessary that the set of axioms used should be computable, the intention being that it should be possible to verify of a formula reputed to be an axiom whether it really is so. We can obtain the same effect with axiomatic sets of axioms in this way. In the rules of procedure describing which are the axioms, we incorporate a method of enumerating them, and we also introduce a rule that in the main part of the deduction, whenever we write down an axiom as such, we must also write down its position in the enumeration. It is possible to verify whether this has been done correctly.

[§] A relation $F(m_1, \ldots, m_r)$ is primitive recursive if it is a necessary and sufficient condition for the vanishing of a primitive recursive function $\phi(m_1, \ldots, m_r)$.

[||] Capital German letters will be used to stand for variable or undetermined formulae in P. An expression such as $\mathfrak{A}[\mathfrak{B}, \mathfrak{C}]$ stands for the result of substituting $\mathfrak{B}$ and $\mathfrak{C}$ for x_0 and y_0 in $\mathfrak{A}$.

The significant formulae in any of our extensions of P are those of the form

$$(x_0)(\exists y_0)\,\mathfrak{A}[x_0, y_0], \tag{8.2}$$

where $\mathfrak{A}[x_0, y_0]$ is a recursion formula, arising from the relation $R(m, n)$ let us say. The corresponding number-theoretic theorem states that for each natural number m there is a natural number n such that $R(m, n)$ is true.

The systems in W which are not valid are those in which a formula of the form (8.2) is provable, but at the same time there is a natural number, m say, such that, for each natural number n, $R(m, n)$ is false. This means to say that $\sim \mathfrak{A}[f^{(m)}0, f^{(n)}0]$ is provable for each natural number n. Since (8.2) is provable, $(\exists x_0)\,\mathfrak{A}[f^{(m)}0, y_0]$ is provable, so that

$$(\exists y_0)\,\mathfrak{A}[f^{(m)}0, y_0], \quad \sim \mathfrak{A}[f^{(m)}0, 0], \quad \sim \mathfrak{A}[f^{(m)}0, f0], \quad \ldots \tag{8.3}$$

are all provable in the system. We may simplify (8.3). For a given m we may prove a formula of the form $\mathfrak{A}[f^{(m)}0, y_0] \equiv \mathfrak{B}[y_0]$ in P, where $\mathfrak{B}[x_0]$ is a recursion formula. Thus we find that a necessary and sufficient condition for a system of W to be valid is that for no recursion formula $\mathfrak{B}[x_0]$ are all of the formulae

$$(\exists x_0)\,\mathfrak{B}[x_0], \quad \sim \mathfrak{B}[0], \quad \sim \mathfrak{B}[f0], \quad \ldots \tag{8.4}$$

provable. An important consequence of this is that, if

$$\mathfrak{A}_1[x_0], \quad \mathfrak{A}_2[x_0], \quad \ldots, \quad \mathfrak{A}_n[x_0]$$

are recursion formulae, if

$$(\exists x_0)\,\mathfrak{A}_1[x_0] \vee \ldots \vee (\exists x_0)\,\mathfrak{A}_n[x_0] \tag{8.5}$$

is provable in C, and C is valid, then we can prove $\mathfrak{A}_r[f^{(a)}0]$ in C for some natural numbers r, a, where $1 \leqslant r \leqslant n$. Let us define $\mathfrak{D}_r$ to be the formula

$$(\exists x_0)\,\mathfrak{A}_1[x_0] \vee \ldots \vee (\exists x_0)\,\mathfrak{A}_r[x_0]$$

and then define $\mathfrak{E}_r[x_0]$ recursively by the condition that $\mathfrak{E}_1[x_0]$ is $\mathfrak{A}_1[x_0]$ and $\mathfrak{E}_{r+1}[x_0]$ be $\mathfrak{E}_{\mathfrak{E}_r,\, \mathfrak{A}_{r+1}}[x_0]$. Now I say that

$$\mathfrak{D}_r \supset (\exists x_0)\,\mathfrak{E}_r[x_0] \tag{8.6}$$

is provable for $1 \leqslant r \leqslant n$. It is clearly provable for $r = 1$: suppose it to be provable for a given r. We can prove

$$(y_0)(\exists x_0)\,\mathrm{Db}[x_0, y_0]$$

and

$$(y_0)(\exists x_0)\,\mathrm{Db}[fx_0, fy_0],$$

from which we obtain

$$\mathfrak{E}_r[y_0] \supset (\exists x_0)\big((\mathrm{Db}[x_0, y_0] \,.\, \mathfrak{E}_r[y_0]) \vee (\mathrm{Db}[fx_0, fy_0] \,.\, \mathfrak{A}_{r+1}[y_0])\big)$$

and

$$\mathfrak{A}_{r+1}[y_0] \supset (\exists x_0)\big((\mathrm{Db}[x_0, y_0] \,.\, \mathfrak{E}_r[y_0]) \vee (\mathrm{Db}[fx_0, fy_0] \,.\, \mathfrak{A}_{r+1}[y_0])\big).$$

These together with (8.1) yield

$$(\exists y_0)\,\mathfrak{E}_r[y_0] \vee (\exists y_0)\,\mathfrak{A}_{r+1}[y_0] \supset (\exists x_0)\,\mathfrak{C}_{(\mathfrak{E}_r,\, \mathfrak{A}_{r+1}}[x_0],$$

which is sufficient to prove (8.6) for $r+1$. Now, since (8.5) is provable in C, $(\exists x_0)\,\mathfrak{E}_n[x_0]$ must also be provable, and, since C is valid, this means that $\mathfrak{E}_n[f^{(m)}0]$ must be provable for some natural number m. From (8.1) and the definition of $\mathfrak{E}_n[x_0]$ we see that this implies that $\mathfrak{A}_r[f^{(a)}0]$ is provable for some natural numbers a and r, $1 \leqslant r \leqslant n$.

To any system C of W we can assign a primitive recursive relation $P_C(m, n)$ with the intuitive meaning "m is the G.R. of a proof of the formula whose G.R. is n". We call the corresponding recursion formula $\mathrm{Proof}_C[x_0, y_0]$ (*i.e.* $\mathrm{Proof}_C[f^{(m)}0, f^{(n)}0]$ is provable when $P_C(m, n)$ is true, and its negation is provable otherwise). We can now explain what is the relation of a system C' to its predecessor C. The set of axioms which we adjoin to P to obtain C' consists of those adjoined in obtaining C, together with all formulae of the form

$$(\exists x_0)\ \mathrm{Proof}_C[x_0, f^{(m)}0] \supset \mathfrak{F}, \qquad (8.7)$$

where m is the G.R. of $\mathfrak{F}$.

We want to show that a contradiction can be obtained by assuming C' to be invalid but C to be valid. Let us suppose that a set of formulae of the form (8.4) is provable in C'. Let $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_k$ be those axioms of C' of the form (8.7) which are used in the proof of $(\exists x_0)\,\mathfrak{B}[x_0]$. We may suppose that none of them is provable in C. Then by the deduction theorem we see that

$$(\mathfrak{A}_1 \,.\, \mathfrak{A}_2 \ldots \mathfrak{A}_k) \supset (\exists x_0)\,\mathfrak{B}[x_0] \qquad (8.8)$$

is provable in C. Let $\mathfrak{A}_l$ be $(\exists x_0)\ \mathrm{Proof}_C[x_0, f^{(m_l)}0] \supset \mathfrak{F}_l$. Then from (8.8) we find that

$$(\exists x_0)\ \mathrm{Proof}_C[x_0, f^{(m_1)}0] \vee \ldots \vee (\exists x_0)\ \mathrm{Proof}_C[x_0, f^{(m_k)}0] \vee (\exists x_0)\,\mathfrak{B}[x_0]$$

is provable in C. It follows from a result which we have just proved that either $\mathfrak{B}[f^{(c)}0]$ is provable for some natural number c, or else $\mathrm{Proof}_C[f^{(n)}0, f^{(m_l)}0]$ is provable in C for some natural number u and some l, $1 \leqslant l \leqslant k$: but this would mean that $\mathfrak{F}_l$ is provable in C (this is one of the points where we assume the validity of C) and therefore also in C', contrary to hypothesis. Thus $\mathfrak{B}[f^{(c)}0]$ must be provable in C'; but we are also assuming $\sim\mathfrak{B}[f^{(c)}0]$ to be provable in C'. There is therefore a contradiction in C'. Let us suppose that the axioms $\mathfrak{A}_1'$, ..., $\mathfrak{A}_{k'}'$, of the form (8.7), when adjoined to C are sufficient to obtain the contradiction and that none of these axioms is that provable in C. Then

$$\sim\mathfrak{A}_1' \vee \sim\mathfrak{A}_2' \vee \ldots \vee \sim\mathfrak{A}_{k'}'$$

is provable in C, and if $\mathfrak{A}_l'$ is $(\exists x_0)\ \mathrm{Proof}_C[x_0, f^{(m_l')}0] \supset \mathfrak{F}_l'$ then

$$(\exists x_0]\ \mathrm{Proof}_C[x_0, f^{(m_1')}0] \vee \ldots \vee (\exists x_0)\ \mathrm{Proof}\,[x_0, f^{(m_{k'}')}0]$$

is provable in C. But, by repetition of a previous argument, this means that $\mathfrak{A}_l'$ is provable for some l, $1 \leqslant l \leqslant k'$, contrary to hypothesis. This is the required contradiction.

We may now construct an ordinal logic in the manner described on pp. 190–193. We shall, however, carry out the construction in rather more detail, and with some modifications appropriate to the particular case. Each system C of our set W may be described by means of a W.F.F. M_C which enumerates the G.R.'s of the axioms of C. There is a W.F.F. E such that, if a is the G.R. of some proposition $\mathfrak{F}$, then $E(M_C, \mathbf{a})$ is convertible to the G.R. of

$$(\exists x_0)\ \mathrm{Proof}_C[x_0, f^{(a)}0] \supset \mathfrak{F}.$$

If $\mathbf{a}$ is not the G.R. of any proposition in P, then $E(M_C, \mathbf{a})$ is to be convertible to the G.R. of $0=0$. From E we obtain a W.F.F. K such that $K(M_C, 2\mathbf{n}+1)$ conv $M_C(\mathbf{n})$, $K(M_C, 2\mathbf{n})$ conv $E(M_C, \mathbf{n})$. The successor system C' is defined by $K(M_C)$ conv M_C'. Let us choose a formula G such that $G(M_C, \mathrm{A})$ conv 2 if and only if the number-theoretic theorem equivalent to "A is dual" is provable in C. Then we define Λ_P by

$$\Lambda_P \to \lambda wa\,.\,\Gamma\Big(\lambda y\,.\,G\Big(\mathrm{Ck}\Big(\mathrm{Tn}(w, y),\ \lambda mn\,.\,m\Big(\varpi(2, n), \varpi(3, n)\Big), K, M_P\Big)\Big), a\Big).$$

This is an ordinal logic provided that P is valid.

Another ordinal logic of this type has in effect been introduced by Church†. Superficially this ordinal logic seems to have no more in common with Λ_P than that they both arise by the method which we have described, which uses C-K ordinal formulae. The initial systems are entirely different. However, in the relation between C and C' there is an interesting analogy. In Church's method the step from C to C' is performed by means of subsidiary axioms of which the most important (Church [**2**], p. 88, 1_m) is almost a direct translation into his symbolism of the rule that we may take any formula of the form (8.4) as an axiom. There are other extra axioms, however, in Church's system, and it is therefore not unlikely that it is in some respects more complete than Λ_P.

There are other types of ordinal logic, apparently quite unrelated to the type that we have so far considered. I have in mind two types of ordinal logic, both of which can be best described directly in terms of ordinal formulae without any reference to C-K ordinal formulae. I shall describe here a specimen Λ_H of one of these types of ordinal logic. Ordinal logics of this kind were first considered by Hilbert (Hilbert [**1**], 183ff), and have also been used by Tarski (Tarski [**1**], 395ff); see also Gödel [**1**], foot-note 48^a.

Suppose that we have selected a particular ordinal formula Ω. We shall construct a modification P_Ω of the system P of Gödel (see foot-note † on p. 193. We shall say that a natural number n is a *type* if it is either even or $2p-1$, where $\Omega(\mathbf{p}, \mathbf{p})$ conv 3. The definition of a variable in P is to be modified by the condition that the only admissible subscripts are to be the types in our sense. Elementary expressions are then defined as in P: in particular the definition of an elementary expression of type 0 is unchanged. An elementary formula is defined to be a sequence of symbols of the form $\mathfrak{A}_m \mathfrak{A}_n$, where $\mathfrak{A}_m$, $\mathfrak{A}_n$ are elementary expressions of types m, n satisfying one of the conditions (a), (b), (c).

(a) m and n are both even and m exceeds n,

(b) m is odd and n is even,

(c) $m = 2p-1$, $n = 2q-1$, and $\Omega(\mathbf{p}, \mathbf{q})$ conv 2.

With these modifications the formal development of P_Ω is the same as that of P. We want, however, to have a method of associating number-theoretic theorems with certain of the formulae of P_Ω. We cannot take over directly the association which we used in P. Suppose that G is a formula in P interpretable as a number-theoretic theorem in the way described in the course of constructing Λ_P (p. 193). Then, if every type suffix in G is doubled, we shall obtain a formula in P_Ω which is to be interpreted as the same number-theoretic theorem. By the method of § 6 we can now obtain from P_Ω a formula L_Ω which is a logic formula if P_Ω is valid; in fact, given $\mathbf{\Omega}$ there is a method of obtaining L_Ω, so that there is a formula Λ_H such that $\Lambda_H(\mathbf{\Omega})$ conv L_Ω for each ordinal formula $\mathbf{\Omega}$.

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† In outline Church [1], 279–280. In greater detail Church [2], Chap. X.

Having now familiarized ourselves with ordinal logics by means of these examples we may begin to consider general questions concerning them.

9. *Completeness questions.*

The purpose of introducing ordinal logics was to avoid as far as possible the effects of Gödel's theorem. It is a consequence of this theorem, suitably modified, that it is impossible to obtain a complete logic formula, or (roughly speaking now) a complete system of logic. We were able, however, from a given system to obtain a more complete one by the adjunction as axioms of formulae, seen intuitively to be correct, but which the Gödel theorem shows are unprovable[†] in the original system; from this we obtained a yet more complete system by a repetition of the process, and so on. We found that the repetition of the process gave us a new system for each C-K ordinal formula. We should like to know whether this process suffices, or whether the system should be extended in other ways as well. If it were possible to determine about a W.F.F. in normal form whether it was an ordinal formula, we should know for certain that it was necessary to make extensions in other ways. In fact for any ordinal formula $\mathbf{\Lambda}$ it would then be possible to find a single logic formula $\mathbf{L}$ such that, if $\mathbf{\Lambda}(\mathbf{\Omega}, \mathbf{A})$ conv 2 for some ordinal formula $\mathbf{\Omega}$, then $\mathbf{L}(\mathbf{A})$ conv 2. Since $\mathbf{L}$ must be incomplete, there must be formulae $\mathbf{A}$ for which $\mathbf{\Lambda}(\mathbf{\Omega}, \mathbf{A})$ is not convertible to 2 for any ordinal formula $\mathbf{\Omega}$. However, in view of the fact, proved in § 7, that there is no method of determining about a formula in normal form whether it is an ordinal formula, the case does not arise, and there is still a possibility that some ordinal logics may be complete in some sense. There is a quite natural way of defining completeness.

Definition of completeness of an ordinal logic. We say that an ordinal logic $\mathbf{\Lambda}$ is complete if corresponding to each dual formula $\mathbf{A}$ there is an ordinal formula $\mathbf{\Omega_A}$ such that $\mathbf{\Lambda}(\mathbf{\Omega_A}, \mathbf{A})$ conv 2.

[†] In the case of P we adjoined all of the axioms $(\exists x_0)$ Proof $[x_0, f^{(m)}0] \supset \mathfrak{F}$, where m is the G.R. of $\mathfrak{F}$; the Gödel theorem shows that *some* of them are unprovable in P.

As has been explained in § 2, the reference in the definition to the existence of $\Omega_{\mathbf{A}}$ for each $\mathbf{A}$ is to be understood in the same naïve way as any reference to existence in mathematics.

There is room for modification in this definition: we might require that there is a formula $\mathbf{X}$ such that $\mathbf{X}(\mathbf{A}) \operatorname{conv} \Omega_{\mathbf{A}}$, $\mathbf{X}(\mathbf{A})$ being an ordinal formula whenever $\mathbf{A}$ is dual. There is no need, however, to discuss the relative merits of these two definitions, because in all cases in which we prove an ordinal logic to be complete we shall prove it to be complete even in the modified sense; but in cases in which we prove an ordinal logic to be incomplete, we use the definition as it stands.

In the terminology of § 6, Λ is complete if the class of logics $\Lambda(\Omega)$ is complete when $\mathbf{\Omega}$ runs through all ordinal formulae.

There is another completeness property which is related to this one. Let us for the moment describe an ordinal logic Λ as *all inclusive* if to each logic formula $\mathbf{L}$ there corresponds an ordinal formula $\Omega_{(\mathbf{L})}$ such that $\Lambda(\mathbf{\Omega}_{(\mathbf{L})})$ is as complete as $\mathbf{L}$. Clearly every all inclusive ordinal logic is complete; for, if $\mathbf{A}$ is dual, then $\delta(\mathbf{A})$ is a logic with $\mathbf{A}$ in its extent. But, if Λ is complete and

$$\mathrm{Ai} \rightarrow \lambda kw\,.\,\Gamma\Big(\lambda ra\,.\,\delta\Big(4,\ \delta\Big(2,\ k\Big(w,\ V\Big(\mathrm{Nm}(r)\Big)\Big)\Big) + \delta\Big(2,\ \mathrm{Nm}(r,\ a)\Big)\Big)\Big),$$

then $\mathrm{Ai}(\Lambda)$ is an all inclusive ordinal logic. For, if $\mathbf{A}$ is in the extent of $\Lambda(\Omega_{\mathbf{A}})$ for each $\mathbf{A}$, and we put $\Omega_{(\mathbf{L})} \rightarrow \Omega_{V(\mathbf{L})}$, then I say that, if $\mathbf{B}$ is in the extent of $\mathbf{L}$, it must be in the extent of $\mathrm{Ai}(\Lambda,\ \mathbf{\Omega}_{(\mathbf{L})})$. In fact, we see that $\mathrm{Ai}(\Lambda,\ \mathbf{\Omega}_{V(\mathbf{L})},\ \mathbf{B})$ is convertible to

$$\Gamma\Big(\lambda ra\,.\,\delta\Big(4,\ \delta\Big(2,\ \Lambda\Big(\Omega_{V(\mathbf{L})},\ V\Big(\mathrm{Nm}(r)\Big)\Big)\Big) + \delta\Big(2,\ \mathrm{Nm}(r,\ a)\Big)\Big),\ \mathbf{B}\Big).$$

For suitable n, $\mathrm{Nm}(\mathbf{n}) \operatorname{conv} \mathbf{L}$ and then

$$\Lambda\Big(\Omega_{V(\mathbf{L})},\ V\Big(\mathrm{Nm}(\mathbf{n})\Big)\Big) \operatorname{conv} 2,$$

$$\mathrm{Nm}(\mathbf{n},\ \mathbf{B}) \operatorname{conv} 2,$$

and therefore, by the properties of Γ and δ

$$\mathrm{Ai}(\Lambda,\ \Omega_{V(\mathbf{L})},\ \mathbf{B}) \operatorname{conv} 2.$$

Conversely $\mathrm{Ai}(\Lambda,\ \Omega_{V(\mathbf{L})},\ \mathbf{B})$ can be convertible to 2 only if both $\mathrm{Nm}(\mathbf{n},\ \mathbf{B})$

and $\Lambda\Big(\Omega_{V(\mathbf{L})}, V\big(\mathrm{Nm}(\mathbf{n})\big)\Big)$ are convertible to 2 for some positive integer n; but, if $\Lambda\Big(\Omega_{V(\mathbf{L})}, V\big(\mathrm{Nm}(\mathbf{n})\big)\Big)$ conv 2, then $\mathrm{Nm}(\mathbf{n})$ must be a logic, and, since $\mathrm{Nm}(\mathbf{n}, \mathbf{B})$ conv 2, $\mathbf{B}$ must be dual.

It should be noticed that our definitions of completeness refer only to number-theoretic theorems. Although it would be possible to introduce formulae analogous to ordinal logics which would prove more general theorems than number-theoretic ones, and have a corresponding definition of completeness, yet, if our theorems are too general, we shall find that our (modified) ordinal logics are never complete. This follows from the argument of § 4. If our "oracle" tells us, not whether any given number-theoretic statement is true, but whether a given formula is an ordinal formula, the argument still applies, and we find that there are classes of problem which cannot be solved by a uniform process even with the help of this oracle. This is equivalent to saying that there is no ordinal logic of the proposed modified type which is complete with respect to these problems. This situation becomes more definite if we take formulae satisfying conditions (a)–(e), (f') (as described at the end of § 12) instead of ordinal formulae; it is then not possible for the ordinal logic to be complete with respect to any class of problems more extensive than the number-theoretic problems.

We might hope to obtain some intellectually satisfying system of logical inference (for the proof of number-theoretic theorems) with some ordinal logic. Gödel's theorem shows that such a system cannot be wholly mechanical; but with a complete ordinal logic we should be able to confine the non-mechanical steps entirely to verifications that particular formulae are ordinal formulae.

We might also expect to obtain an interesting classification of number-theoretic theorems according to "depth". A theorem which required an ordinal α to prove it would be deeper than one which could be proved by the use of an ordinal β less than α. However, this presupposes more than is justified. We now define

Invariance of ordinal logics. An ordinal logic Λ is said to be *invariant up to* an ordinal α if, whenever Ω, Ω' are ordinal formulae representing the same ordinal less than α, the extent of $\Lambda(\Omega)$ is identical with the extent of $\Lambda(\Omega')$. An ordinal logic is *invariant* if it is invariant up to each ordinal represented by an ordinal formula.

Clearly the classification into depths presupposes that the ordinal logic used is invariant.

Among the questions that we should now like to ask are

(*a*) Are there any complete ordinal logics?

(*b*) Are there any complete invariant ordinal logics?

To these we might have added "are all ordinal logics complete?"; but this is trivial; in fact, there are ordinal logics which do not suffice to prove any number-theoretic theorems whatever.

We shall now show that (*a*) must be answered affirmatively. In fact, we can write down a complete ordinal logic at once. Put

$$\mathrm{Od} \rightarrow \lambda a\,.\,\{\lambda fmn\,.\,\mathrm{Dt}\big(f(m), f(n)\big)\}\Big(\lambda s\,.\,\mathscr{P}\Big(\lambda r\,.\,r\big(I, a(s)\big), 1, s\Big)\Big)$$

and $$\mathrm{Comp} \rightarrow \lambda wa\,.\,\delta\big(w, \mathrm{Od}(a)\big).$$

I shall show that Comp is a complete ordinal logic.

For if, Comp $(\boldsymbol{\Omega}, \mathbf{A})$ conv 2, then

$$\boldsymbol{\Omega} \text{ conv } \mathrm{Od}\ \ (\mathbf{A})$$

$$\text{conv } \lambda mn\,.\,\mathrm{Dt}\Big(\mathscr{P}\Big(\lambda r\,.\,r\big(I, \mathbf{A}(m)\big), 1, m\Big),\ \mathscr{P}\Big(\lambda r\,.\,r\big(I, \mathbf{A}(n)\big), 1, n\Big)\Big).$$

$\boldsymbol{\Omega}(\mathbf{m}, \mathbf{n})$ has a normal form if $\boldsymbol{\Omega}$ is an ordinal formula, so that then

$$\mathscr{P}\Big(\lambda r\,.\,r\big(I; \mathbf{A}(\mathbf{m})\big), 1\Big)$$

has a normal form; this means that $\mathbf{r}\big(I, \mathbf{A}(\mathbf{m})\big)$ conv 2 some *r*, *i.e.* $\mathbf{A}(\mathbf{m})$ conv 2. Thus, if Comp$(\boldsymbol{\Omega}, \mathbf{A})$ conv 2 and $\boldsymbol{\Omega}$ is an ordinal formula, then **A** is dual. Comp is therefore an ordinal logic. Now suppose conversely that **A** is dual. I shall show that Od(**A**) is an ordinal formula representing the ordinal ω. For

$$\mathscr{P}\Big(\lambda r\,.\,r\big(I, \mathbf{A}(\mathbf{m})\big), 1, \mathbf{m}\Big) \text{ conv } \mathscr{P}\big(\lambda r\,.\,r(I, 2), 1, \mathbf{m}\big)$$

$$\text{conv } 1(\mathbf{m}) \text{ conv } \mathbf{m},$$

$$\mathrm{Od}(\mathbf{A}, \mathbf{m}, \mathbf{n}) \text{ conv } \mathrm{Dt}(\mathbf{m}, \mathbf{n}),$$

i.e. Od(**A**) is an ordinal formula representing the same ordinal as Dt. But

$$\mathrm{Comp}\big(\mathrm{Od}(\mathbf{A}), \mathbf{A}\big) \text{ conv } \delta\big(\mathrm{Od}(\mathbf{A}), \mathrm{Od}(\mathbf{A})\big) \text{ conv } 2.$$

This proves the completeness of Comp.

Of course Comp is not the kind of complete ordinal logic that we should really wish to use. The use of Comp does not make it any easier to see that $\mathbf{A}$ is dual. In fact, if we really want to use an ordinal logic a proof, of completeness for that particular ordinal logic will be of little value; the ordinals given by the completeness proof will not be ones which can easily be seen intuitively to be ordinals. The only value in a completeness proof of this kind would be to show that, if any objection is to be raised against an ordinal logic, it must be on account of something more subtle than incompleteness.

The theorem of completeness is also unexpected in that the ordinal formulae used are all formulae representing ω. This is contrary to our intentions in constructing Λ_P for instance; implicitly we had in mind large ordinals expressed in a simple manner. Here we have small ordinals expressed in a very complex and artificial way.

Before trying to solve the problem (*b*), let us see how far Λ_P and Λ_H are invariant. We should certainly not expect Λ_P to be invariant, since the extent of $\Lambda_P(\Omega)$ will depend on whether Ω is convertible to a formula of the form $H(\mathbf{A})$: but suppose that we call an ordinal logic Λ "C-K invariant up to α" if the extent of $\Lambda\big(H(\mathbf{A})\big)$ is the same as the extent of $\Lambda\big(H(\mathbf{B})\big)$ whenever $\mathbf{A}$ and $\mathbf{B}$ are C-K ordinal formulae representing the same ordinal less than α. How far is Λ_P C-K invariant? It is not difficult to see that it is C-K invariant up to any finite ordinal, that is to say up to ω. It is also C-K invariant up to $\omega+1$, as follows from the fact that the extent of

$$\Lambda_P\Big(H\big(\lambda ufx\,.\,u(\mathbf{R})\big)\Big)$$

is the set-theoretic sum of the extents of

$$\Lambda_P\Big(H\big(\lambda ufx\,.\,\mathbf{R}(1)\big)\Big),\quad \Lambda_P\Big(H\big(\lambda ufx\,.\,\mathbf{R}(2)\big)\Big),\quad \ldots .$$

However, there is no obvious reason for believing that it is C-K invariant up to $\omega+2$, and in fact it is demonstrable that this is not the case (see the end of this section). Let us find out what happens if we try to prove that the extent of

$$\Lambda_P\Big(H\Big(\mathrm{Suc}\big(\lambda ufx\,.\,u(\mathbf{R}_1)\big)\Big)\Big)$$

is the same as the extent of

$$\Lambda_P\Big(H\Big(\mathrm{Suc}\big(\lambda ufx\,.\,u(\mathbf{R}_2)\big)\Big)\Big),$$

where $\lambda ufx \, . \, u(\mathbf{R}_1)$ and $\lambda ufx \, . \, u(\mathbf{R}_2)$ are two C-K ordinal formulae representing ω. We should have to prove that a formula interpretable as a number-theoretic theorem is provable in $C\Big[\mathrm{Suc}\Big(\lambda ufx \, . \, u(\mathbf{R}_1)\Big)\Big]$ if, and only if. it is provable in $C\Big[\mathrm{Suc}\Big(\lambda ufx \, . \, u(\mathbf{R}_2)\Big)\Big]$. Now $C\Big[\mathrm{Suc}\Big(\lambda ufx \, . \, u(\mathbf{R}_1)\Big)\Big]$ is obtained from $C[\lambda ufx \, . \, u(\mathbf{R}_1)]$ by adjoining all axioms of the form

$$(\exists x_0)\ \mathrm{Proof}_{C[\lambda ufx \, . \, u(\mathbf{R}_1)]}[x_0, f^{(m)}0) \supset \mathfrak{F}, \qquad (9.1)$$

where m is the G.R. of $\mathfrak{F}$, and $C\Big[\mathrm{Suc}\Big(\lambda ufx \, . \, u(\mathbf{R}_2)\Big)\Big]$ is obtained from $C[\lambda ufx \, . \, u(\mathbf{R}_2)]$ by adjoining all axioms of the form

$$(\exists x_0)\ \mathrm{Proof}_{C[\lambda ufx \, . \, u(\mathbf{R}_2)]}[x_0, f^{(m)}0] \supset \mathfrak{F}. \qquad (9.2)$$

The axioms which must be adjoined to P to obtain $C[\lambda ufx \, . \, u(\mathbf{R}_1)]$ are essentially the same as those which must be adjoined to obtain the system $C[\lambda ufx \, . \, u(\mathbf{R}_2)]$: however the *rules of procedure which have to be applied before these axioms can be written down are in general quite different in the two cases.* Consequently (9.1) and (9.2) are quite different axioms, and there is no reason to expect their consequences to be the same. A proper understanding of this will make our treatment of question (*b*) much more intelligible. See also footnote ‡ on page 193.

Now let us turn to Λ_H. This ordinal logic is invariant. Suppose that Ω, Ω' represent the same ordinal, and suppose that we have a proof of a number-theoretic theorem G in P_Ω. The formula expressing the number-theoretic theorem does not involve any odd types. Now there is a one-one correspondence between the odd types such that if $2m-1$ corresponds to $2m'-1$ and $2n-1$ to $2n'-1$ then $\Omega(\mathbf{m}, \mathbf{n})$ conv 2 implies $\Omega'(\mathbf{m}', \mathbf{n}')$ conv 2. Let us modify the odd type-subscripts occurring in the proof of G, replacing each by its mate in the one-one correspondence. There results a proof in $P_{\Omega'}$ with the same end formula G. That is to say that if G is provable in P_Ω it is provable in $P_{\Omega'}$. Λ_H is invariant.

The question (*b*) must be answered negatively. Much more can be proved, but we shall first prove an even weaker result which can be established very quickly, in order to illustrate the method.

I shall prove that an ordinal logic Λ cannot be invariant and have the property that the extent of $\Lambda(\Omega)$ is a strictly increasing function of the ordinal represented by Ω. Suppose that Λ has these properties; then we shall obtain a contradiction. Let $\mathbf{A}$ be a W.F.F. in normal form and without free variables, and consider the process of carrying out conversions on $\mathbf{A}(1)$ until we have shown it convertible to 2, then converting $\mathbf{A}(2)$ to 2, then $\mathbf{A}$ (3) and so on: suppose that after r steps we are still performing the conversion on $\mathbf{A}(\mathbf{m}_r)$. There is a formula Jh such that $\mathrm{Jh}(\mathbf{A}, \mathbf{r})$ conv $\mathbf{m}_r$ for each positive integer r. Now let Z be a formula such that, for each positive integer n, $Z(\mathbf{n})$ is an ordinal formula representing ω^n, and suppose $\mathbf{B}$ to be a member of the extent of $\Lambda\Big(\mathrm{Suc}\big(\mathrm{Lim}(Z)\big)\Big)$ but not of the extent of $\Lambda\big(\mathrm{Lim}(Z)\big)$. Put

$$\mathbf{K}^* \to \lambda a . \Lambda\Bigg(\mathrm{Suc}\Big(\mathrm{Lim}\Big(\lambda r . Z\big(\mathrm{Jh}(a, r)\big)\Big)\Big), \ \mathbf{B}\Bigg);$$

then $\mathbf{K}^*$ is a complete logic. For, if $\mathbf{A}$ is dual, then

$$\mathrm{Suc}\Big(\mathrm{Lim}\Big(\lambda r . Z\big(\mathrm{Jh}(\mathbf{A}, r)\big)\Big)\Big)$$

represents the ordinal $\omega^\omega + 1$, and therefore $\mathbf{K}^*(\mathbf{A})$ conv 2; but, if $\mathbf{A}(\mathbf{c})$ is not convertible to 2, then

$$\mathrm{Suc}\Big(\mathrm{Lim}\Big(\lambda r . Z\big(\mathrm{Jh}(\mathbf{A}, r)\big)\Big)\Big)$$

represents an ordinal not exceeding $\omega^c + 1$, and $\mathbf{K}^*(\mathbf{A})$ is therefore not convertible to 2. Since there are no complete logic formulae, this proves our assertion.

We may now prove more powerful results.

Incompleteness theorems. (A) If an ordinal logic Λ is invariant up to an ordinal α, then for any ordinal formula Ω representing an ordinal β, $\beta < \alpha$, the extent of $\Lambda(\Omega)$ is contained in the (set-theoretic) sum of the extents of the logics $\Lambda(\mathbf{P})$, where $\mathbf{P}$ is finite.

(B) If an ordinal logic $\mathbf{\Lambda}$ is C-K invariant up to an ordinal α, then for any C-K ordinal formula $\mathbf{A}$ representing an ordinal β, $\beta < \alpha$, the extent of $\Lambda\big(H(\mathbf{A})\big)$ is contained in the (set-theoretic) sum of the extents of the logics $\Lambda\big(H(\mathbf{F})\big)$, where $\mathbf{F}$ is a C-K ordinal formula representing an ordinal less than ω^2.

Proof of (A). It is sufficient to prove that, if Ω represents an ordinal γ, $\omega \leqslant \gamma < \alpha$, then the extent of $\Lambda(\Omega)$ is contained in the set-theoretic sum of the extents of the logics $\Lambda(\Omega')$, where Ω' represents an ordinal less than γ. The ordinal γ must be of the form $\gamma_0 + \rho$, where ρ is finite and represented by $\mathbf{P}$ say, and γ_0 is not the successor of any ordinal and is not less than ω. There are two cases to consider; $\gamma_0 = \omega$ and $\gamma_0 \geqslant 2\omega$. In each of them we shall obtain a contradiction from the assumption that there is a W.F.F.

B such that $\Lambda(\Omega, \mathbf{B})$ conv 2 whenever Ω represents γ, but is not convertible to 2 if Ω represents a smaller ordinal. Let us take first the case $\gamma_0 \geqslant 2\omega$. Suppose that $\gamma_0 = \omega + \gamma_1$, and that Ω_1 is an ordinal formula representing γ_1. Let **A** be any W.F.F. with a normal form and no free variables, and let Z be the class of those positive integers which are exceeded by all integers n for which $\mathbf{A}(\mathbf{n})$ is not convertible to 2. Let E be the class of integers $2p$ such that $\Omega(\mathbf{p}, \mathbf{n})$ conv 2 for some n belonging to Z. The class E, together with the class Q of all odd integers, is constructively enumerable. It is evident that the class can be enumerated with repetitions, and since it is infinite the required enumeration can be obtained by striking out the repetitions. There is, therefore, a formula En such that $\mathrm{En}(\Omega, \mathbf{A}, \mathbf{r})$ runs through the formulae of the class $E+Q$ without repetitions as r runs through the positive integers. We define

$$\mathrm{Rt} \rightarrow \lambda wamn\,.\,\mathrm{Sum}\Big(\mathrm{Dt},\ w,\ \mathrm{En}\ (w,\ a,\ m),\ \mathrm{En}(w,\ a,\ n)\Big).$$

Then $\mathrm{Rt}(\Omega_1, \mathbf{A})$ is an ordinal formula which represents γ_0 if **A** is dual, but a smaller ordinal otherwise. In fact

$$\mathrm{Rt}(\Omega_1,\ \mathbf{A},\ \mathbf{m},\ \mathbf{n}) \text{ conv } \{\mathrm{Sum}(\mathrm{Dt},\ \Omega_1)\}\Big(\mathrm{En}(\Omega_1,\ \mathbf{A},\ \mathbf{m}),\ \mathrm{En}(\Omega_1,\ \mathbf{A},\ \mathbf{n})\Big).$$

Now, if **A** is dual, $E+Q$ includes all integers m for which

$$\{\mathrm{Sum}(\mathrm{Dt},\ \Omega_1)\}\,(\mathbf{m},\ \mathbf{m}) \text{ conv } 3.$$

(This depends on the particular form that we have chosen for the formula Sum.) Putting "$\mathrm{En}(\Omega_1, \mathbf{A}, \mathbf{p})$ conv $\mathbf{q}$" for $M(p, q)$, we see that condition (7.4) is satisfied, so that $\mathrm{Rt}(\Omega_1, \mathbf{A})$ is an ordinal formula representing γ_0. But, if **A** is not dual, the set $E+Q$ consists of all integers m for which

$$\{\mathrm{Sum}(\mathrm{Dt},\ \Omega_1)\}\,(\mathbf{m},\ \mathbf{r}) \text{ conv } 2,$$

where r depends only on **A**. In this case $\mathrm{Rt}(\Omega_1, \mathbf{A})$ is an ordinal formula representing the same ordinal as $\mathrm{Inf}\Big(\mathrm{Sum}\,(\mathrm{Dt},\ \Omega_1),\ \mathbf{r}\Big)$, and this is smaller than γ_0. Now consider **K**:

$$\mathbf{K} \rightarrow \lambda a\,.\,\Lambda\Big(\mathrm{Sum}\,\Big(\mathrm{Rt}(\Omega_1,\ \mathbf{A}),\ \mathbf{P}\Big),\ \mathbf{B}\Big).$$

If **A** is dual, $\mathbf{K}(\mathbf{A})$ is convertible to 2 since $\mathrm{Sum}\Big(\mathrm{Rt}(\Omega_1,\ \mathbf{A}),\ \mathbf{P}\Big)$ represents γ. But, if **A** is not dual, it is not convertible to 2, since $\mathrm{Sum}\Big(\mathrm{Rt}(\Omega_1,\ \mathbf{A}),\ \mathbf{P}\Big)$ then represents an ordinal smaller than γ. In **K** we therefore have a complete logic formula, which is impossible.

Now we take the case $\gamma_0 = \omega$. We introduce a W.F.F. Mg such that if n is the D.N. of a computing machine $\mathcal{M}$, and if by the m-th complete configuration of $\mathcal{M}$ the figure 0 has been printed, then $\mathrm{Mg}(\mathbf{n}, \mathbf{m})$ is convertible to $\lambda pq . \mathrm{Al}\big(4(P, 2p+2q), 3, 4\big)$ (which is an ordinal formula representing the ordinal 1), but if 0 has not been printed it is convertible to $\lambda pq . p(q, I, 4)$ (which represents 0). Now consider

$$\mathbf{M} \to \lambda n . \Lambda\Big(\mathrm{Sum}\Big(\mathrm{Lim}\big(\mathrm{Mg}(n)\big), \mathbf{P}\Big), \mathbf{B}\Big).$$

If the machine never prints 0, then $\mathrm{Lim}\big(\lambda r . \mathrm{Mg}(\mathbf{n}, r)\big)$ represents ω and $\mathrm{Sum}\Big(\mathrm{Lim}\big(\mathrm{Mg}(\mathbf{n})\big), \mathbf{P}\Big)$ represents γ. This means that $\mathbf{M}(\mathbf{n})$ is convertible to 2. If, however, $\mathcal{M}$ never prints 0, $\mathrm{Sum}\Big(\mathrm{Lim}\big(\mathrm{Mg}(\mathbf{n})\big), \mathbf{P}\Big)$ represents a finite ordinal and $\mathbf{M}(\mathbf{n})$ is not convertible to 2. In $\mathbf{M}$ we therefore have means of determining about a machine whether it ever prints 0, which is impossible† (Turing [1], §8). This completes the proof of (A).

Proof of (B). It is sufficient to prove that, if $\mathbf{C}$ represents an ordinal γ, $\omega^2 \leqslant \gamma < \alpha$, then the extent of $\Lambda\big(H(\mathbf{C})\big)$ is included in the set-theoretic sum of the extents of $\Lambda\big(H(\mathbf{G})\big)$, where $\mathbf{G}$ represents an ordinal less than γ. We obtain a contradiction from the assumption that there is a formula $\mathbf{B}$ which is in the extent of $\Lambda\big(H(\mathbf{G})\big)$ if $\mathbf{G}$ represents γ, but not if it represents any smaller ordinal. The ordinal γ is of the form $\delta+\omega^2+\xi$, where $\xi < \omega^2$. Let $\mathbf{D}$ be a C-K ordinal formula representing δ and $\lambda ufx . \mathbf{Q}\big(u, f, \mathbf{A}(u, f, x)\big)$ one representing $\alpha+\xi$ whenever $\mathbf{A}$ represents α.

We now define a formula Hg. Suppose that $\mathbf{A}$ is a W.F.F. in normal form and without free variables; consider the process of carrying out conversions on $\mathbf{A}(1)$ until it is brought into the form 2, then converting $\mathbf{A}(2)$ to 2, then $\mathbf{A}(3)$, and so on. Suppose that at the r-th step of this process we are doing the n_r-th step in the conversion of $\mathbf{A}(\mathbf{m}_r)$. Thus, for instance, if $\mathbf{A}$ is not convertible to 2, m_r can never exceed 3. Then $\mathrm{Hg}(\mathbf{A}, \mathbf{r})$ is to be convertible to $\lambda f . f(\mathbf{m}_r, \mathbf{n}_r)$ for each positive integer r. Put

$$\mathrm{Sq} \to \lambda dmn . n\bigg(\mathrm{Suc}, m\Big(\lambda aufx . u\Big(\lambda y . y\big(\mathrm{Suc}, a(u, f, x)\big)\Big), d(u, f, x)\Big)\bigg),$$

$$\mathbf{M} \to \lambda aufx . \mathbf{Q}\Big(u, f, u\Big(\lambda y . \mathrm{Hg}\big(a, y, \mathrm{Sq}(\mathbf{D})\big)\Big)\Big),$$

$$\mathbf{K}_1 \to \lambda a . \Lambda\big(\mathbf{M}(a), \mathbf{B}\big),$$

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† This part of the argument can equally well be based on the impossibility of determining about two W.F.F. whether they are interconvertible. (Church [3], 363.)

then I say that $\mathbf{K}_1$ is a complete logic formula. $\mathrm{Sq}\,(\mathbf{D}, \mathbf{m}, \mathbf{n})$ is a C-K ordinal formula representing $\delta + m\omega + n$, and therefore $\mathrm{Hg}\big(\mathbf{A}, \mathbf{r}, \mathrm{Sq}(\mathbf{D})\big)$ represents an ordinal ζ_r which increases steadily with increasing r, and tends to the limit $\delta + \omega^2$ if $\mathbf{A}$ is dual. Further

$$\mathrm{Hg}\big(\mathbf{A}, \mathbf{r}, \mathrm{Sq}(\mathbf{D})\big) < \mathrm{Hg}\big(\mathbf{A}, S(\mathbf{r}), \mathrm{Sq}(\mathbf{D})\big)$$

for each positive integer r. Therefore $\lambda ufx\,.\,u\Big(\lambda y\,.\,\mathrm{Hg}\big(\mathbf{A}, y, \mathrm{Sq}(\mathbf{D})\big)\Big)$ is a C-K ordinal formula and represents the limit of the sequence $\zeta_1, \zeta_2, \zeta_3, \ldots$. This is $\delta + \omega^2$ if $\mathbf{A}$ is dual, but a smaller ordinal otherwise. Likewise $\mathbf{M}(\mathbf{A})$ represents γ if $\mathbf{A}$ is dual, but is a smaller ordinal otherwise. The formula $\mathbf{B}$ therefore belongs to the extent of $\Lambda\Big(H\big(\mathbf{M}(\mathbf{A})\big)\Big)$ if and only if $\mathbf{A}$ is dual, and this implies that $\mathbf{K}_1$ is a complete logic formula, as was asserted. But this is impossible and we have the required contradiction.

As a corollary to (A) we see that Λ_H is incomplete and in fact that the extent of $\Lambda_H(\mathrm{Dt})$ contains the extent of $\Lambda_H(\Omega)$ for any ordinal formula Ω. This result, suggested to me first by the solution of question (*b*), may also be obtained more directly. In fact, if a number-theoretic theorem can be proved in any particular P_Ω, it can also be proved in $P_{\lambda mn\,.\,m(n, I, 4)}$. The formulae describing number-theoretic theorems in P do not involve more than a finite number of types, type 3 being the highest necessary. The formulae describing the number-theoretic theorems in any P_Ω will be obtained by doubling the type subscripts. Now suppose that we have a proof of a number-theoretic theorem G in P_Ω and that the types occurring in the proof are among $0, 2, 4, 6, t_1, t_2, t_3, \ldots$. We may suppose that they have been arranged with all the even types preceding all the odd types, the even types in order of magnitude and the type $2m-1$ preceding $2n-1$ if $\Omega(\mathbf{m}, \mathbf{n})$ conv 2. Now let each t_r be replaced by $10+2r$ throughout the proof of G. We thus obtain a proof of G in $P_{\lambda mn\,.\,(n, I, 4)}$.

As with problem (*a*), the solution of problem (*b*) does not require the use of high ordinals [*e.g.* if we make the assumption that the extent of $\Lambda(\Omega)$ is a steadily increasing function of the ordinal represented by Ω we do not have to consider ordinals higher than $\omega + 2$]. However, if we restrict what we are to call ordinal formulae in some way, we shall have corresponding modified problems (*a*) and (*b*); the solutions will presumably be essentially the same, but will involve higher ordinals. Suppose, for example, that Prod is a W.F.F. with the property that $\mathrm{Prod}(\Omega_1, \Omega_2)$ is an ordinal formula representing $\alpha_1 \alpha_2$ when Ω_1, Ω_2 are ordinal formulae representing α_1, α_2 respectively, and suppose that we call a W.F.F. a 1-ordinal formula when it is convertible to the form $\mathrm{Sum}\Big(\mathrm{Prod}(\mathbf{\Omega}, \mathrm{Dt}), \mathbf{P}\Big)$, where $\mathbf{\Omega}$, $\mathbf{P}$ are ordinal formulae of which $\mathbf{P}$ represents a finite ordinal. We may define 1-ordinal logics, 1-completeness and 1-invariance in an obvious way, and obtain a solution of problem (*b*) which differs from the solution in the ordinary case in that the ordinals less than ω^2 take the place of the finite ordinals. More generally the cases that I have in mind are covered by the following theorem.

Suppose that we have a class V of formulae representing ordinals in some manner which we do not propose to specify definitely, and a subset† U of the class V such that:

(i) There is a formula $\mathbf{\Phi}$ such that if $\mathbf{T}$ enumerates a sequence of members of U representing an increasing sequence of ordinals, then $\mathbf{\Phi}(\mathbf{T})$ is a member of U representing the limit of the sequence.

(ii) There is a formula $\mathbf{E}$ such that $\mathbf{E}(\mathbf{m}, \mathbf{n})$ is a member of U for each pair of positive integers m, n and, if it represents $\epsilon_{m,n}$, then $\epsilon_{m,n} < \epsilon_{m',n'}$ if either $m < m'$ or $m = m'$, $n < n'$.

(iii) There is a formula $\mathbf{G}$ such that, if $\mathbf{A}$ is a member of U, then $\mathbf{G}(\mathbf{A})$ is a member of U representing a larger ordinal than does $\mathbf{A}$, and such that $\mathbf{G}\Big(\mathbf{E}(\mathbf{m}, \mathbf{n})\Big)$ always represents an ordinal not larger than $\epsilon_{m,n+1}$.

We define a V-ordinal logic to be a W.F.F. Λ such that $\Lambda(\mathbf{A})$ is a logic whenever $\mathbf{A}$ belongs to V. Λ is V-invariant if the extent of $\Lambda(\mathbf{A})$ depends only on the ordinal represented by $\mathbf{A}$. Then it is not possible for a V-ordinal logic Λ to be V-invariant and have the property that, if $\mathbf{C}_1$ represents a greater ordinal than $\mathbf{C}_2$ ($\mathbf{C}_1$ and $\mathbf{C}_2$ both being members of U), then the extent of $\Lambda(\mathbf{C}_1)$ is greater than the extent of $\Lambda(\mathbf{C}_2)$.

We suppose the contrary. Let $\mathbf{B}$ be a formula belonging to the extent of $\Lambda\Big(\quad\Big(\mathbf{\Phi}\Big(\lambda r \,.\, \mathbf{E}(r, 1)\Big)\Big)\Big)$ but not to the extent of $\Lambda\Big(\mathbf{\Phi}\Big(\lambda r \,.\, \mathbf{E}(r, 1)\Big)\Big)$,

and let
$$\mathbf{K}' \rightarrow \lambda a \,.\, \Lambda\Big(\mathbf{G}\Big(\mathbf{\Phi}\Big(\lambda r \,.\, \mathrm{Hg}(a, r, \mathbf{E})\Big)\Big), \mathbf{B}\Big).$$

Then $\mathbf{K}'$ is a complete logic. For

$$\mathrm{Hg}(\mathbf{A}, \mathbf{r}, \mathbf{E}) \text{ conv } \mathbf{E}(\mathbf{m}_{\mathbf{r}}, \mathbf{n}_{\mathbf{r}}).$$

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† The subset U wholly supersedes V in what follows. The introduction of V serves to emphasise the fact that the set of ordinals represented by members of U may have gaps.

$\mathrm{E}(\mathbf{m}_r, \mathbf{n}_r)$ is a sequence of V-ordinal formulae representing an increasing sequence of ordinals. Their limit is represented by $\Phi\big(\lambda r . \mathrm{Hg}(\mathbf{A}, r, \mathbf{E})\big)$; let us see what this limit is. First suppose that $\mathbf{A}$ is dual: then m_r tends to infinity as r tends to infinity, and $\Phi\big(\lambda r . \mathrm{Hg}(\mathbf{A}, r, \mathbf{E})\big)$ therefore represents the same ordinal as $\Phi\big(\lambda r . \mathrm{E}(r, 1)\big)$. In this case we must have

$$\mathbf{K}'(\mathbf{A}) \text{ conv } 2.$$

Now suppose that $\mathbf{A}$ is not dual: m_r is eventually equal to some constant number, a say, and $\Phi\big(\lambda r . \mathrm{Hg}(\mathbf{A}, r, \mathbf{E})\big)$ represents the same ordinal as $\Phi\big(\lambda r . \mathrm{E}(\mathrm{a}, r)\big)$, which is smaller than that represented by $\Phi\big(\lambda r . \mathrm{E}(r, 1)\big)$. $\mathbf{B}$ cannot therefore belong to the extent of $\Lambda\Big(\mathrm{G}\Big(\Phi\Big(\lambda r . \mathrm{Hg}(\mathbf{A}, r, \mathbf{E})\Big)\Big)\Big)$, and $\mathbf{K}'(\mathbf{A})$ is not convertible to 2. We have proved that $\mathbf{K}'$ is a complete logic, which is impossible.

This theorem can no doubt be improved in many ways. However, it is sufficiently general to show that, with almost any reasonable notation for ordinals, completeness is incompatible with invariance.

We can still give a certain meaning to the classification into depths with highly restricted kinds of ordinals. Suppose that we take a particular ordinal logic Λ and a particular ordinal formula Ψ representing the ordinal α say (preferably a large one), and that we restrict ourselves to ordinal formulae of the form $\mathrm{Inf}(\Psi, \mathbf{a})$. We then have a classification into depths, but the extents of all the logics which we so obtain are contained in the extent of a single logic.

We now attempt a problem of a rather different character, that of the completeness of Λ_P. It is to be expected that this ordinal logic is complete. I cannot at present give a proof of this, but I can give a proof that it is complete as regards a simpler type of theorem than the number-theoretic theorems, viz. those of form "$\theta(x)$ vanishes identically", where $\theta(x)$ is primitive recursive. The proof will have to be much abbreviated since we do not wish to go into the formal details of the system P. Also there is a certain lack of definiteness in the problem as at present stated, owing to the fact that the formulae G, E, M_P were not completely defined. Our attitude here is that it is open to the sceptical reader to give detailed definitions for these formulae and then verify that the remaining details of the proof can be filled in, using his definition. It is not asserted that these details can be filled in whatever be the definitions of G, E, M_P consistent with the properties already required of them, only that they can be filled in with the more natural definitions.

I shall prove the completeness theorem in the following form. If $\mathfrak{B}[x_0]$ is a recursion formula and if $\mathfrak{B}[0]$, $\mathfrak{B}[f0]$, ... are all provable in P, then there is a C-K ordinal formula $\mathbf{A}$ such that $(x_0)\mathfrak{B}[x_0]$ is provable in the system $P^{\mathbf{A}}$ of logic obtained from P by adjoining as axioms all formulae whose G.R.'s are of the form

$$\mathbf{A}\Big(\lambda mn\,.\,m\big(\varpi(2, n), \varpi(3, n)\big), K, M_P, \mathbf{r}\Big)$$

(provided they represent propositions).

First let us define the formula $\mathbf{A}$. Suppose that $\mathbf{D}$ is a W.F.F. with the property that $\mathbf{D}(\mathbf{n})$ conv 2 if $\mathfrak{B}[f^{(n-1)}0]$ is provable in P, but $\mathbf{D}(\mathbf{n})$ conv 1 if $\sim\mathfrak{B}[f^{(n-1)}0]$ is provable in P (P is being assumed consistent). Let Θ be defined by

$$\Theta \rightarrow \Big\{\lambda vu\,.\,u\big(v(v, u)\big)\Big\}\Big(\lambda vu\,.\,u\big(v(v, u)\big)\Big),$$

and let Vi be a formula with the properties

$$\mathrm{Vi}(2) \text{ conv } \lambda u\,.\,u(\mathrm{Suc}, U),$$

$$\mathrm{Vi}(1) \text{ conv } \lambda u\,.\,u\big(I, \Theta(\mathrm{Suc})\big).$$

The existence of such a formula is established in Kleene [**1**], corollary on p. 220. Now put

$$\mathbf{A}^* \rightarrow \lambda ufx\,.\,u\Big(\lambda y\,.\,\mathrm{Vi}\big(\mathbf{D}(y), y, u, f, x\big)\Big),$$

$$\mathbf{A} \rightarrow \mathrm{Suc}(\mathbf{A}^*).$$

I assert that $\mathbf{A}^*$, $\mathbf{A}$ are C-K ordinal formulae whenever it is true that $\mathfrak{B}[0]$, $\mathfrak{B}[f0]$, ... are all provable in P. For in this case $\mathbf{A}^*$ is $\lambda ufx\,.\,u(\mathbf{R})$, where

$$\mathbf{R} \rightarrow \lambda y\,.\,\mathrm{Vi}\big(\mathbf{D}(y), y, u, f, x\big),$$

and then

$$\begin{aligned} \lambda ufx\,.\,\mathbf{R}(\mathbf{n}) &\text{ conv } \lambda ufx\,.\,\mathrm{Vi}\big(\mathbf{D}(\mathbf{n}), \mathbf{n}, u, f, x\big) \\ &\text{ conv } \lambda ufx\,.\,\mathrm{Vi}(2, \mathbf{n}, u, f, x) \\ &\text{ conv } \lambda ufx\,.\,\{\lambda n\,.\,n(\mathrm{Suc}, U)\}(\mathbf{n}, u, f, x) \\ &\text{ conv } \lambda ufx\,.\,\mathbf{n}(\mathrm{Suc}, U, u, f, x), \end{aligned}$$

which is a C-K ordinal formula, and

$$\lambda ufx\,.\,S(\mathbf{n}, \mathrm{Suc}, U, u, f, x) \text{ conv } \mathrm{Suc}\big(\lambda ufx\,.\,\mathbf{n}(\mathrm{Suc}, U, u, f, x)\big).$$

These relations hold for an arbitrary positive integer n and therefore $\mathbf{A}^*$ is a C-K ordinal formula [condition (9) p. 181]: it follows immediately that $\mathbf{A}$ is also a C-K ordinal formula. It remains to prove that $(x_0)\mathfrak{B}[x_0]$ is provable in $P^{\mathbf{A}}$. To do this it is necessary to examine the structure of $\mathbf{A}^*$ in the case in which $(x_0)\mathfrak{B}[x_0]$ is false. Let us suppose that $\sim\mathfrak{B}[f^{(a-1)}0]$ is true, so that $\mathbf{D}(\mathbf{a})$ conv 1, and let us consider $\mathbf{B}$ where

$$\mathbf{B} \rightarrow \lambda ufx \,.\, \mathrm{Vi}\Big(\mathbf{D}(\mathbf{a}),\ \mathbf{a},\ u,\ f,\ x\Big).$$

If $\mathbf{A}^*$ was a C-K ordinal formula, then $\mathbf{B}$ would be a member of its fundamental sequence; but

$$\begin{aligned}
\mathbf{B}\ &\mathrm{conv}\ \lambda ufx \,.\, \mathrm{Vi}(1,\ \mathbf{a},\ u,\ f,\ x)\\
&\mathrm{conv}\ \lambda ufx \,.\, \Big\{\lambda u \,.\, u\Big(I,\ \Theta(\mathrm{Suc})\Big)\Big\}(\mathbf{a},\ u,\ f,\ x)\\
&\mathrm{conv}\ \lambda ufx \,.\, \Theta(\mathrm{Suc},\ u,\ f,\ x)\\
&\mathrm{conv}\ \lambda ufx \,.\, \Big\{\lambda u \,.\, u\Big(\Theta(u)\Big)\Big\}(\mathrm{Suc},\ u,\ f,\ x)\\
&\mathrm{conv}\ \lambda ufx \,.\, \mathrm{Suc}\Big(\Theta(\mathrm{Suc}),\ u,\ f,\ x\Big)\\
&\mathrm{conv}\ \mathrm{Suc}\Big(\lambda ufx \,.\, \Theta(\mathrm{Suc},\ u,\ f,\ x)\Big)\\
&\mathrm{conv}\ \mathrm{Suc}(\mathbf{B}). \qquad (9.3)
\end{aligned}$$

This, of course, implies that $\mathbf{B} < \mathbf{B}$ and therefore that $\mathbf{B}$ is no C-K ordinal formula. This, although fundamental in the possibility of proving our completeness theorem, does not form an actual step in the argument. Roughly speaking, our argument amounts to this. The relation (9.3) implies that the system $P^{\mathbf{B}}$ is inconsistent and therefore that $P^{\mathbf{A}^*}$ is inconsistent and indeed we can prove in P (and *a fortiori* in $P^{\mathbf{A}}$) that $\sim(x_0)\mathfrak{B}[x_0]$ implies the inconsistency of $P^{\mathbf{A}^*}$. On the other hand in $P^{\mathbf{A}}$ we can prove the consistency of $P^{\mathbf{A}^*}$. The inconsistency of $P^{\mathbf{B}}$ is proved by the Gödel argument. Let us return to the details.

The axioms in $P^{\mathbf{B}}$ are those whose G.R.'s are of the form

$$\mathbf{B}\Big(\lambda mn \,.\, m\Big(\varpi(2,\ n),\ \varpi(3,\ n)\Big),\ K,\ M_P,\ \mathbf{r}\Big).$$

When we replace $\mathbf{B}$, by $\mathrm{Suc}(\mathbf{B})$, this becomes

$$\mathrm{Suc}\Big(\mathbf{B},\ \lambda mn\,.\,m\big(\varpi(2,n),\ \varpi(3,n)\big),\ K,\ M_P,\ \mathbf{r}\Big)$$

$$\mathrm{conv}\ K\Big(\mathbf{B}\big(\lambda mn\,.\,m\big(\varpi(2,n),\ \varpi(3,n)\big),\ K,\ M_P,\ \mathbf{r}\big)\Big)$$

$$\mathrm{conv}\ \mathbf{B}\Big(\lambda mn\,.\,m\big(\varpi(2,n),\ \varpi(3,n)\big),\ K,\ M_P,\ \mathbf{p}\Big)$$

if $\mathbf{r}$ conv $2\mathbf{p}+1$,

$$\mathrm{conv}\ E\Big(\mathbf{B}\big(\lambda mn\,.\,m\big(\varpi(2,n),\ \varpi(3,n)\big),\ K,\ M_P\big),\ \mathbf{p}\Big)$$

if $\mathbf{r}$ conv $2\mathbf{p}$.

When we remember the essential property of the formula E, we see that the axioms of $P^{\mathbf{B}}$ include all formulae of the form

$$(\exists x_0)\ \mathrm{Proof}_{P^{\mathbf{B}}}[x_0,\ f^{(q)}0] \supset \mathfrak{F},$$

where q is the G.R. of the formula $\mathfrak{F}$.

Let b be the G.R. of the formula $\mathfrak{A}$.

$$\sim(\exists x_0)(\exists y_0)\{\mathrm{Proof}_{P^{\mathbf{B}}}[x_0,\ y_0]\,.\,\mathrm{Sb}[z_0,\ z_0,\ y_0]\}. \tag{$\mathfrak{A}$}$$

$\mathrm{Sb}[x_0,\ y_0,\ z_0]$ is a particular recursion formula such that $\mathrm{Sb}[f^{(l)}0,\ f^{(m)}0, f^{(n)}0]$ holds if and only if n is the G.R. of the result of substituting $f^{(m)}0$ for z_0 in the formula whose G.R. is l at all points where z_0 is free. Let p be the G.R. of the formula $\mathfrak{C}$.

$$\sim(\exists x_0)(\exists y_0)\{\mathrm{Proof}_{P^{\mathbf{B}}}[x_0,\ y_0]\,.\,\mathrm{Sb}\,[f^{(b)}0,\ f^{(b)}0,\ y_0]\}. \tag{$\mathfrak{C}$}$$

Then we have as an axiom in P

$$(\exists x_0)\ \mathrm{Proof}_{P^{\mathbf{B}}}[x_0,\ f^{(p)}0] \supset \mathfrak{C},$$

and we can prove in $P^{\mathbf{A}}$

$$(x_0)\{\mathrm{Sb}\,[f^{(b)}0,\ f^{(b)}0,\ x_0] \equiv x_0 = f^{(p)}0\}, \tag{9.4}$$

since $\mathfrak{C}$ is the result of substituting $f^{(b)}0$ for z_0 in $\mathfrak{A}$; hence

$$\sim(\exists y_0)\ \mathrm{Proof}_{P^{\mathbf{B}}}[y_0,\ f^{(p)}0] \tag{9.5}$$

is provable in P. Using (9.4) again, we see that $\mathfrak{C}$ can be proved in $P^{\mathbf{B}}$. But, if we can prove $\mathfrak{C}$ in $P^{\mathbf{B}}$, then we can prove its provability in $P^{\mathbf{B}}$, the proof being in P; *i.e.* we can prove

$$(\exists x_0)\,\mathrm{Proof}_{P^B}\,[x_0,\, f^{(p)}0]$$

in P (since p is the G.R. of $\mathfrak{C}$). But this contradicts (9.5), so that, if

$$\sim \mathfrak{B}\,[f^{(a-1)}0]$$

is true, we can prove a contradiction in P^B or in P^{A*}. Now I assert that the whole argument up to this point can be carried through formally in the system P, in fact, that, if c is the G.R. of $\sim(0=0)$, then

$$\sim (x_0)\,\mathfrak{B}[x_0] \supset (\exists v_0)\,\mathrm{Proof}_{P^{A*}}\,[v_0,\, f^{(c)}0] \qquad (9.6)$$

is provable in P. I shall not attempt to give any more detailed proof of this assertion.

The formula

$$(\exists x_0)\,\mathrm{Proof}_{P^{A*}}\,[x_0,\, f^{(c)}0] \supset \sim (0=0) \qquad (9.7)$$

is an axiom in P^A. Combining (9.6), (9.7) we obtain $(x_0)\,\mathfrak{B}[x_0]$ in P^A.

This completeness theorem as usual is of no value. Although it shows, for instance, that it is possible to prove Fermat's last theorem with Λ_P (if it is true) yet the truth of the theorem would really be assumed by taking a certain formula as an ordinal formula.

That Λ_P is not invariant may be proved easily by our general theorem; alternatively it follows from the fact that, in proving our partial completeness theorem, we never used ordinals higher than $\omega+1$. This fact can also be used to prove that Λ_P is not C-K invariant up to $\omega+2$.

10. *The continuum hypothesis. A digression.*

The methods of § 9 may be applied to problems which are constructive analogues of the continuum hypothesis problem. The continuum hypothesis asserts that $2^{\aleph_0} = \aleph_1$, in other words that, if ω_1 is the smallest ordinal α greater than ω such that a series with order type α cannot be put into one-one correspondence with the positive integers, then the ordinals less than ω_1 can be put into one-one correspondence with the subsets of the positive integers. To obtain a constructive analogue of this proposition we may replace the ordinals less than ω_1 either by the ordinal formulae, or by the ordinals represented by them; we may replace the subsets of the positive integers either by the computable sequences of figures 0, 1, or by the description numbers of the machines which compute these sequences. In the manner in which the correspondence is to be set up there is also more than one possibility. Thus, even when we use only one kind of ordinal formula, there is still great ambiguity concerning what the constructive analogue of the continuum hypothesis should be. I shall prove a single result in this connection†. A number of others may be proved in the same way.

We ask "Is it possible to find a computable function of ordinal formulae determining a one-one correspondence between the ordinals represented by ordinal formulae and the computable sequences of figures 0, 1?" More accurately, "Is there a formula $\mathbf{F}$ such that if $\mathbf{\Omega}$ is an ordinal formula and n a positive integer then $\mathbf{F}(\mathbf{\Omega}, \mathbf{n})$ is convertible to 1 or to 2, and such that $\mathbf{F}(\mathbf{\Omega}, \mathbf{n})$ conv $\mathbf{F}(\mathbf{\Omega}', \mathbf{n})$ for each positive integer n, if and only if $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ represent the same ordinal?" The answer is "No", as will be seen to be a consequence of the following argument: there is no formula $\mathbf{F}$ such that $\mathbf{F}(\mathbf{\Omega})$ enumerates one sequence of integers (each being 1 or 2) when $\mathbf{\Omega}$ represents ω and enumerates another sequence when $\mathbf{\Omega}$ represents 0. If there is such an $\mathbf{F}$, then there is an a such that $\mathbf{F}(\mathbf{\Omega}, \mathbf{a})$ conv $(\mathrm{Dt}, \mathbf{a})$ if $\mathbf{\Omega}$ represents ω but $\mathbf{F}(\mathbf{\Omega}, \mathbf{a})$ and $\mathbf{F}(\mathrm{Dt}, \mathbf{a})$ are convertible to different integers (1 or 2) if $\mathbf{\Omega}$ represents 0. To obtain a contradiction from this we introduce a W.F.F. Gm not unlike Mg. If the machine $\mathcal{M}$ whose D.N. is n has printed 0 by the time the m-th complete configuration is reached then

$$\mathrm{Gm}(\mathbf{n}, \mathbf{m}) \text{ conv } \lambda mn\,.\,m(n, I, 4);$$

otherwise $\mathrm{Gm}(\mathbf{n}, \mathbf{m}) \text{ conv } \lambda pq\,.\,\mathrm{Al}\big(4(P, 2p+2q), 3, 4\big)$. Now consider $\mathbf{F}(\mathrm{Dt}, \mathbf{a})$ and $\mathbf{F}\Big(\mathrm{Lim}\big(\mathrm{Gm}(\mathbf{n})\big), \mathbf{a}\Big)$. If $\mathcal{M}$ never prints 0, $\mathrm{Lim}\big(\mathrm{Gm}(\mathbf{n})\big)$ represents the ordinal ω. Otherwise it represents 0. Consequently these two formulae are convertible to one another if and only if $\mathcal{M}$ never prints 0. This gives us a means of determining about any machine whether it ever prints 0, which is impossible.

Results of this kind have of course no real relevance for the classical continuum hypothesis.

11. *The purpose of ordinal logics.*

Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two faculties‡, which we may call *intuition* and *ingenuity*. The activity of the intuition consists in making spontaneous judgments which are not the result of conscious trains of reasoning. These judgments are often but by no means invariably correct (leaving aside the question what is meant by "correct"). Often it is possible to find some other way of verifying the correctness of an intuitive judgment. We may, for instance, judge that all positive integers are uniquely factorizable into primes; a detailed mathematical argument leads to the same result. This argument will also involve intuitive judgments, but they will be less open to criticism than the original judgment about factorization. I shall not attempt to explain this idea of "intuition" any more explicitly.

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† A suggestion to consider this problem came to me indirectly from F. Bernstein. A related problem was suggested by P. Bernays.

‡ We are leaving out of account that most important faculty which distinguishes topics of interest from others; in fact, we are regarding the function of the mathematician as simply to determine the truth or falsity of propositions.

The exercise of ingenuity in mathematics consists in aiding the intuition through suitable arrangements of propositions, and perhaps geometrical figures or drawings. It is intended that when these are really well arranged the validity of the intuitive steps which are required cannot seriously be doubted.

The parts played by these two faculties differ of course from occasion to occasion, and from mathematician to mathematician. This arbitrariness can be removed by the introduction of a formal logic. The necessity for using the intuition is then greatly reduced by setting down formal rules for carrying out inferences which are always intuitively valid. When working with a formal logic, the idea of ingenuity takes a more definite shape. In general a formal logic, will be framed so as to admit a considerable variety of possible steps in any stage in a proof. Ingenuity will then determine which steps are the more profitable for the purpose of proving a particular proposition. In pre-Gödel times it was thought by some that it would probably be possible to carry this programme to such a point that all the intuitive judgments of mathematics could be replaced by a finite number of these rules. The necessity for intuition would then be entirely eliminated.

In our discussions, however, we have gone to the opposite extreme and eliminated not intuition but ingenuity, and this in spite of the fact that our aim has been in much the same direction. We have been trying to see how far it is possible to eliminate intuition, and leave only ingenuity. We do not mind how much ingenuity is required, and therefore assume it to be available in unlimited supply. In our metamathematical discussions we actually express this assumption rather differently. We are always able to obtain from the rules of a formal logic a method of enumerating the propositions proved by its means. We then imagine that all proofs take the form of a search through this enumeration for the theorem for which a proof is desired. In this way ingenuity is replaced by patience. In these heuristic discussions, however, it is better not to make this reduction.

In consequence of the impossibility of finding a formal logic which wholly eliminates the necessity of using intuition, we naturally turn to "non-constructive" systems of logic with which not all the steps in a proof are mechanical, some being intuitive. An example of a non-constructive logic is afforded by any ordinal logic. When we have an ordinal logic, we are in a position to prove number-theoretic theorems by the intuitive steps of recognizing formulae as ordinal formulae, and the mechanical steps of carrying out conversions. What properties do we desire a non-constructive logic to have if we are to make use of it for the expression of mathematical proofs? We want it to show quite clearly when a step makes use of intuition, and when it is purely formal. The strain put on the intuition should be a minimum. Most important of all, it must be beyond all reasonable doubt that the logic leads to correct results whenever the intuitive steps are correct†. It is also desirable that the logic shall be adequate for the expression of number-theoretic theorems, in order that it may be used in metamathematical discussions (cf. § 5).

Of the particular ordinal logics that we have discussed, Λ_H and Λ_P certainly will not satisfy us. In the case of Λ_H we are in no better position than with a constructive logic. In the case of Λ_P (and for that matter also Λ_H) we are by no means certain that we shall never obtain any but true results, because we do not know whether all the number-theoretic theorems provable in the system P are true. To take Λ_P as a fundamental non-constructive logic for metamathematical arguments would be most unsound. There remains the system of Church which is free from these objections. It is probably complete (although this would not necessarily mean much) and it is beyond reasonable doubt that it always leads to correct results‡. In the next section I propose to describe another ordinal logic, of a very different type, which is suggested by the work of Gentzen and which should also be adequate for the formalization of number-theoretic theorems. In particular it should be suitable for proofs of metamathematical theorems (cf. § 5).

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† This requirement is very vague. It is not of course intended that the criterion of the correctness of the intuitive steps be the correctness of the final result. The meaning becomes clearer if each intuitive step is regarded as a judgment that a particular proposition is true. In the case of an ordinal logic it is always a judgment that a formula is an ordinal formula, and this is equivalent to judging that a number-theoretic proposition is true. In this case then the requirement is that the reputed ordinal logic *is* an ordinal logic.

‡ This ordinal logic arises from a certain system C_0 in essentially the same way as Λ_P arose from P. By an argument similar to one occurring in § 8 we can show that the ordinal logic leads to correct results if and only if C_0 is valid; the validity of C_0 is proved in Church [**1**], making use of the results of Church and Rosser [**1**].

12. *Gentzen type ordinal logics.*

In proving the consistency of a certain system of formal logic Gentzen (Gentzen [1]) has made use of the principle of transfinite induction for ordinals less than ϵ_0, and has suggested that it is to be expected that transfinite induction carried sufficiently far would suffice to solve all problems of consistency. Another suggestion of basing systems of logic on transfinite induction has been made by Zermelo (Zermelo [1]). In this section I propose to show how this method of proof may be put into the form of a formal (non-constructive) logic, and afterwards to obtain from it an ordinal logic.

We can express the Gentzen method of proof formally in this way. Let us take the system P and adjoin to it an axiom $\mathfrak{A}_\Omega$ with the intuitive meaning that the W.F.F. Ω is an ordinal formula, whenever we feel certain that Ω *is* an ordinal formula. This is a non-constructive system of logic which may easily be put into the form of an ordinal logic. By the method of § 6 we make correspond to the system of logic consisting of P with the axiom $\mathfrak{A}_\Omega$ adjoined a logic formula L_Ω: L_Ω is an effectively calculable function of Ω, and there is therefore a formula $\Lambda_G{}^1$ such that $\Lambda_G{}^1(\Omega)$ conv L_Ω for each formula Ω. $\Lambda_G{}^1$ is certainly not an ordinal logic unless P is valid, and therefore consistent. This formalization of Gentzen's idea would therefore not be applicable for the problem with which Gentzen himself was concerned, for he was proving the consistency of a system weaker than P. However, there are other ways in which the Gentzen method of proof can be formalized. I shall explain one, beginning by describing a certain logical calculus.

The symbols of the calculus are f, x, ^{1}, $_1$, 0, S, R, Γ, Δ, E, $|$, $\odot$, $!$, $($, $)$, $=$, and the comma ",". For clarity we shall use various sizes of brackets (,) in the following. We use capital German letters to stand for variable or undetermined sequences of these symbols.

It is to be understood that the relations that we are about to define hold only when compelled to do so by the conditions that we lay down. The conditions should be taken together as a simultaneous inductive definition of all the relations involved.

Suffixes.

$_1$ is a suffix. If $\mathfrak{S}$ is a suffix then $\mathfrak{S}_1$ is a suffix.

Indices.

^{1} is an index. If $\mathfrak{J}$ is an index then $\mathfrak{J}^1$ is an index.

Numerical variables.

If $\mathfrak{S}$ is a suffix then $x\mathfrak{S}$ is a numerical variable.

Functional variables.

If $\mathfrak{S}$ is a suffix and $\mathfrak{I}$ is an index, then $f\mathfrak{S}\mathfrak{I}$ is a functional variable of index $\mathfrak{I}$.

Arguments.

$(,)$ is an argument of index ^{1}. If $(\mathfrak{A})$ is an argument of index $\mathfrak{I}$ and $\mathfrak{T}$ is a term, then $(\mathfrak{A}\mathfrak{T},)$ is an argument of index $\mathfrak{I}^{1}$.

Numerals.

0 is a numeral.

If $\mathfrak{N}$ is a numeral, then $S(,\ \mathfrak{N},)$ is a numeral.

In metamathematical statements we shall denote the numeral in which S occurs r times by $S^{(r)}(,\ 0,)$.

Expressions of a given index.

A functional variable of index $\mathfrak{I}$ is an expression of index $\mathfrak{I}$.

R, S are expressions of index ^{111}, ^{11} respectively.

If $\mathfrak{N}$ is a numeral, then it is also an expression of index ^{1}.

Suppose that $\mathfrak{G}$ is an expression of index $\mathfrak{I}$, $\mathfrak{H}$ one of index $\mathfrak{I}^{1}$ and $\mathfrak{K}$ one of index $\mathfrak{I}^{111}$; then $(\Gamma\mathfrak{G})$ and $(\Delta\mathfrak{G})$ are expressions of index $\mathfrak{I}$, while $(E\mathfrak{G})$ and $(\mathfrak{G}\,|\,\mathfrak{H})$ and $(\mathfrak{G}\odot\mathfrak{K})$ and $(\mathfrak{G}\,!\,\mathfrak{H}\,!\,\mathfrak{K})$ are expressions of index $\mathfrak{I}^{1}$.

Function constants.

An expression of index $\mathfrak{I}$ in which no functional variable occurs is a function constant of index $\mathfrak{I}$. If in addition R does not occur, the expression is called a *primitive function constant.*

Terms.

0 is a term.

Every numerical variable is a term.

If $\mathfrak{G}$ is an expression of index $\mathfrak{I}$ and $(\mathfrak{A})$ is an argument of index $\mathfrak{I}$, then $\mathfrak{G}(\mathfrak{A})$ is a term.

Equations.

If $\mathfrak{T}$ and $\mathfrak{T}'$ are terms, then $\mathfrak{T} = \mathfrak{T}'$ is an equation.

Provable equations.

We define what is meant by the provable equations relative to a given set of equations as axioms.

(*a*) The provable equations include all the axioms. The axioms are of the form of equations in which the symbols Γ, Δ, E, $|$, $\odot$, ! do not appear.

(*b*) If $\mathfrak{G}$ is an expression of index $\mathfrak{I}^{11}$ and $(\mathfrak{A})$ is an argument of index $\mathfrak{I}$, then

$$(\Gamma\mathfrak{G})(\mathfrak{A}x_1,\, x_{11},) = \mathfrak{G}(\mathfrak{A}x_{11},\, x_1,)$$

is a provable equation.

(*c*) If $\mathfrak{G}$ is an expression of index $\mathfrak{I}^1$, and $(\mathfrak{A})$ is an argument of index $\mathfrak{I}$, then

$$(\Delta\mathfrak{G})(\mathfrak{A}x_1,) = \mathfrak{G}(,\, x_1\,\mathfrak{A})$$

is a provable equation.

(*d*) If $\mathfrak{G}$ is an expression of index $\mathfrak{I}$, and $(\mathfrak{A})$ is an argument of index $\mathfrak{I}$, then

$$(E\mathfrak{G})(\mathfrak{A}x_1,) = \mathfrak{G}(\mathfrak{A})$$

is a provable equation.

(*e*) If $\mathfrak{G}$ is an expression of index $\mathfrak{I}$ and $\mathfrak{H}$ is one of index $\mathfrak{I}^1$, and $(\mathfrak{A})$ is an argument of index $\mathfrak{I}$, then

$$(\mathfrak{G}\,|\,\mathfrak{H})(\mathfrak{A}) = \mathfrak{H}\Big(\mathfrak{A}\mathfrak{G}(\mathfrak{A}),\Big)$$

is a provable equation.

(*f*) If $\mathfrak{N}$ is an expression of index ^{1}, then $\mathfrak{N}(,) = \mathfrak{N}$ is a provable equation.

(*g*) If $\mathfrak{G}$ is an expression of index $\mathfrak{I}$ and $\mathfrak{K}$ one of index $\mathfrak{I}^{111}$, and $(\mathfrak{A})$ an argument of index $\mathfrak{I}^1$, then

$$(\mathfrak{G} \odot \mathfrak{K})\,(\mathfrak{A}0,) = \mathfrak{G}(\mathfrak{A})$$

and
$$(\mathfrak{G} \odot \mathfrak{K})\Big(\mathfrak{A}S(,\, x_1,),\Big) = \mathfrak{K}\Big(\mathfrak{A}x_1,\; S(,\, x_1,),\; (\mathfrak{G} \odot \mathfrak{K})(\mathfrak{A}x_1,),\Big)$$

are provable equations. If in addition $\mathfrak{H}$ is an expression of index $\mathfrak{I}^1$ and

$$R\Big(,\mathfrak{G}\Big(\mathfrak{A}S(,x_1,),\Big),x_1,\Big) = 0$$

is provable, then

$$(\mathfrak{G}\,!\,\mathfrak{K}\,!\,\mathfrak{H})\,(\mathfrak{A}0,) = \mathfrak{G}(\mathfrak{A})$$

and

$$\begin{aligned}&(\mathfrak{G}!\,\mathfrak{K}!\,\mathfrak{H})\Big(\mathfrak{A}S(,x_1,),\Big)\\ &\quad = \mathfrak{K}\Big(\Big(\mathfrak{A}\mathfrak{H}(\mathfrak{A}S(,x_1,),\Big),S(,x_1,),(\mathfrak{G}!\,\mathfrak{K}!\,\mathfrak{H})\Big(\mathfrak{A}\mathfrak{H}\Big(\mathfrak{A}S(,x_1,),\Big),\Big),\Big)\end{aligned}$$

are provable.

(*h*) If $\mathfrak{T} = \mathfrak{T}'$ and $\mathfrak{U} = \mathfrak{U}'$ are provable, where $\mathfrak{T}$, $\mathfrak{T}'$, $\mathfrak{U}$ and $\mathfrak{U}'$ are terms, then $\mathfrak{U}' = \mathfrak{U}$ and the result of substituting $\mathfrak{U}'$ for $\mathfrak{U}$ at any particular occurrence in $\mathfrak{T} = \mathfrak{T}'$ are provable equations.

(*i*) The result of substituting any term for a particular numerical variable throughout a provable equation is provable.

(*j*) Suppose that $\mathfrak{G}$, $\mathfrak{G}'$ are expressions of index $\mathfrak{J}^1$, that $(\mathfrak{A})$ is an argument of index $\mathfrak{J}$ not containing the numerical variable $\mathfrak{X}$ and that $\mathfrak{G}(\mathfrak{A}0,) = \mathfrak{G}'(\mathfrak{A}0,)$ is provable. Also suppose that, if we add

$$\mathfrak{G}(\mathfrak{A}\mathfrak{X},) = \mathfrak{G}'(\mathfrak{A}\mathfrak{X},)$$

to the axioms and restrict (*i*) so that it can never be applied to the numerical variable $\mathfrak{X}$, then

$$\mathfrak{G}\Big(\mathfrak{A}S(,\mathfrak{X},),\Big) = \mathfrak{G}'\Big(\mathfrak{A}S(,\mathfrak{X}),\Big)$$

becomes a provable equation; in the hypothetical proof of this equation this rule (*j*) itself may be used provided that a different variable is chosen to take the part of $\mathfrak{X}$.

Under these conditions $\mathfrak{G}(\mathfrak{A}\mathfrak{X},) = \mathfrak{G}'(\mathfrak{A}\mathfrak{X},)$ is a provable equation.

(*k*) Suppose that $\mathfrak{G}$, $\mathfrak{G}'$, $\mathfrak{H}$ are expressions of index $\mathfrak{J}^1$, that $(\mathfrak{A})$ is an argument of index $\mathfrak{J}$ not containing the numerical variable $\mathfrak{X}$ and that

$$\mathfrak{G}(\mathfrak{A}0,) = \mathfrak{G}'(\mathfrak{A}0,) \quad \text{and} \quad R\Big(,\mathfrak{H}\big(\mathfrak{A}S(,\mathfrak{X},),\big), S(,\mathfrak{X},),\Big) = 0$$

are provable equations. Suppose also that, if we add

$$\mathfrak{G}\Big(\mathfrak{A}\mathfrak{H}\big(\mathfrak{A}S(,\mathfrak{X},),\big)\Big) = \mathfrak{G}'\Big(\mathfrak{A}\mathfrak{H}\big(\mathfrak{A}S(,\mathfrak{X},),\big)\Big)$$

to the axioms, and again restrict (*i*) so that it does not apply to $\mathfrak{X}$, then

$$\mathfrak{G}(\mathfrak{A}\mathfrak{X},) = \mathfrak{G}'(\mathfrak{A}\mathfrak{X},) \tag{12.1}$$

becomes a provable equation; in the hypothetical proof of (12.1) the rule (*k*) may be used if a different variable takes the part of $\mathfrak{X}$.

Under these conditions (12.1) is a provable equation.

We have now completed the definition of a provable equation relative to a given set of axioms. Next we shall show how to obtain an ordinal logic from this calculus. The first step is to set up a correspondence between some of the equations and number-theoretic theorems, in other words to show how they can be interpreted as number-theoretic theorems.

Let $\mathfrak{G}$ be a primitive function constant of index ^{111}. $\mathfrak{G}$ describes a certain primitive recursive function $\phi(m, n)$, determined by the condition that, for all natural numbers m, n, the equation

$$\mathfrak{G}\Big(, S^{(m)}(,0,), S^{(n)}(,0,),\Big) = S^{(\phi(m,n))}(,0,)$$

is provable without using the axioms (a). Suppose also that $\mathfrak{H}$ is an expression of index $\mathfrak{I}$. Then to the equation

$$\mathfrak{G}\Big(, x_1, \mathfrak{H}(, x_1,),\Big) = 0$$

we make correspond the number-theoretic theorem which asserts that for each natural number m there is a natural number n such that $\phi(m, n) = 0$. (The circumstance that there is more than one equation to represent each number-theoretic theorem could be avoided by a trivial but inconvenient modification of the calculus.)

Now let us suppose that some definite method is chosen for describing the sets of axioms by means of positive integers, the null set of axioms being described by the integer 1. By an argument used in § 6 there is a W.F.F. Σ such that, if r is the integer describing a set A of axioms, then $\Sigma(\mathbf{r})$ is a logic formula enabling us to prove just those number-theoretic theorems which are associated with equations provable with the above described calculus, the axioms being those described by the number r.

I explain two ways in which the construction of the ordinal logic may be completed.

In the first method we make use of the theory of general recursive functions (Kleene [**2**]). Let us consider all equations of the form

$$R\Big(, S^{(m)}(,0,), S^{(n)}(,0,),\Big) = S^{(p)}(,0,) \tag{12.2}$$

which are obtainable from the axioms by the use of rules (h), (i). It is a consequence of the theorem of equivalence of λ-definable and general recursive functions (Kleene [**3**]) that, if $r(m, n)$ is any λ-definable function of two variables, then we can choose the axioms so that (12.2) with $p = r(m, n)$ is obtainable in this way for each pair of natural numbers m, n, and no equation of the form

$$S^{(m)}(,0,) = S^{(n)}(,0,) \quad (m \neq n) \tag{12.3}$$

is obtainable. In particular, this is the case if $r(m, n)$ is defined by the condition that

$$\Omega(\mathbf{m}, \mathbf{n}) \text{ conv } S(\mathbf{p}) \quad \text{implies} \quad p = r(m, n),$$

$$r(0, n) = 1, \quad \text{all} \quad n > 0, \quad r(0, 0) = 2,$$

where Ω is an ordinal formula. There is a method for obtaining the axioms given the ordinal formula, and consequently a formula Rec such that, for any ordinal formula Ω, Rec (Ω) conv $\mathbf{m}$, where m is the integer describing the set of axioms corresponding to Ω. Then the formula

$$\Lambda_G{}^2 \to \lambda w \,.\, \Sigma\Big(\mathrm{Rec}\,(w)\Big)$$

is an ordinal logic. Let us leave the proof of this aside for the present.

Our second ordinal logic is to be constructed by a method not unlike the one which we used in constructing Λ_P. We begin by assigning ordinal formulae to all sets of axioms satisfying certain conditions. For this purpose we again consider that part of the calculus which is obtained by restricting "expressions" to be functional variables or R or S and restricting the meaning of "term" accordingly; the new provable equations are given by conditions (a), (h), (i), together with an extra condition (l).

(l) The equation

$$R\Big(, 0,\ S(, x_1,),\Big) = 0$$

is provable.

We could design a machine which would obtain all equations of the form (12 . 2), with $m \neq n$, provable in this sense, and all of the form (12 . 3), except that it would cease to obtain any more equations when it had once obtained one of the latter "contradictory" equations. From the description of the machine we obtain a formula Ω such that

$$\Omega(\mathbf{m},\ \mathbf{n}) \text{ conv } 2 \quad \text{if} \quad R\Big(,\ S^{(m-1)}(,\ 0,),\ S^{(n-1)}(, 0,),\Big) = 0$$

is obtained by the machine,

$$\Omega(\mathbf{m},\ \mathbf{n}) \text{ conv } 1 \quad \text{if} \quad R\Big(,\ S^{(n-1)}(, 0,),\ S^{(m-1)}(, 0,),\Big) = 0$$

is obtained by the machine, and

$$\Omega(\mathbf{m},\ \mathbf{m}) \text{ conv } 3 \text{ always.}$$

The formula Ω is an effectively calculable function of the set of axioms, and therefore also of m: consequently there is a formula M such that $M(\mathbf{m})$ conv Ω when m describes the set of axioms. Now let Cm be a formula such that, if b is the G.R. of a formula $M(\mathbf{m})$, then Cm(b) conv $\mathbf{m}$, but otherwise Cm(b) conv 1. Let

$$\Lambda_G{}^3 \to \lambda wa \,.\, \Gamma\Big(\lambda n \,.\, \Sigma\Big(\mathrm{Cm}(\mathrm{Tn}(w,\ n))\Big),\ a\Big).$$

Then $\Lambda_G{}^3$ (Ω, A) conv 2 if and only if Ω conv $M(\mathbf{m})$, where m describes a set of axioms which, taken with our calculus, suffices to prove the equation which is, roughly speaking, equivalent to "A is dual". To prove that $\Lambda_G{}^3$ is an ordinal logic, it is sufficient to prove that the calculus with the axioms described by m proves only true number-theoretic theorems when $\boldsymbol{\Omega}$ is an ordinal formula. This condition on m may also be expressed in this way. Let us put $m \leqslant n$ if we can prove $R\Big(, S^{(m)}(,0,),\ S^{(n)}(,0,),\Big) = 0$ with (a), (h), (i), (l): the condition is that $m \leqslant n$ is a well-ordering of the natural numbers and that no contradictory equation (12.3) is provable with the same rules (a), (h), (i), (l). Let us say that such a set of axioms is *admissible.* $\Lambda_G{}^3$ is an ordinal logic if the calculus leads to none but true number-theoretic theorems when an admissible set of axioms is used.

In the case of $\Lambda_G{}^2$, Rec(Ω) describes an admissible set of axioms whenever Ω is an ordinal formula. $\Lambda_G{}^2$ therefore is an ordinal logic if the calculus leads to correct results when admissible axioms are used.

To prove that admissible axioms have the required property, I do not attempt to do more than show how interpretations can be given to the equations of the calculus so that the rules of inference (a)–(k) become intuitively valid methods of deduction, and so that the interpretation agrees with our convention regarding number-theoretic theorems.

Each expression is the name of a function, which may be only partially defined. The expression S corresponds simply to the successor function. If $\mathfrak{S}$ is either R or a functional variable and has $p+1$ symbols in its index, then it corresponds to a function g of p natural numbers defined as follows. If

$$\mathfrak{S}\Big(, S^{(r_1)}(,0,),\ S^{(r_2)}(,0,),\ \ldots,\ S^{(r_p)}(,0,),\Big) = S^{(l)}(,0,)$$

is provable by the use of (a), (h), (i), (l) only, then $g(r_1, r_2, \ldots, r_p)$ has the value p. It may not be defined for all arguments, but its value is always unique, for otherwise we could prove a "contradictory" equation and $M(\mathbf{m})$ would then not be an ordinal formula. The functions corresponding to the other expressions are essentially defined by (b)–(f). For example, if g is the function corresponding to $\mathfrak{S}$ and g' that corresponding to $(\Gamma\mathfrak{S})$, then

$$g'(r_1, r_2, \ldots, r_p, l, m) = g(r_1, r_2, \ldots, r_p, m, l).$$

The values of the functions are clearly unique (when defined at all) if given by one of (b)–(e). The case (f) is less obvious since the function defined appears also in the definiens. I do not treat the case of $(\mathfrak{S} \odot \mathfrak{K})$, since this is the well-known definition by primitive recursion, but I shall show that the values of the function corresponding to $(\mathfrak{S}!\ \mathfrak{K}!\ \mathfrak{H})$ are unique. Without loss of generality we may suppose that $(\mathfrak{A})$ in (f) is of index ^{1}. We have then to show that, if $h(m)$ is the function corresponding to $\mathfrak{H}$ and $r(m, n)$ that corresponding to R, and $k(u, v, w)$ is a given function and a a given natural number, then the equations

$$l(0) = a, \tag{α}$$

$$l(m+1) = k\Big(h(m+1),\ m+1,\ l\big(h(m+1)\big)\Big) \tag{β}$$

do not ever assign two different values for the function $l(m)$. Consider those values of r for which we obtain more than one value of $l(r)$, and suppose that there is at least one such. Clearly 0 is not one, for $l(0)$ can be defined only by (α). Since the relation $\leqslant$ is a well ordering, there is an integer r_0 such that $r_0 > 0$, $l(r_0)$ is not unique, and if $s \neq r_0$ and $l(s)$ is not unique then $r_0 \leqslant s$. We may put $s = h(r_0)$, for, if $l\big(h(r_0)\big)$ were unique, then $l(r_0)$, defined by (β), would be unique. But $r\big(h(r_0), r_0\big) = 0$ *i.e.* $s \leqslant r_0$. There is, therefore, no integer r for which we obtain more than one value for the function $l(r)$.

Our interpretation of expressions as functions gives us an immediate interpretation for equations with no numerical variables. In general we interpret an equation with numerical variables as the (infinite) conjunction of all equations obtainable by replacing the variables by numerals. With this interpretation (h). (i) are seen to be valid methods of proof. In (j) the provability of

$$\mathfrak{G}\big(\mathfrak{A}S(,x_1,),\big) = \mathfrak{G}'\big(\mathfrak{A}S(,x_1,),\big)$$

when $\mathfrak{G}(\mathfrak{A}x_1,) = \mathfrak{G}'(\mathfrak{A}x_1,)$ is assumed to be interpreted as meaning that the implication between these equations holds for all substitutions of numerals for x_1. To justify this, one should satisfy oneself that these implications always hold when the hypothetical proof can be carried out. The rule of procedure (j) is now seen to be simply mathematical induction. The rule (k) is a form of transfinite induction. In proving the validity of (k) we may again suppose ($\mathfrak{A}$) is of index ^{1}. Let $r(m, n)$, $g(m)$, $g_1(m)$, $h(n)$ be the functions corresponding respectively to R, $\mathfrak{G}$, $\mathfrak{G}'$, $\mathfrak{H}$. We shall prove that, if $g(0) = g'(0)$ and $r\big(h(n), n\big) = 0$ for each positive integer n and if $g(n+1) = g'(n+1)$ whenever $g\big(h(n+1)\big) = g'\big(h(n+1)\big)$, then $g(n) = g'(n)$ for each natural number n. We consider the class of natural numbers for which $g(n) = g'(n)$ is not true. If the class is not void it has a positive member n_0 which precedes all other members in the well ordering $\leqslant$. But $h(n_0)$ is another member of the class, for otherwise we should have

$$g\big(h(n_0)\big) = g'\big(h(n_0)\big)$$

and therefore $g(n_0) = g'(n_0)$, *i.e.* n_0 would not be in the class. This implies $n_0 \leqslant h(n_0)$ contrary to $r\big(h(n_0),\ n_0\big) = 0$. The class is therefore void.

It should be noticed that we do not really need to make use of the fact that Ω is an ordinal formula. It suffices that Ω should satisfy conditions (a)–(e) (p. 179) for ordinal formulae, and in place of (f) satisfy (f').

(f') There is no formula **T** such that $\mathbf{T}(\mathbf{n})$ is convertible to a formula representing a positive integer for each positive integer n, and such that $\Omega\big(\mathbf{T}(\mathbf{n}), \mathbf{n}\big)$ conv 2, for each positive integer n for which $\Omega(\mathbf{n}, \mathbf{n})$ conv 3.

The problem whether a formula satisfies conditions (a)–(e), (f') is number-theoretic. If we use formulae satisfying these conditions instead of ordinal formulae with $\Lambda_G{}^2$ or $\Lambda_G{}^3$, we have a non-constructive logic with certain advantages over ordinal logics. The intuitive judgments that must be made are all judgments of the truth of number theoretic-theorems. We have seen in § 9 that the connection of ordinal logics with the classical theory of ordinals is quite superficial. There seem to be good reasons, therefore, for giving attention to ordinal formulae in this modified sense.

The ordinal logic $\Lambda_G{}^3$ appears to be adequate for most purposes. It should, for instance, be possible to carry out Gentzen's proof of consistency of number theory, or the proof of the uniqueness of the normal form of a well-formed formula (Church and Rosser [**1**]) with our calculus and a fairly simple set of axioms. How far this is the case can, of course, only be determined by experiment.

One would prefer a non-constructive system of logic based on transfinite induction rather simpler than the system which we have described. In particular, it would seem that it should be possible to eliminate the necessity of stating explicitly the validity of definitions by primitive recursions, since this principle itself can be shown to be valid by transfinite induction. It is possible to make such modifications in the system, even in such a way that the resulting system is still complete, but no real advantage is gained by doing so. The effect is always, so far as I know, to restrict the class of formulae provable with a given set of axioms, so that we obtain no theorems but trivial restatements of the axioms. We have therefore to compromise between simplicity and comprehensiveness.

Index of definitions.

No attempt is being made to list heavy type formulae since their meanings are not always constant throughout the paper. Abbreviations for definite well-formed formulae are listed alphabetically.

Page

Ai 199
Al 188
Bd 188
Ck 192
Cm 222
Comp 201
Dt 167
E 196
form 166
G 196
Gm 214
Gr 166
H 183, 186
H_1 186
Hf 188
Hg 206
I 163
Inf 188
Jh 204
K 196
Lim 188
Ls 187
M 222
M_P 196
Mg 205
Nm 177
Od 201
P 188

Page

Prod 207
Q 176
Rec 222
Rt 205
S 164
Sum... 188
Sq 206
Tn 192
Ug 188
V 177
Vi 210
W 175
W' 176
X 175
Z 204

Γ 177
δ 162
Θ 210
Λ_G^1 217
Λ_G^2 222
Λ_G^3 222
Λ_H 198
Λ_P 196
ϖ 167
Σ 221
1, 2, 3, ... 164
$\mathfrak{P}$ 170

(*The following refer to* §§1–10 *only.*)

All-inclusive (logic formula) 199
Axiomatic (class or property)... 167
Circle-free ... ... (Turing [**1**], 233)
Computable function 166
Completeness, of class of logics 176
    of logic 177
    of ordinal logic 198
Convertible 163

Description number (D.N.) ... (Turing [1], 240)
Dual (W.F.F.) ... 170
Effectively calculable function ... 166
Enumerate (to) ... 165
Formally definable function ... 165
General recursive function ... 166
Gödel representation (G.R.) ... 165, 166
Immediately convertible ... 162
Invariance (of ordinal logics) ... 200
(see also 202, 203)
Limit system ... 190
Logic formula, Logic ... 174
Normal form ... 162, 165
Number-theoretic (theorem or problem) ... 168
Oracle ... 172
Ordinal ... 178
Ordinal formula ... 179, 180
C-K ordinal formula ... 181
Ordinal logic ... 189
Primitive recursive (function or relation) ... 168, 193
Recursion formula ... 193
Representation of ordinals, by ordinal formulae ... 179
by C-K ordinal formulae ... 182
Standardized logic ... 175
Type ... 197
Validity of system ... 189
Well-formed formula (W.F.F.) ... 162
Well ordered series ... 178

Miscellaneous (in order of appearance).

$\rightarrow$ ... 163
λ-definable function ... 165
$G(x, y)$, $D(x)$ ... 173
"$<$" between W.F.F. ... 181
$\Xi_{\mathbf{A}}$ ... 182
Class C, systems W ... 189
$C[\mathbf{A}]$ ($\mathbf{A}$ a C-K ordinal formula) ... 190
System P ... (foot-note †) 193
$\text{Proof}_C[x_0, y_0]$ ... 195
Systems $P\Omega$ ... 197
Systems $P^{\mathbf{A}}$ ... 210

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BEZIEHUNGEN ZWISCHEN SYSTEMEN SELBSTREZIPROKER FUNKTIONEN

(*Dem Andenken Edmund Landaus gewidmet.*)

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1. Die Funktionen $f(x)$, $g(x)$, $h(x)$ sollen zum Raume $\mathfrak{L}_2$, dem Raume der von Null bis Unendlich quadratisch integrierbaren Funktionen gehören, $T[\psi]$ die Transformation bedeuten: $g = Tf$,

$$\int_0^x g(\xi)\,d\xi = \int_0^\infty \psi(tx)\,f(t)\,dt \quad \left(\frac{\psi(x)}{x} \subset \mathfrak{L}_2\right), \tag{1a}$$

$U[\psi]$ die Transformation

$$\int_0^x g(\xi)\,d\xi = \int_0^\infty \psi\left(\frac{x}{t}\right) f(t)\,dt \quad \left(\frac{\psi(x)}{x} \subset \mathfrak{L}_2\right), \tag{1b}$$

die Mellinsche Transformierte $H(t)$ von $h(x)$, $H = Mh$, die Funktion

$$H(t) = Mh = \underset{n\to\infty}{\text{l.i.m.}} \int_{1/n}^{n} h(x)\,x^{-\frac{1}{2}+it}\,dx \quad (-\infty < t < \infty).$$

Alle auftretenden Funktionen dürfen komplexwertig sein.

Sei $T[\chi]$ von der Art (1*a*) und involutorisch, d.h. $T^2[\chi]f = f$ für alle f aus dem Bereiche† $\mathfrak{D}$ von T, $\mathfrak{R} = \mathfrak{R}[\chi]$ die Menge der in Bezug auf $T[\chi]$ selbstreziproken Funktionen aus $\mathfrak{L}_2$, $\mathfrak{R}' = \mathfrak{R}[-\chi]$ die Menge der schiefreziproken Funktionen, so sollen hier Operationen behandelt werden, die, an selbstreziproken Funktionen vorgenommem, wieder selbstreziproke

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† Bereich: Menge der Funktionen $f(x)$, für die $g = Tf$ Sinn hat; Vorrat: Menge dieser zugeordneten g. Eine Transformation der Art (1*a*) oder (1*b*) ist dann und nur dann stetig, wenn ihr Bereich ganz $\mathfrak{L}_2$, oder wenn $\omega(t) = (\frac{1}{2} - it)\,M(\psi(x)/x)$ in $(-\infty, \infty)$ fast überall, d.h. bis auf eine Menge vom Masse Null, beschränkt ist, ist umkehrbar, d.h. ein-eindeutig, wenn $\omega(t) \neq 0$ f.ü. ist, schlechthin umkehrbar, d.h. ein-eindeutig mit dem Vorrate $\mathfrak{L}_2$, wenn $|\omega(t)| \geqslant c > 0$ f. ü. ist [Verf. I und II, *Quart. J. of Math.*, 8 (1937), 172–185, und 33 (1938), 41–52]; $T[\psi]$ von der Art (1*a*) gehört zur "Watson-Klasse", wenn $T[\psi]$ stetig und involutorisch ist; G. N. Watson hat den Fall von reellwertigem ψ behandelt, *Proc. London Math. Soc.* (2), 35 (1933), 156–199.

Funktionen desselben oder eines anderen Systems ergeben. Eine Fülle von Verfahren dieser Art sind wohl bekannt, so die Regel: Die Operation

$$g(x) = \int_0^\infty f(t)\, e^{-tx}\, dt \tag{1.2}$$

führt jede selbstreziproke Funktion der Fourier-cos-Transformation in eine solche der Fourier-sin-Transformation über und umgekehrt†, auch wenn $f(x)$ gewissen von $\mathfrak{L}_2$ verschiedenen Funktionenräumen angehört.

Alle diese Regeln lassen sich nun aus einfachen Gesichtspunkten im Falle des Raumes $\mathfrak{L}_2$ betrachten, z.B. aus folgendem: Seien $\mathfrak{R}_1$ und $\mathfrak{R}_2$ zwei Systeme selbstreziproker Funktionen, die auch identisch sein dürfen. Die zu untersuchenden Operationen sind Transformationen aus $\mathfrak{R}_1$ nach $\mathfrak{R}_2$. Beide Systeme sind Hilbertsche Räume‡. Seien V_1 bezw. V_2 fest gewählte lineare, stetige, ein-eindeutige Transformationen mit dem Bereich $\mathfrak{L}_2$, den Vorräten $\mathfrak{R}_1$ bezw. $\mathfrak{R}_2$, A eine Transformation in $\mathfrak{L}_2$, d.h. aus $\mathfrak{L}_2$ nach $\mathfrak{L}_2$, so ist $V_1^{-1} A V_2$ eine Transformation aus $\mathfrak{R}_1$ nach $\mathfrak{R}_2$ und durchläuft alle Transformationen der verlangten Art, wenn A alle Transformationen in $\mathfrak{L}_2$ durchläuft, ist dann und nur dann linear oder stetig (beschränkt) oder linear abgeschlossen oder umkehrbar, wenn A es bezüglich ist, und hat dann und nur dann ganz $\mathfrak{R}_1$ bezw. ganz $\mathfrak{R}_2$ als Bereich bezw. Vorrat, wenn A ganz $\mathfrak{L}_2$ als Bereich bezw. Vorrat hat. Die Transformationen in $\mathfrak{L}_2$ sind aber ein bekannter Problemkreis, und Transformationen V_1, V_2 sind leicht angebbar§.

Die zu untersuchenden Transformationen sollen nun hier von vorneherein die Forderung erfüllen, dass sie gleichzeitig mit der Abbildung von $\mathfrak{R}_1 = \mathfrak{R}[\chi_1]$ auf $\mathfrak{R}_2 = \mathfrak{R}[\chi_2]$ auch die von $\mathfrak{R}_1' = \mathfrak{R}[-\chi_1]$ auf $\mathfrak{R}_2' = \mathfrak{R}[-\chi_2]$ leisten, und dann auf stetige Transformationen der Art (1*a*) bezw. (1*b*) spezialisiert werden, welche immer dieser Forderung genügen, wenn sie überhaupt $\mathfrak{R}_1$ auf $\mathfrak{R}_2$ abbilden (Satz III): alle bisher in der Literatur behandelten Beispiele sind von dieser speziellen Art; so erhält man einfache allgemeine Gesetze (Satz I bis IV, Abschn. 2).

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† G. H. Hardy and E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 33 (1931), 225–232. E. C. Titchmarsh, *Introduction to the theory of Fourier integrals* (Oxford, 1937), dort auch Literaturangabe. W. N. Bailey, *Journal London Math. Soc.*, 6 (1931), 242–247. B. M. Mehrotra, *Proc. London Math. Soc.* (2), 34 (1932), 231–240; *Bull. Calcutta Math. Soc.*, 34 (1932), 231–40; *Proc. Edinburgh Math. Soc.* (2), 4 (1934), 53 ff.; *Tôhoku Math. J.*, 40 (1935), 451–485, zitiert als Mehrotra I bis IV.

‡ Verf. I, 179, und Verf. III, *Proc. London Math. Soc.* (2), 44 (1938), 453–465, Satz IV. "T ist aus $\mathfrak{R}_1$ nach $\mathfrak{R}_2$" bedeute, dass der Bereich von T in $\mathfrak{R}_1$, der Vorrat in $\mathfrak{R}_2$ enthalten ist.

§ M. H. Stone, *Linear transformations in Hilbert space and their applications to analysis* (New York, 1932).—Wenn T zur Watson-Klasse gehört, wähle z.B. $\phi = V_1 f$ als die Transformation: $\phi = M^{-1} h(t)$, wobei $h(t) = f(t)$ bezw. $f(-t)\, \omega(t)$ für $t > 0$ bezw. $t < 0$ ist, Verf. I, 178.

Im dritten Abschnitt werden Beispiele behandelt, im vierten die Fälle $\mathfrak{R}_2 = \mathfrak{R}_1$ und $\mathfrak{R}_2 = \mathfrak{R}_1'$ besprochen. W. N. Bailey (*loc. cit.*, Theor. I) hat bereits eine Klasse von Transformationen gefunden, allerdings nicht mit strengem, sonder nur mit formalem Beweise, die jedes $\mathfrak{R}_1$ auf sich selbst abbilden. Hier wird die Frage nach *sämtlichen* linearen abgeschlossenen Transformationen gestellt, die das leisten, und folgende weitergehende Antwort gefunden:

SATZ V. *Wenn T alle involutorischen Transformationen, die überdies von der Art* (1a) *sind, durchläuft, bildet eine lineare, abgeschlossene oder stetige Transformation W das zu jedem T gehörige System selbstreziproker Funktionen dann und nur dann auf einen—echten oder unechten—Teil desselben Systems bezw. des zugehörigen Raumes schiefreziproker Funktionen ab, wenn sie von der Art* (1b) *ist und die Funktion $\Omega(t) = (\frac{1}{2} - it) M\big(\psi(x)\, x\big)$ in $(-\infty, \infty)$ beschränkt*[†] *und überdies gerade bezw. ungerade ist.*

ZUSATZ. *Die Abbildung ist dann und nur dann vollständig, bedeckt also das jeweils zweite System dann und nur dann ganz, wenn $|\Omega(t)| \geqslant c > 0$ in $(-\infty, \infty)$ ist; ihre Umkehrung ist dann eine Transformation genau derselben Art.*

Im letzten Abschnitte werden Beispiele gegeben. Bei den Untersuchungen soll auch der bisher kaum beachtete Gesichtspunkt der Vollständigkeit bezw. Unvollständigkeit der Abbildungen berücksichtigt werden.

2. *Allgemeine Sätze.*

SATZ I. *Seien T_1 und T_2 lineare involutorische Transformationen mit den Bereichen $\mathfrak{D}_1$ bezw. $\mathfrak{D}_2$, $\mathfrak{R}_1$ bezw. $\mathfrak{R}_2$ die zugehörigen Räume selbstreziproker, $\mathfrak{R}_1'$ bezw. $\mathfrak{R}_2'$ die schiefreziproker Funktionen. Eine lineare Transformation W bildet $\mathfrak{R}_1$ auf $\mathfrak{R}_2$*[‡] *und zugleich $\mathfrak{R}_1'$ auf $\mathfrak{R}_2'$ dann nur dann ab, wenn ihr Bereich $\mathfrak{D}_1$ enthält und für alle f aus $\mathfrak{D}_1$ auch $T_2 Wf$ sinnvoll und*

$$T_2 Wf = WT_1 f$$

ist.

[†] Natürlich bis auf eine Menge vom Masse Null. Entsprechendes gilt für $\Omega(t)$ im Zusatz.

[‡] Das bedeute stets: Ganz $\mathfrak{R}_1$ wird auf einen echten oder unechten Teil von $\mathfrak{R}_2$ abgebildet, im letzten Falle heisse die Abbildung vollständig.

Beweis. Jede Funktion f aus $\mathfrak{D}_1$ ist wegen $T_1^2 f = f$ und der Linearität von T als Summe aus einer selbst- und einer schiefreziproken Funktion darstellbar (Verf. III, Satz II′), also

$$f = \phi + \psi, \quad T_1 \phi = \phi, \quad T_1 \psi = -\psi. \tag{2.1}$$

Da W nun $\mathfrak{R}_1$ auf $\mathfrak{R}_2$, $\mathfrak{R}_1'$ auf $\mathfrak{R}_2'$ abbilden soll, ist

$$T_2(W\phi) = W\phi, \quad T_2(W\psi) = -W\psi.$$

Da $W\phi$ und $W\psi$ Sinn haben sollen, muss auch Wf für alle f aus $\mathfrak{D}_1$ Sinn haben, der Bereich $\mathfrak{D}_W$ also $\mathfrak{D}_1$ enthalten; aus der Linearität folgt:

$$WT_1 f = W(T_1\phi + T_1\psi) = W(\phi - \psi) = T_2 W\phi + T_2 W\psi = T_2 Wf,$$

q.e.d. Umgekehrt folgt aus $T_2 Wf = WT_1 f$ für $f = \phi \subset \mathfrak{R}_1$ bezw. für $f = \psi \subset \mathfrak{R}_2$ *sofort* $T_2(W\phi) = W\phi$, $T_2(W\psi) = -W\psi$.

SATZ IA. *Seien T_1 und T_2 Transformationen der "Watson-Klasse" (s. Fussnote 1), sei W stetig und von der Art* (1*a*) *oder* (1*b*); *dann und nur dann bildet W den Raum $\mathfrak{R}_1$ auf $\mathfrak{R}_2$ ab, wenn $T_2 W = WT_1$ ist.*

Beweis. Sei f aus $\mathfrak{L}_2$, so ist $f + T_1 f$ selbstreziprok in Bezug auf T_1, $W(f + T_1 f)$ in Bezug auf T_2, also wegen $T_1 T_1 f = T_1^2 f = f$

$$WT_1(f + T_1 f) = W(f + T_1 f) = T_2 W(f + T_1 f),$$

$$WT_1 f - T_2 Wf + WT_1 T_1 f - T_2 WT_1 f = 0,$$

$$WT_1 - T_2 W + (WT_1 - T_2 W) T_1 = 0.$$

Sei nun $WT_1 - T_2 W = V$, so ist folglich $VT_1 = -V$.

Wenn nun W zur "Produktklasse" (1*b*) gehört, so ist V eine stetige Transformation der Art (1*a*), VT_1 als Produkt zweier Transformationen der Art (1*a*) kann also nur dann gleich $-V$, also selbst von der Art (1*a*) sein, wenn VT_1 und somit V die Transformation Null ist (Verf. I, Abschn. 5 u. 6). Ist aber W von der Art (1*a*), so gehört V zur Produktklasse (1*b*), VT_1 ist von der Art (1*a*), also kann $VT_1 = -V$ nur für $V = 0$ sein. So ist $WT_1 = T_2 W$. Die Umkehrung folgt aus Satz I. Die Stetigkeit von W braucht für die notwendigen Bedingungen nicht vorausgesetzt zu werden†.

SATZ II$_A$. *Seien T_1, T_2 und W von der Art* (1*a*) *und stetig, dann sind folgende beiden Beziehungen äquivalent:*

$$T_2 W = WT_1 \quad \textit{und} \quad T_1 W = WT_2.$$

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† Sie lässt sich als eine Folge aufzeigen, ebenso in Satz III, IV$_A$, IV$_B$, durch Überlegungen ähnlich denen im Anfang des dritten Abschnittes.

Sei nämlich R die Transformation $g(x)=Rf(x)=x^{-1}f(x^{-1})$, so ist R stetig und von der Art (1*a*). Aus $T_2W=WT_1$ z.B. folgt $T_2WR=WT_1R$, hieraus nach einer Vertauschbarkeitsregel (Verf. 1, Abschn. 5)

$$RWT_2=RT_1W,$$

aus der Umkehrbarkeit von R also $WT_2=T_1W$.

SATZ II_B. *Seien T_1 und T_2 von der Art* (1*a*), *$U_0=U[\psi_0]$ von der Art* (1*b*), *T_1, T_2 und U_0 stetig, sei $U_{00}=U[\psi_{00}]$ bestimmt durch*

$$\int_0^x \psi_{00}\left(\frac{1}{u}\right)du = x\int_0^{1/x}\psi_0\left(\frac{1}{u}\right)du \quad (x>0), \tag{2.2}$$

dann sind folgende beiden Beziehungen äquivalent:

$$T_2U_0=U_0T_1 \quad \textit{und} \quad T_1U_{00}=U_{00}T_2.$$

Es ist nämlich $U_0=T[\psi_0]R$, $U_{00}=RT[\psi_0]=RU_0R$ (Verf. 1, Abschn. 5, Beispiel 4 und Abschn. 6), der Beweis erfolgt jetzt ähnlich wie der von Satz II.

Aus den Sätzen I, I_A, II_A, II_B folgt sofort:

SATZ III. *Wenn T_1 und T_2 zur Watson-Klasse* (*serste Fussnote*) *gehören, W stetig und von der Art* (1*a*) *ist und $\mathfrak{R}_1$ auf $\mathfrak{R}_2$ abbildet, dann bildet W auch $\mathfrak{R}_1'$ auf $\mathfrak{R}_2'$, $\mathfrak{R}_2$ auf $\mathfrak{R}_1$, $\mathfrak{R}_2'$ auf $\mathfrak{R}_1'$ ab; falls $U[\psi_0]$ von der Art* (1*b*) *und stetig ist und $\mathfrak{R}_1$ auf $\mathfrak{R}_2$ abbildet, so bildet $U[\psi_0]$ auch $\mathfrak{R}_1'$ auf $\mathfrak{R}_2'$, $U[\psi_{00}]$ aber $\mathfrak{R}_2$ auf $\mathfrak{R}_1$ und $\mathfrak{R}_2'$ auf $\mathfrak{R}_1'$ ab.*

Ferner lassen sich noch unschwer folgende Sätze beweisen:

SATZ IV_A. *Mögen T_1, T_2, T_3, T_4 zur Watson-Klasse gehören, sei W stetig, umkehrbar, von der Art* (1*a*) *bezw.* (1*b*), *und möge W ferner $\mathfrak{R}_1$ auf $\mathfrak{R}_2$ abbilden. Dann und nur dann bildet W auch $\mathfrak{R}_3$ auf $\mathfrak{R}_4$ ab, wenn*

$$T_3T_1=T_2T_4 \textit{ bezw. } T_1T_3=T_2T_4 \textit{ ist.}$$

SATZ IV_B. *Mögen T_1 und T_2 aus der Watson-Klasse, W stetig umkehrbar und von der Art* (1*a*) *bezw.* (1*b*) *sein, und möge W ferner $\mathfrak{R}_1$ auf $\mathfrak{R}_2$ abbilden. Eine stetige Transformation W' der Art* (1*a*) *bezw.* (1*b*) *bildet dann und nur dann $\mathfrak{R}_1$ auf $\mathfrak{R}_2$ ab, wenn $WW'=W'W$ bezw. $WRW'=W'RW$ ist.* (R ist im Satz II_A definiert).

SATZ IV_C. *Wenn T_1 und T_2 zur Watson-Klasse gehören, so sind*

$$W_0f=T_2T_1f+f \quad \textit{bezw.} \quad W_{00}f=T_1T_2f+f$$

stetige Transformationen der Art (1*b*), *die $\mathfrak{R}_1$ auf $\mathfrak{R}_2$ und $\mathfrak{R}_1'$ auf $\mathfrak{R}_2'$ bezw. $\mathfrak{R}_2$ auf $\mathfrak{R}_1$ und $\mathfrak{R}_2'$ auf $\mathfrak{R}_1'$ abbilden.*

3. *Beispiele.*

Vorbemerkung. Seien $T[\chi_1]$, $T[\chi_2]$ und $W[\chi_0]$ von der Art ($1a$) und stetig, also die Funktionen

$$\omega_i(t) = (\tfrac{1}{2} - it)\, M\big(\chi_i(x)/x\big) \quad (i = 0,\ 1,\ 2)$$

beschränkt, dann ist die Beziehung $T_2 W = W T_1$ gleichwertig mit

$$\omega_2(t)\, \omega_0(-t) = \omega_1(-t)\, \omega_0(t). \tag{3.1}$$

Das folgt leicht aus der Gleichwertigkeit† der Transformation $g = T_i f$ mit

$$G(t) = \omega_i(t)\, F(-t) \quad [G(t) = Mg(x),\ \ F(t) = Mf(x)]. \tag{3.11}$$

Falls $W[\chi_0]$ von der Art ($1b$) und stetig ist, tritt anstelle von (3.11) bezw. (3.1)

$$G(t) = \omega_0(t)\, F(t) \quad (\text{nur für } i = 0;\ \text{für } i = 1,\ 2 \text{ gilt } 3.11), \tag{3.21}$$

$$\omega_2(t)\, \omega_0(-t) = \omega_1(t)\, \omega_0(t). \tag{3.2}$$

Wenn $W[\chi_0]$ stetig und von der Art ($1a$) oder ($1b$) ist, T_1 und T_2 zur Watson-Klasse gehören und W ferner $\mathfrak{R}_1$ auf $\mathfrak{R}_2$ abbildet, erfasst diese Abbildung sicher alle Funktionen des abzubildenden Raumes $\mathfrak{R}_1$, da ja der Bereich von W ganz $\mathfrak{L}_2$ ist, liefert aber eine vollständige Überdeckung des Raumes $\mathfrak{R}_2$ dann und nur dann, wenn $W[\chi_0]$ schlechthin umkehrbar, also $|\omega_0(t)| \geqslant c > 0$ in $(-\infty, \infty)$ ist; dann nämlich wird durch $h = Wf$ auch jedem h ein f zugeordnet, jedem $h \subset \mathfrak{R}_2$ ein $f \subset \mathfrak{R}_1$, andernfalls aber nur denjenigen h aus $\mathfrak{L}_2$, für welche $\omega_0^{-1}(t)\, Mh$ zu $\mathfrak{L}^2(-\infty, \infty)$ (Verf. II, Beweis K) gehört, und unter den hier ausgeschlossenen Funktionen befinden sich sicher selbstreziproke Funktionen von T_2, da jede beliebig aus $\mathfrak{L}^2$ $(0, \infty)$ oder $\mathfrak{L}^2(-\infty, 0)$ gewählte Funktion so fortgesetzt werden kann, dass sie $\mathfrak{L}^2$ $(-\infty, \infty)$ und die Mellinsche Transformierte einer in Bezug auf T_2 selbstreziproken Funktion wird (Verf. I, Abschn. 4).

Beispiel 1. Selbstreziproke Funktionen bei Hankelschen Transformationen.

$$\chi_1(x) = \int_0^x J_\mu(\xi)\sqrt{\xi}\, d\xi, \quad \chi_2(x) = \int_0^x J_\nu(\xi)\sqrt{\xi}\, d\xi, \quad \mathfrak{R}(\mu) > -1, \quad \mathfrak{R}(\nu) > -1.$$

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† I. W. Busbridge, *Journal London Math. Soc.*, 9 (1934), 179–187; *s.a.* Verf. I.

Zufolge (3 . 1) erhält man, wenn $W[\chi_0]$ von der Art (1a) sein soll,

$$(3.12)\qquad \frac{\omega_0(t)}{\omega_0(-t)} = 2^{it}\,\frac{\Gamma\{\frac{1}{2}(\nu+1+it)\}}{\Gamma\{\frac{1}{2}(\nu+1-it)\}}\;2^{it}\,\frac{\Gamma\{\frac{1}{2}(\mu+1+it)\}}{\Gamma(\frac{1}{2}(\mu+1-it)\}}.$$

Wählt man $\omega_0(t) = 2^{it}\,\Gamma\{\frac{1}{2}(\nu+1+it)\}\,\Gamma\{\frac{1}{2}(\mu+1+it)\}$, so ist

$$\omega_0(t) = O(|t|^{\frac{1}{2}\Re(\mu+\nu)}e^{-\frac{1}{2}\pi|t|}) \quad \text{für} \quad t \to \pm\infty,$$

$\omega_0(t)$ beschränkt, man erhält schliesslich (s. Verf. I, Satz 2_A, 3, 3_i) für $W[\chi_0]$ die Form

$$(3.3)\qquad g(x) = 2^{1-\frac{1}{2}(\mu+\nu)}\int_0^\infty (tx)^{\frac{1}{2}(\mu+\nu+1)}K_{\frac{1}{2}(\nu-\mu)}(tx)f(t)\,dt.$$

Diese wohlbekannte† Transformation bildet nach den Sätzen I_A und III wohl $\mathfrak{R}_1$ auf $\mathfrak{R}_2$, $\mathfrak{R}_2$ auf $\mathfrak{R}_1$, $\mathfrak{R}_1'$ auf $\mathfrak{R}_2'$, $\mathfrak{R}_2'$ auf $\mathfrak{R}_1'$ ab, hat aber den Nachteil, dass sie keine vollständige Überdeckung des Raumes liefert, auf den abgebildet wird, da $\omega_0(t) \to 0$ strebt für $t \to \pm\infty$.

Beispiel 2. χ_1, χ_2 wie eben, $\mu = -\frac{1}{2}$, $\nu = \frac{1}{2}$, also

$$\frac{d}{dx}\chi_1(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\cos x, \quad \frac{d}{dx}\chi_2(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sin x.$$

Wähle

$$(3.4)\quad \omega_0(t) = 2^{it}\,\Gamma(\tfrac{3}{4}+\tfrac{1}{2}it)\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}it)(2\pi)^{-\frac{1}{2}}\cosh\tfrac{1}{2}\pi t = \Gamma(\tfrac{1}{2}+it)\cosh\tfrac{1}{2}\pi t,$$

also
$$\chi_0(x) = \sin(x-\tfrac{1}{4}\pi)+\sin\tfrac{1}{4}\pi = \int_0^x \cos(\xi-\tfrac{1}{4}\pi)\,d\xi.$$

Die Transformation

$$g(x) = \underset{n\to\infty}{\mathrm{l.i.m.}}\int_0^n \cos(tx-\tfrac{1}{4}\pi)f(t)\,dt$$

bildet den ganzen Raum der selbst-bezw. schiefreziproken Funktionen der Fourier-cos-Transformation ein-eindeutig auf den ganzen Raum der selbst-bezw. schiefreziproken Funktionen der Fourier sin-Transformation ab und umgekehrt; die Vollständigkeit der Abbildung folgt nämlich aus

$$|\omega_0(t)|^2 = \pi\cosh^2\tfrac{1}{2}\pi t/\cosh\pi t = \tfrac{1}{2}\pi+\pi/(e^{\pi t}+e^{-\pi t}) \geqslant \tfrac{1}{2}\pi > 0.$$

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† H.-T. (Hardy-Titchmarsh, *loc. cit.*), Satz II; $\mu = -\frac{1}{2}$, $\nu = \frac{1}{2}$ ergibt (1.2), abgesehen von einem konstanten Faktor.

Beispiel 3. Seien χ_1 und χ_2 wie im Beispiel 1, $W[\psi_0]$ von der Art (1*b*), $T_2 W = WT_1$, also mit Rücksicht auf (3.2)

$$\frac{\omega_0(t)}{\omega_0(-t)} = \frac{\omega_2(t)}{\omega_1(t)} = \frac{\Gamma\{\frac{1}{2}(\nu+1+it)\}\,\Gamma\{\frac{1}{2}(\mu+1-it)\}}{\Gamma\{\frac{1}{2}(\nu+1-it)\}\,\Gamma\{\frac{1}{2}(\mu+1+it)\}}. \tag{3.22}$$

Man wähle

$$\omega_0(t) = \tfrac{1}{4}\Gamma\{\tfrac{1}{2}(\nu+1+it)\}\,\Gamma\{\tfrac{1}{2}(\mu+1-it)\} = \Omega_2(t)\,\Omega_1(-t),$$

wobei $\Omega_1(t) = \frac{1}{2}\Gamma\{\frac{1}{2}(\mu+1+it)\}$, $\Omega_2(t) = \frac{1}{2}\Gamma\{\frac{1}{2}(\nu+1+it)\}$ zu folgenden stetigen Transformationen S_1, S_2 der Art (1*a*) gehören:

$$g(x) = \int_0^\infty (tx)^{\mu+\frac{1}{2}} e^{-t^2x^2} f(t)\,dt \quad \text{bezw.} \quad g(x) = \int_0^\infty (tx)^{\nu+\frac{1}{2}} e^{-t^2x^2} f(t)\,dt.$$

Daher ist $W[\chi_0]$ das Produkt $S_2 S_1$ (Verf. I, Abschn. 5 u. 6) und bildet $\mathfrak{R}_1$ auf $\mathfrak{R}_2$, $\mathfrak{R}_1'$ auf $\mathfrak{R}_2'$ ab. Die umgekehrte Abbildung erfolgt nach Satz III durch

$$W_{00} = RWR = RS_2 S_1 R = RRS_1 S_2 = S_1 S_2$$

mit Rücksicht auf die Vertauschbarkeitsregel (Verf. I, Abschn. 5). Die Überdeckung des jeweils zweiten Raumes, auf den abgebildet wird, ist nicht vollständig wegen

$$\omega_0(t) \to 0, \quad \omega_{00}(t) = \Omega_1(t)\,\Omega_2(-t) \to 0 \quad \text{für} \quad t \to \pm\infty.$$

Die Transformation $W[\psi_0]$ lässt sich darstellen in der Form†

$$g(x) = \frac{d}{dx}\int_0^\infty \psi_0\left(\frac{x}{t}\right) f(t)\,dt = a\int_0^\infty \frac{t^{\mu+\frac{1}{2}}\,x^{\nu+\frac{1}{2}} f(t)}{(x^2+t^2)^{1+\frac{1}{2}(\mu+\nu)}}\,dt;$$

$$\psi_0(x) = a\int_0^x \xi^{\nu+\frac{1}{2}}(1+\xi^2)^{-1-\frac{1}{2}(\mu+\nu)}\,d\xi,$$

$$[a = \tfrac{1}{2}\Gamma\{1+\tfrac{1}{2}(\mu+\nu)\}],$$

W_{00} erhält man durch Vertauschung von μ und ν. Ist $\mu = \nu$, $\mathfrak{R}(\mu) > -1$, μ sonst beliebig, so ist $\omega_0(t)$ gerade, $W = W_{00}$; W bildet dann $\mathfrak{R}_1$ und $\mathfrak{R}_1'$ je auf sich selbst ab, W leistet aber dasselbe auch für jede Transformation der Watson-Klasse; denn wenn ein W von der Art (1*b*) es für irgendeine Transformation der Watson-Klasse tut, so leistet W es für jede, mit Rücksicht auf (3.2), s. auch Abschnitt 4.

Beispiel 4. χ_1 und χ_2 wie eben, $\mathfrak{R}(\nu) > \mathfrak{R}(\mu) > -1$. Wähle in (3.22)

$$\omega_0(t) = \tfrac{1}{2}\Gamma\{\tfrac{1}{2}(\nu-\mu)\}\,\frac{\Gamma\{\frac{1}{2}(\mu+1-it)\}}{\Gamma\{\frac{1}{2}(\nu+1-it)\}},$$

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† H.-T., Satz VI: $\mu = -\frac{1}{2}$, $\nu = \frac{1}{2}$ und $\mu = \nu = -\frac{1}{2}$; Mehrotra I, 235, (ii), $\mu = \nu = -\frac{1}{2}$, 237 (i), $\mu = -\frac{1}{2}$, $\nu = \frac{1}{2}$; Bailey, *loc. cit.*, 245.

so wird $\omega_{00}(t) = \omega_0(-t)$ (s. Bew. v. VIa), und für $0 < x \leqslant 1$ bezw. $x > 1$

$$\psi_0(x) = 0 \quad \text{bezw.} \quad \int_1^x \xi^{\frac{1}{2}-\nu}(\xi^2-1)^{\frac{1}{2}(\nu-\mu)-1}\,d\xi,$$

$$\psi_{00}(x) = \int_0^x \xi^{\frac{1}{2}+\mu}(1-\xi^2)^{\frac{1}{2}(\nu-\mu)-1}\,d\xi \quad \text{bezw.} \quad \psi_{00}(1),$$

$$W_0 : g(x) = \int_0^x \frac{x^{\frac{1}{2}-\nu}\,t^{\frac{1}{2}+\mu}f(t)}{(x^2-t^2)^{1+\frac{1}{2}(\mu-\nu)}}\,dt, \qquad W_{00} : g(x) = \int_x^\infty \frac{x^{\frac{1}{2}+\mu}\,t^{\frac{1}{2}-\nu}f(t)}{(t^2-x^2)^{1+\frac{1}{2}(\mu-\nu)}}\,dt,$$

W_0 bildet $\mathfrak{R}_1$ auf $\mathfrak{R}_2$, W_{00} aber $\mathfrak{R}_2$ auf $\mathfrak{R}_1$ ab, beide unvollständig[†].

Alle in diesem Abschnitte auftretenden Transformationen waren stetig, weil $\omega(t)$ immer beschränkt in $(-\infty, \infty)$ war, bei allen in der Literatur und hier behandelten $W[\psi]$ war $\psi(x)$ ein Integral, das Integral $\int \psi'(tx)f(t)\,dt$ bezw. $\int \psi'(xt^{-1})f(t)\,t^{-1}\,dt$ konvergierte, ausser im Beispiel 2, nicht nur im Quadratmittel (Verf. I, Satz 2_A), sondern auch absolut, wie man leicht sieht.

Aus (3.12) und (3.22) folgt, wenn $\mathfrak{R}_1$ und $\mathfrak{R}_2$ die Bedeutungen wie in den Beispielen 1, 3, und 4 haben:

Die allgemeinste Transformation W_0 der Art (1*a*) *bezw.* (1*b*), *welche* $\mathfrak{R}_1$ *auf* $\mathfrak{R}_2$ *abbildet, wird bestimmt durch* (*cf.* IV_B)

$$\omega_0(t) = 2^{it}\,\Gamma\{\tfrac{1}{2}(\mu+1+it)\}\,\Gamma\{\tfrac{1}{2}(\nu+1+it)\}\,\phi(t)$$

bezw. $$\omega_0(t) = \Gamma\{\tfrac{1}{2}(\mu+1-it)\}\,\Gamma\{\tfrac{1}{2}(\nu+1+it)\}\,\phi(t),$$

wobei $\phi(t)$ bis auf eine Menge vom Masse Null eine gerade, in jedem endlichen Intervalle bechränkte und messbare Funktion von t ist und gleich

$$O(|t|^{-\frac{1}{2}\mathfrak{R}(\mu+\nu)}e^{\frac{1}{2}\pi|t|}) \quad \textit{für} \quad t \to \pm\infty\,[‡].$$

[†] W_0: H.-T., Satz VI, $\mu = -\frac{1}{2}$, $\nu = \frac{1}{2}$; Mehrotra I, 237 (ii); W_0 und W_{00}: Bailey, 246 und 247. Im Falle $\mathfrak{R}(\mu) = \mathfrak{R}(\nu)$ erhält man eine vollständige Abbildung, für $\mu = \nu$ wähle man $\omega_0 = 1$, im Falle $\mathfrak{R}(\nu) < \mathfrak{R}(\mu)$, $\mathfrak{R}(\mu-\nu) < 1$, erhält man unstetige, aber schlechthin und stetig umkehrbare Transformationen W_0, W_{00}, die einen echten Teil von $\mathfrak{R}_1$ bezw. $\mathfrak{R}_2$ auf ganz $\mathfrak{R}_2$ bezw. ganz. $\mathfrak{R}_1$ abbilden (*cf.* Verf. II).

[‡] Art (1*a*): H.-T., $\mu = -\frac{1}{2}$, $\nu = \frac{1}{2}$, Satz IV; *cf.* Mehrotra IV, Theor. I. Art (1*b*): Mehrotra I, 232, man erhält formal den Übergang durch

$$\frac{d}{dx}\,\psi_0(x) = x^{-1}P(x^{-1}), \quad it + \tfrac{1}{2} = s\,;$$

cf. Mehrotra III, Theor. II und III.

Die in der Literatur behandelten Fälle lassen sich, natürlich nur für Funktionen aus dem Raume $\mathfrak{L}_2$, aus den Formeln dieses Abschnittes unschwer ableiten.

4. Satz V, s. Abschnitt 1. Möge $T[\chi]$ beliebig aus der Watson-Klasse gewählt sein, $W[\psi]$ stetig und von der Art (1*b*) sein, $\omega(t)$ bezw. $\Omega(t)$ die zu T bezw. W gehörigen Funktionen; sei ferner $\Omega(t)$ gerade. Dann ist sicher $\Omega(-t)\,\omega(t) = \Omega(t)\,\omega(t)$, also auf Grund der Vorbemerkung von Abschnitt 3 (s. 3.2) $TW = WT$, daher bildet W nach Satz I den Raum selbstreziproker Funktionen $\mathfrak{R}[\chi]$ von T auf sich selbst ab; auf den Raum $\mathfrak{R}' = \mathfrak{R}[-\chi]$ schiefreziproker Funktionen sicher, wenn $\Omega(t)$ ungerade ist, da dann $\Omega(-t)\,\omega(t) = -\Omega(t)\,\omega(t)$, also wegen (3.2) $T[\chi]\,W = WT[-\chi]$ ist. Diese Überlegungen lassen sich unschwer auf den Fall unstetiger involutorischer Transformationen T aus der Klasse (1*a*) ausdehnen. Die Bedingungen des Satzes V sind also hinreichend. Zum Beweise der Notwendigkeit genügt die Betrachtung nur gewisser Transformationen der Watson-Klasse: W muss natürlich, wenn es alle $\mathfrak{R}$ z.B. auf sich selbst abbildet, bei allen $\mathfrak{R}'$ dasselbe leisten.

Hilfssatz V_a. *Sei $\alpha > 0$, $\chi_\alpha(x) = 0$ bezw. α für $x < \alpha$ bezw. $x > \alpha$, $\mathfrak{T}$ die Menge aller Transformationen T_α der Form*

$$\int_0^x g(\xi)\,d\xi = \int_0^\infty \chi_\alpha(tx) f(t)\,\frac{dt}{t}, \tag{4.1}$$

dann gehört jede Transformation aus $\mathfrak{T}$ zur Watson-Klasse und hat die Gestalt $T_\alpha f = \alpha^{\frac{1}{2}} x^{-1} f(\alpha x^{-1})$.

Denn aus (4.1) folgt

$$g(x) = \frac{d}{dx}\int_{\alpha/x}^\infty \alpha^{\frac{1}{2}} f(t)\,\frac{dt}{t} = \alpha^{\frac{1}{2}} x^{-1} f(\alpha x^{-1}),$$

$$T_\alpha{}^2 f = T_\alpha g = \alpha^{\frac{1}{2}} x^{-1} g(\alpha x^{-1}) = f(x),$$

$$\int_0^\infty |T_\alpha f|^2\,dx = \alpha \int_0^\infty \frac{dx}{x^2}\left|f\left(\frac{\alpha}{x}\right)\right|^2 = \int_0^\infty |f(x)|^2\,dx,$$

T_α ist also von der Art (1*a*), involutorisch und stetig, sogar unitär.

Hilfssatz V_b. *Wenn eine lineare abgeschlossene Transformation W mit jeder Transformation aus $\mathfrak{T}$ vertauschbar bezw. schief vertauschbar—d.h. $WT_\alpha = -T_\alpha W$—ist, so ist W von der Art (1b) und stetig, die zugehörige Funktion $\Omega(t)$ gerade bezw. ungerade.*

Möge T_1 die Transformation $Rf(x) = x^{-1} f(x^{-1})$ bedeuten, so ist T_1 aus $\mathfrak{T}$,

$$T_1 T_\alpha f(x) = T_1 \{\alpha^{\frac{1}{2}} x^{-1} f(\alpha x^{-1})\} = \alpha^{\frac{1}{2}} f(\alpha x),$$

$$W T_1 T_\alpha f = \pm T_1 W T_\alpha f = T_1 T_\alpha W f$$

mit Rücksicht auf die Voraussetzung. Sei $Wf = g$, dann ist für jedes $\alpha > 0$

$$W\{\alpha^{\frac{1}{2}} f(\alpha x)\} = W T_1 T_\alpha f = T_1 T_\alpha W f = T_1 T_\alpha g = \alpha^{\frac{1}{2}} g(\alpha x),$$

$$Wf(\alpha x) = g(\alpha x) \quad \text{für} \quad Wf = g. \tag{4.2}$$

Da f beliebig aus $\mathfrak{L}_2$ ist, hat W den Bereich $\mathfrak{L}_2$. Aber eine lineare abgeschlossene Transformation mit der Eigenschaft (4.2) und dem Bereiche $\mathfrak{L}_2$ ist von der Art (1*b*) und stetig†. Die Vertauschbarkeit bezw. schiefe Vertauschbarkeit von $W[\psi]$ mit $T_1 = R$ ergibt wegen $\omega_1(t) = 1$ mit Rücksicht auf (3.2) für $\omega_2 = \pm\omega_1 = \pm 1$

$$\Omega(t) = \Omega(-t) \quad \text{bezw.} \quad \Omega(t) = -\Omega(-t).$$

Aus Satz I und Hilfssatz V_b folgt die Notwendigkeit der Bedingungen, Satz V ist bewiesen. Der erste Teil des Zusatzes folgt aus der Vorbemerkung von Abschnitt 3, der zweite Teil nach Verf. I, Satz 6A.

HILFSSATZ VI_A. *Sei* $W[\psi]$ *von der Art* (1*b*) *und stetig, die zugehörige Funktion* Ω. *Die Bedingungen* $\Omega(t) = \pm\Omega(-t)$ *sind bezüglich gleichwertig mit*

$$\int_0^y \psi\left(\frac{1}{u}\right) du = \pm y \int_0^{1/y} \psi\left(\frac{1}{u}\right) du \quad (y > 0). \tag{4.3}$$

Sei nämlich $S[\psi]$ von der Art (1*a*), also $W[\psi] = S[\psi]R$. Die Transformation $W[\psi_{00}] = RS[\psi]$ hat die zugehörige Funktion $\Omega(-t)$, wie man leicht durch zweimalige Anwendung von (3.11) erhält wegen $\omega_R(t) = 1$. Also ist $\Omega(t) = \pm\Omega(-t)$ gleichwertig mit $W[\psi_{00}] = \pm W[\psi]$, mit Rücksicht auf den Beweisgang von Satz II_B folgt die Behauptung.

HILFSSATZ VI_b. *Sei* $W[\psi]$ *von der Art* (1*b*) *und stetig,*

$$\psi(x) = \lim \int_\delta^x k(\xi)\, d\xi \quad \textit{für} \quad \delta \to 0;$$

dann ist die Bedingung $\Omega(t) = \pm\Omega(-t)$ *gleichwertig mit* $k(x) = \pm x^{-1} k(x^{-1})$.

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† M. Plancherel, *Proc. Cambridge Phil. Soc.*, 23 (1937), 413–418; Verf. II, Satz E_4 und Abschn. 4.

Durch zweimaliges Differenzieren folgt aus (4.3) $k(x) = \pm x^{-1} k(x^{-1})$, hieraus folgt umgekehrt

$$\pm\int_{1/a}^{1/y} k(u)\,du = \int_{1/a}^{1/y} k\left(\frac{1}{u}\right)\frac{du}{u} = \int_{y}^{a} k(u)\,\frac{du}{u} = \int_{y}^{a}\frac{d\psi(u)}{u} = \frac{\psi(u)}{u}\Big|_{y}^{a} + \int_{y}^{a}\psi(u)\,\frac{du}{u^{2}}$$

$$= \frac{\psi(a)}{a} - \frac{\psi(y)}{y} + \int_{1/a}^{1/y}\psi\left(\frac{1}{u}\right)du,$$

$$\pm\left(\psi\left(\frac{1}{y}\right) - \psi\left(\frac{1}{a}\right)\right) = \int_{1/a}^{1/y}\psi\left(\frac{1}{u}\right)du - \frac{\psi(y)}{y} + \frac{\psi(a)}{a}.$$

Wegen $\psi(\epsilon) \to 0$ für $\epsilon \to 0$ und wegen $\psi(1/u) \subset \mathfrak{L}_2$ folgt die Existenz von $\lim a^{-1}\psi(a)$ für $a \to \infty$, dieser Grenzwert ist Null wegen $u^{-1}\psi(u) \subset \mathfrak{L}_2$. Daher

$$\pm\psi\left(\frac{1}{y}\right) = \int_{0}^{1/y}\psi\left(\frac{1}{u}\right)du - \frac{\psi(y)}{y}.$$

Hieraus folgt (4.3) durch Integrieren und partielles Integrieren.

SATZ VI. *Sei*

$$\psi(x) = \lim_{\delta \to 0}\int_{\delta}^{x} k(\xi)\,d\xi, \quad \psi(x)/x \subset \mathfrak{L}_2;$$

sei $\Omega(t) = (\tfrac{1}{2} - it)\,M\{\psi(x)/x\}$ *beschränkt in* $(-\infty, \infty)$, $k(x) = \pm x^{-1} k(x^{-1})$, *dann bildet die Transformation*

$$g(x) = \underset{n \to \infty}{\text{l.i.m.}}\int_{1/n}^{n} k\left(\frac{x}{t}\right) f(t)\,\frac{dt}{t} = \pm\underset{n \to \infty}{\text{l.i.m.}}\,\frac{1}{x}\int_{1/n}^{n} k\left(\frac{t}{x}\right) f(t)\,dt$$

für jede involutorische Transformation der Art (1a) *den Raum* $\mathfrak{R}$ *auf sich selbst bezw. auf* $\mathfrak{R}'$ *ab*†.

Der Beweis folgt sofort aus Satz V und Hilfssatz VIb, s. auch Verf. I, Abschn. 6 und Satz 2A mit entsprechender Umgestaltung für (6.1) dort.

SATZ VI$_A$. *Baileys Formel mit der Abänderung*: $a \neq 0$, *Beschränkung auf* $\mathfrak{L}_2$. *Sei a fest*, $0 < a \leqslant 1$, $F(t)$ *in* $(a, 1/a)$ *integrabel und gleich* $\pm F(t^{-1})$, *so führt*

$$g(x) = Wf(x), \quad g(x) = \pm\int_{a}^{1/a} t^{-\frac{1}{2}} F(t) f(tx)\,dt$$

jedes $\mathfrak{R}$ *in sich selbst bezw. in* $\mathfrak{R}_1$ *über.*

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† Bailey, Theor. IIIa, Mehrotra I, 234, für den Fall der Hankel-Transformationen formal bezw. unter anderen Bedingungen ausgesprochen; *cf.* Mehrotra, IV.

Beweis. Sei $k(t) = t^{-\frac{1}{2}} F(t)$ in $(a, 1/a)$, $k(t) = 0$ in $(0, a)$ und $(1/a, \infty)$, also

$$\pm k(t) = t^{-1} k(t^{-1}), \quad \psi(x) = \int_0^x k(\xi)\, d\xi = \int_a^x k(\xi)\, d\xi \quad (x > a).$$

Wegen $\psi(x)/x = 0$ für $0 < x < a$ und $\psi = O(1)$ für $x \to \infty$ ist $\psi(x)/x \subset \mathfrak{L}_2$, das zu $W[\psi]$ gehörige $\Omega(t)$ (Verf. I, Satz 3i)

$$\Omega(t) = \int_0^\infty k(x)\, x^{-\frac{1}{2}+it}\, dx = \int_a^{1/a} F(x)\, x^{-1+it}\, dx = O\left(\int_a^{1/a} |\, F(x)\,|\, \frac{dx}{x}\right) = O(1),$$

also beschränkt, also leistet nach Satz VI die Transformation $W[\psi]$ das Verlangte,

$$g(x) = \pm \underset{n\to\infty}{\text{l.i.m.}} \int_{1/n}^{n} k\left(\frac{t}{x}\right) f(t)\, \frac{dt}{x} = \pm \underset{n\to\infty}{\text{l.i.m.}} \int_{1/(nx)}^{n/x} k(t)\, f(tx)\, dt$$

$$= \pm \int_a^{1/a} t^{-\frac{1}{2}} F(t)\, f(tx)\, dt.$$

Beispiel 1. $k(x) = \sin x \pm x^{-1} \sin x^{-1} = \pm x^{-1} k(x^{-1})$. Wegen

$$\lim_{\delta\to 0} \int_\delta^x \frac{du}{u} \sin\frac{1}{u} = \lim_{U=\delta^{-1}\to\infty} \int_{1/x}^{U} \frac{du}{u} \sin u$$

ist ein $\psi(x)$ im Sinne von Satz VI definiert, es folgt leicht $\psi(x) = O(x)$ bezw. $O(1)$ für $x \to 0$ bezw. $x \to \infty$, also ist (Verf. I, Satz 3i)

$$\Omega(t) = \lim_{\delta\to 0} \int_\delta^{1/\delta} k(x)\, x^{-\frac{1}{2}+it}\, dx = (\tfrac{1}{2}\pi)^{\frac{1}{2}} \left\{ 2^{it} \frac{\Gamma(\frac{3}{4}+\frac{1}{2}it)}{\Gamma(\frac{3}{4}-\frac{1}{2}it)} \pm 2^{-it} \frac{\Gamma(\frac{3}{4}-\frac{1}{2}it)}{\Gamma(\frac{3}{4}+\frac{1}{2}it)} \right\},$$

$|\,\Omega(t)\,| \leqslant (2\pi)^{\frac{1}{2}}$. Die Transformation

$$g(x) = \underset{n\to\infty}{\text{l.i.m.}} \int_{1/n}^{n} \left\{ \frac{1}{t} \sin\frac{x}{t} \pm \frac{1}{x} \sin\frac{t}{x} \right\} f(t)\, dt$$

ist wohl umkehrbar, da auf Grund des analytischen Charakters $\Omega(t)$ in $(-\infty, \infty)$ höchstens auf einer Menge vom Masse Nulle verschwindet, aber nicht schlechthin, die Überdeckung des jeweils zweiten Raumes ist also unvollständig; denn $\Omega(t)$ kann beliebig oft beliebig klein werden: Nach Stirlings Formel ist:

$$(2\pi)^{-\frac{1}{2}}\, \Omega(t) = \genfrac{}{}{0pt}{}{\cos}{i\,\sin}\, (t \log t - t + \tfrac{1}{4}\pi) + O\left(\frac{1}{t}\right) \quad (t \to \infty).$$

Ganz entsprechende Ergebnisse erhält man für den Fall

$$k(x) = \cos x \pm x^{-1} \cos x^{-1}.$$

Beispiel 2. Sei $b > 2$, $1 \geqslant a > 0$, sei bezüglich $\psi(t) = 0$, $a^{-\frac{1}{2}}$, $b+a^{-\frac{1}{2}}$, $b+a^{-\frac{1}{2}}+a^{\frac{1}{2}}$ in $(0, a^{-1})$, $(a^{-1}, 1)$, $(1, a)$, (a, ∞), so bildet $g = Wf$,

$$g(x) = a^{-\frac{1}{2}} f(x/a) + bf(x) + a^{\frac{1}{2}} f(ax)$$

jedes System $\mathfrak{R}[\chi]$ ein-eindeutig und vollständig auf sich selbst ab, wegen

$$\Omega(t) = (\tfrac{1}{2} - it) \int_0^\infty \psi(x)\, x^{-\frac{3}{2}+it}\, dx = b + 2\cos(t \log a) = \Omega(-t) \geqslant b-2 > 0.$$

Beispiel 3. Dasselbe leistet die Transformation† $g = Wf$

$$g(x) = f(x) + \frac{1}{\pi} \int_0^\infty \frac{f(t)}{t+x}\, dt = \frac{d}{dx} \int_0^\infty \psi\left(\frac{x}{t}\right) f(t)\, dt,$$

wobei

$$\psi(x) = \frac{1}{\pi} \log(1+x) + \begin{matrix} 0 & \text{für} & 0 < x < 1, \\ 1 & \text{für} & x > 1, \end{matrix}$$

$$\Omega(t) = 1 + 1/\cosh \pi t = \Omega(-t), \quad 1 < \Omega(t) \leqslant 2.$$

Beispiel 4. Dasselbe leistet die Umkehrung $f = W^{-1} g$ der letzten Transformation. Die zugehörige Funktion Ω^* ist (Verf. I, Satz 6_A)

$$\Omega^*(t) = \Omega^{-1}(t) = \tfrac{1}{2} \cosh \pi t / \cosh^2 \tfrac{1}{2}\pi \mathrm{t}, \quad \tfrac{1}{2} \leqslant \Omega^*(t) < 1,$$

$$\psi^*(x) = \frac{2}{\pi^2} \int_0^x \frac{du\, u^{\frac{1}{2}} \log u}{1-u^2} + \begin{matrix} 0 & \text{für} & 0 < x < 1, \\ 1 & \text{für} & x > 1, \end{matrix}$$

$$f(x) = \frac{d}{dx} \int_0^\infty \psi^*\left(\frac{x}{t}\right) g(t)\, dt = g(x) + \frac{2}{\pi^2} \int_0^\infty \frac{(tx)^{\frac{1}{2}} \log x/t}{t^2 - x^2}\, g(t)\, dt.$$

Breslau 18,
Frobenstr. 9.

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† Die—unvollständige—Abbildung durch W $[\psi = \log(1+x)]$, $\Omega(t) = \pi/\cosh \pi t$, bildet auch jedes $\mathfrak{R}$ auf sich selbst ab, natürlich auch für die Fourier-cos-Transformation, H.-T., 230.

SIMPLICIAL SPACES, NUCLEI AND m-GROUPS

By J. H. C. WHITEHEAD.

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1. *Introductory.*

This paper centres round a generalization of the notion of a group which may be briefly described as follows. We start with the definition of a symbolic complex K, as a set of sets, called simplexes, which contains each sub-set of any simplex in K[†]. Restricting ourselves to complexes in which the dimensionality of each simplex is finite, though the dimensionalities of the simplexes need not have an upper bound, we separate complexes into mutually exclusive equivalence classes by means of certain elementary transformations of "order m" and associate an abstract "m-group" with each class, where $m = -1, 0, 1, \ldots$. The interest begins with $m = 2$, and it appears that two connected complexes, finite or infinite, have the same 2-group if, and only if, they have the same fundamental group. Moreover, any group is isomorphic to the fundamental group of some complex. If an abstract group is taken to be an object associated with a class of mutually isomorphic groups[‡], we may therefore identify 2-groups with abstract groups. That is to say, a geometric, or set-theoretic, representation of an abstract group by a complex may be regarded as equivalent to an algebraic representation by a set of elements with a multiplication table. From this point of view an m-group is seen to be an automatic generalization of an abstract group.

Two complexes have the same m-group for each value of m if they have the same homotopy type. In particular the m-group of a geometrical complex is a topological invariant. The converse applies to complexes of bounded dimensionality. In fact, two complexes of at most n dimensions are of the same homotopy type if they have the same $(n+1)$-group.

[†] *Cf.* P. Alexandroff und H. Hopf, *Topologie*, 1 (Berlin, 1935), p. 155.

[‡] This is how one normally uses the word in referring, for example, to the 6-group.

The simplicial spaces and nuclei, referred to in the title, are also defined in terms of elementary transformations. There is nothing new about the former. Two symbolic complexes determine the same simplicial space if, and only if, they are combinatorially equivalent in the sense of J. W. Alexander† and M. H. A. Newman‡. Nuclei are defined in terms of what we call formal deformations. An elementary sub-division is a formal deformation, and the nucleus of a complex is therefore a combinatorial invariant, meaning that two complexes which are combinatorially equivalent have the same nucleus. Whether or no the nucleus is a topological invariant remains an open question, except in the case of a finite complex whose fundamental group satisfies a certain condition, described in § 11. Two finite complexes whose fundamental groups satisfy this condition have the same nucleus if they are of the same homotopy type.

In a section on manifolds it is proved that any complex K, imbedded in a manifold M, has a "regular neighbourhood" in M and that any two regular neighbourhoods of the same complex in the same manifold are combinatorially equivalent. Moreover, if p is large enough, regular neighbourhoods of two finite complexes in Euclidean p-space R^p are combinatorially equivalent if the complexes have the same nucleus. In particular, a regular neighbourhood of a finite n-dimensional complex K^n, in R^p ($p \geqslant 2n+5$), is a p-element if the (multiplicative) fundamental group of K^n is unity and all its (additive) homology groups are zero. If the fundamental group and homology groups of K^n are the same as those of an n-sphere its regular neighbourhood in R^p is the topological product of an n-sphere and a $(p-n)$-element.

The presentation may be summarized as follows. With one or two exceptions everything in §§ 3–9 is needed for the proof of Theorem 17, which states that two (finite) complexes are of the same homotopy type if, and only if, they have the same m-group for each value of m. The main exception is Theorem 12, stating that two connected complexes have the same fundamental group if, and only if, they have the same 2-group. Sections 10 and 11 lead up to Theorem 21, that, subject to the condition on the fundamental group stated in § 11, two finite complexes of the same homotopy type have the same nucleus. Section 12 is concerned with regular neighbourhoods of complexes in manifolds, and § 13 is an appendix to the sections on finite complexes. Section 14 is concerned with the combinatorial, and § 15 with the topological theory of infinite complexes.

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† *Annals of Math.*, 31 (1930), 292–320.

‡ *Akad. Wet. Amsterdam*, 29 (1926), 611–626; 627–641.

In the final § 16 it is shown how many of the earlier results may be extended from finite to infinite complexes.

Nuclei and m-groups are closely related to the homotopy groups discovered by W. Hurewicz[†], and §§ 8 and 10 below may overlap with the more complete account which was announced in the first of his notes on homotopy groups. In particular, I learn from S. Eilenberg that Theorem 15, and from Shaun Wylie that the group of automorphisms $\psi_n(g)$ in § 11 are known to Hurewicz and others. But, since these are auxiliary to the main theorems, I have given full details without further reference except to what has already been published.

2. *Nuclei and m-groups.*

We start with an infinite aggregate of undefined vertices (*Eckpunktbereich*) $a, b, c, \ldots$. Any set of $n+1$ vertices $(n \geqslant -1)$ will be called an *open* (*symbolic*) *n-simplex*[‡]. A *closed* (*symbolic*) *n-simplex* will be the closure of an open n-simplex, consisting of a set of $n+1$ vertices together with all its subsets, including the empty set or (-1)-simplex. The closure $Cl(\Sigma)$, of a set of simplexes Σ, will consist of the closures of the simplexes in Σ, and Σ will be described as *closed* if $\Sigma = Cl(\Sigma)$. By a *symbolic complex* we shall mean any closed set of simplexes. As usual a complex will be described as finite or infinite according as it contains a finite or an infinite number of simplexes. Until the end of § 13 it is to be understood that, except where the contrary is stated, all the complexes referred to are finite. Our formalism is similar to that of J. W. Alexander except that, instead of his "mod 2" or nilpotent algebra, we use the idempotent algebra of logic. Thus $\Sigma_1+\Sigma_2$, $\Sigma_1-\Sigma_2$ and $\Sigma_1 . \Sigma_2$ will denote respectively the set of simplexes in either of two given sets Σ_1 and Σ_2, the set in Σ_1 but not in Σ_2, and the set common to both. We shall continue to use multiplication without the dot $\Sigma_1\Sigma_2$ to stand for the join of the two sets; that is to say, for the totality of simplexes $(a_0, \ldots, a_n, b_0, \ldots, b_m)$, where[§] $(a_0, \ldots, a_n) \in \Sigma_1$ and $(b_0, \ldots, b_m) \in \Sigma_2$. This operation is also to be idempotent, so that

$$\Sigma+\Sigma = \Sigma . \Sigma = \Sigma\Sigma = \Sigma.$$

[†] *Akad. Wet. Amsterdam*, 38 (1935), 112–119; 521–528; 39 (1936), 117–125; 215–223.

[‡] The word "symbolic" will be omitted except when a contrast with geometrical simplexes and complexes is necessary. Also when it is obvious which is meant, or irrelevant, we shall refer to either an open or a closed simplex simply as a simplex, and shall use the same kind of letter, namely, A, B or C, to stand for both.

[§] If A is an open simplex belonging to a set Σ we write $A \in \Sigma$, but for a closed simplex we write $A \subset \Sigma$. We use $\bar{\in}$ to stand for "not $\in$."

The empty, or (−1)-dimensional, simplex plays a part analogous to zero with respect to the multiplication $\Sigma_1 . \Sigma_2$ and to unity with respect to $\Sigma_1 \Sigma_2$. Following Alexander we shall denote it by 1. Thus[†]

$$\Sigma . 1 = 1, \quad \Sigma 1 = \Sigma.$$

Though the simplex 1 belongs to every complex, we shall say that two sets of simplexes meet each other if, and only if, they have a k-simplex in common, where $k \geqslant 0$.

The boundaries which appear in this paper are calculated with residue classes mod 2 as coefficients, and we follow Alexandroff and Hopf in using[‡] $\dot{K}$ to stand for the boundary of a complex K. We adopt Alexander's convention that the boundary of a 0-simplex is 1 and the boundary of 1 is 0, the empty set of simplexes. The latter satisfies the conditions

$$\Sigma \pm 0 = \Sigma, \quad \Sigma . 0 = \Sigma 0 = 0,$$

where Σ is any set of simplexes, and is therefore analogous to zero in its relation to all four operations.

We now associate three kinds of abstract object, a *simplicial space*, a *nucleus*, and an *m-group* ($m = -1, 0, 1, \ldots$) with every complex. By analogy with polyhedra we shall describe a complex as a *triangulation* of the corresponding simplicial space, but we shall refer to the nucleus and the m-group, like the fundamental group and other classical invariants, as properties of the complex. Thus, under appropriate conditions, we shall say that two complexes have the same nucleus or m-group.

1. *Simplicial spaces.* Two complexes will be described as triangulations of the same simplicial space if, and only if, they are combinatorially equivalent. Here we adopt Alexander's definition of combinatorial equivalence in terms of elementary sub-divisions. We recall that an elementary sub-division[§] (A^k, a), of order k, is a transformation of the form

$$K = A^k P + Q \rightarrow a \dot{A}^k P + Q,$$

where A^k is a closed k-simplex in K, a is not in K, and $P = K_{A^k}$, the complement† of A^k in K. Two (finite) complexes are said to be combinatorially equivalent if, and only if, one is transformable into the other by a finite sequence of elementary sub-divisions and their inverses. A property which is unaltered by an elementary sub-division or its inverse is called a combinatorial invariant.

[†] In general, $\Sigma + 1 \neq \Sigma$, but $K + 1 = K$ if K is a complex.

[‡] We shall also use $\dot{A}$ to stand for the boundary of an open simplex A (*i.e.* for the boundary of its closure).

[§] Superscripts will invariably denote dimensionality.

2. *Nuclei.* If

$$K_1 = K_0 + aA,$$

where A is a closed simplex such that $a\dot{A} \subset K_0$, $A \not\subset K_0$, the transformation $K_0 \to K_1$ will be called an *elementary expansion*, and $K_1 \to K_0$ will be called an *elementary contraction*‡. As a matter of convention we admit the identical transformation $K_0 \to K_0$ both as an elementary expansion and as an elementary contraction§. An elementary expansion or contraction will be called an *elementary deformation*, and the resultant of a finite sequence of elementary deformations a *formal deformation*. We shall denote a formal deformation by the letter D. Two (finite) complexes will be said to have the same *nucleus* if, and only if, one is transformable into the other by a formal deformation.

If the simplex aA is m-dimensional, we shall describe

$$K_0 \to K_1 = K_0 + aA,$$

or $K_1 \to K_0$, as an elementary expansion, or contraction, of *order m*.

3. *m-groups.* If

$$K_1 = K_0 + A^k \quad (k \geqslant 0),$$

where A^k is a k-simplex such that $\dot{A}^k \subset K_0$, $A^k \not\subset K_0$, we shall describe the transformation $K_0 \to K_1$ as a *filling of order k* and $K_1 \to K_0$ as a *perforation* of order k. Two complexes will be said to have the same m-group if, and only if, one is transformable into the other by a finite sequence of elementary deformations and, possibly, fillings and perforations whose orders exceed m. Thus two complexes with the same nucleus have the same m-group for each value of m, and if $n > m$ two complexes with the same n-group have the same m-group.

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† *Cf.* Alexander, *loc. cit.* We shall always use K_A to stand for the complement of a (closed or open) simplex A in K, and, when we write K in the form $AP+Q$, it is to be understood that A is closed and $P = K_A$. As in Alexander's paper, vertices will always be denoted by small Roman letters.

‡ Transformations of this kind have been previously studied by I. Johansson, *Avhand. Norske Vidensk.-Akad.* (1932), No. 1.

§ The cases $K_0 = K_1 = 0$ or 1 are possible. The identity $K_0 \to K_0$ is the only elementary transformation of any kind which is applicable to the empty complex. The only elementary transformation which is applicable to 1 is a filling of order -1, defined below, which transforms 1 into a closed 0-simplex. It is to be assumed throughout that no " given complex " is the complex 0, which will appear only, in special cases, as a term in a calculation.

3. *Formal deformations.*

If a complex L can be transformed into K by a sequence of elementary expansions we shall say that L *expands* into K and that K *contracts* into L. If L is a single vertex, we shall describe K as *collapsible*. The open simplexes aA and A can be removed by an elementary contraction of K if, and only if, $aA \in K$ and aA is the only simplex in K having A on its boundary. If, in a contraction of K_0, an elementary contraction $K_i \rightarrow K_{i+1} = K_i - aA^p - A^p$ is immediately followed by $K_{i+1} \rightarrow K_{i+1} - bB^q - B^q$, where $q > p$, it follows that these two elementary contractions are interchangeable. For $bB^q \in K_{i+1} \subset K_i$, and $B^q \bar{\in} (aA^p)^{\cdot}$ since $q > p$. Therefore bB^q is the only simplex in K_i with B^q on its boundary, and bB^q and B^q may be removed first and then A^p and aA^p. When we repeat this argument, it follows that the elementary contractions in a given contraction of K_0 may be so arranged that all those of order q precede those of order p if $p < q$. In particular, if K_0 is collapsible it follows that K_0 contracts first into a linear graph containing all the vertices[†] of K_0 and then into a vertex. The graph, being collapsible, is obviously a tree and therefore contracts into a given one of its vertices. Thus any collapsible complex contracts into a given one of its vertices.

LEMMA 1. *If* $K . L_0 \subset L_q$ *and if* L_0 *contracts into* L_q, *then* $K + L_0$ *contracts into* $K + L_q$.

Let the transformation $L_0 \rightarrow L_q$ be the resultant of elementary contractions $L_i \rightarrow L_{i+1} = L_i - a_i A_i - A_i$ $(i = 0, \ldots, q-1)$, where A_i and $a_i A_i$ are open simplexes. Since $K . L_0 \subset L_q \subset L_{i+1}$, the simplex A_i is not on the boundary of any simplex in K and it follows that the transformation $K + L_i \rightarrow K + L_{i+1}$ is an elementary contraction of $K + L_i$. Therefore the transformation $K + L_0 \rightarrow K + L_q$ is a contraction and the lemma is established.

[†] Strictly speaking, we should refer to the 0-simplexes, rather than to the vertices of a complex. But we shall usually refer to a 0-simplex as a vertex, and a vertex may mean either an open or a closed 0-simplex according to the context [if a is an open 0-simplex $Cl(a) = a + 1$]. The distinction is not a trivial one, since the join aK does or does not contain K according as the 0-simplex a is closed or open.

LEMMA 2. *If $L\subset K$ and A is any closed simplex which does not meet K ($A\neq 1$, possibly $L=1$), the complex AK contracts into AL.*

First let A be a single vertex a. If $K=L$ there is nothing to prove. Otherwise let B be a principal† open simplex of $K-L$. Then the closure of aB meets $a(K-B)$ in $a\dot{B}$. Therefore aK contracts into $a(K-B)$ and the special case of the lemma follows from induction on the number of simplexes in $K-L$. In general, let $A=aA_1$. Then it follows from what we have already proved that $AK(=aA_1K)$ contracts into $aA_1L(=AL)$ and the lemma is established.

COROLLARY. *AK is collapsible ($A\neq 1$).*

For if $A=aA_1$ (possibly $A_1=1$) it follows from the lemma, with $L=1$ and K replaced by A_1K, that aA_1K contracts into a.

LEMMA 3. *If $K=AL_0+Q$ ($L_0\neq 1$) and L_0 contracts into L, then K contracts into $AL+\dot{A}L_0+Q$.*

If $L_0=L$, there is nothing to prove. Otherwise let

$$L_0=L_1+bB \quad (b\dot{B}\subset L_1),$$

where bB is a closed simplex and $L_0\rightarrow L_1$ is the first step in some process of contracting L_0 into L. Then the simplex AbB meets $AL_1+\dot{A}L_0+Q$ in

$$Ab\dot{B}+\dot{A}bB=b(AB)^{\cdot},$$

and

$$\begin{aligned} AL_0+Q &= AL_0+\dot{A}L_0+Q \\ &= A(L_1+bB)+\dot{A}L_0+Q \\ &= AL_1+\dot{A}L_0+Q+AbB. \end{aligned}$$

Therefore the transformation

$$AL_0+Q\rightarrow AL_1+\dot{A}L_0+Q$$

is an elementary contraction and the lemma follows from induction on the number of simplexes in L_0-L.

COROLLARY. *If L_0 is collapsible AL_0+Q contracts into $\dot{A}L_0+Q$.*

For AL_0+Q contracts into $Ab+\dot{A}L_0+Q$, where b is any vertex in L_0, and so into $\dot{A}L_0+Q$ by Lemma 2 and Lemma 1.

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† *I.e.* B is not on the boundary of any other simplex in K, though it need not be a simplex of maximum dimensionality.

Notice two special cases of the corollary: first, if L_0 is collapsible, aL_0+Q contracts into L_0+Q, where a is any vertex not in L_0+Q; secondly, if L_0 is a single closed simplex B, then $AB+Q$ contracts into $\dot{A}B+Q$.

THEOREM 1. *The nucleus of a complex is a combinatorial invariant.*

It is enough to show that K_0 and K_1 have the same nucleus, where K_1 is derived from K_0 by an elementary sub-division (A, a). Let

$$K_0 = AP+Q, \quad K_1 = a\dot{A}P+Q \quad (A \not\subset Q, a \not\subset K_0).$$

By Lemma 2 the star aAP, with a as centre, contracts into $a\dot{A}P$. Since $A \not\subset Q$ we have $AP \,.\, Q \subset \dot{A}P$, and it follows from Lemma 1 that the complex $aAP+Q$ contracts into K_1. But AP is collapsible, by the corollary to Lemma 2. Therefore $aAP+Q$ also contracts into K_0, by the corollary to Lemma 3, and the theorem follows.

By a *contractible neighbourhood* of a complex L we mean a complex N which contains L as a sub-complex and satisfies the conditions:

1. *N is a normal simplicial neighbourhood of L, meaning that every principal closed simplex in N meets L, but no open simplex in $N-L$ has all its vertices in L,*

2. *$L \,.\, N_A$ is collapsible, where A is any closed simplex in N which does not meet L.*

THEOREM 2. *Any contractible neighbourhood of L contracts into L.*

Let N be a contractible neighbourhood of L, and let R be the complex consisting of the closed simplexes in N which do not meet L. If $R = 1$ every vertex lies in L and, since N is a normal neighbourhood, it follows that $N = L$ and there is nothing to prove. Otherwise let A be a principal closed simplex of R and let $N = AP+Q$. If some simplex in P was not in L it would contain at least one vertex a, in R. We should then have $aA \subset R$, contrary to the fact that A is a principal simplex of R. Therefore $P \subset L$ and, since N is a contractible neighbourhood, P is collapsible. Therefore N contracts into $N^* = \dot{A}P+Q$, by the corollary to Lemma 3. Let $B \subset L \,.\, N_C$, where $C \subset R$ and $C \neq A$. Then $A \not\subset BC$, since $A \not\subset C$, and it follows that $BC \subset N^*$, whence $L \,.\, N_C{}^* = L \,.\, N_C$. Therefore the second condition for a contractible neighbourhood, and obviously the first, is satisfied by N^*, and the theorem follows from induction on the number of simplexes in R.

Let L be any sub-complex of a given complex K, let K_1 be the complex consisting of all the closed simplexes in K which do not meet L and let $R = K_1 . N(L, K)$, where $N(L, K)$ stands for the set of all closed simplexes in K which meet L. The transformation $K \rightarrow K^* = aR + K_1$, where a is any vertex not in K_1, will be described as the operation of *shrinking* L *into a point*, namely the vertex a. It is the result of formally identifying the vertices in L with the vertex a, which may or may not belong to L.

THEOREM 3. *If L is collapsible and $N(K, L)$ is a contractible neighbourhood of L the operation of shrinking L into a point is a formal deformation.*

Let $K^* = aR + K_1$, where R and K_1 mean the same as before and $a \not\subset K_1$. Since an elementary subdivision of order zero is a formal deformation we may assume $a \not\subset K$, and the proof is similar to the proof of Theorem 1. For the complex $N(L, K)$ is collapsible, since it contracts into the collapsible complex L, and the complex $aN(L, K) + K_1$ contracts both into K^* and into K. Therefore $K^* = D(K)$ and the theorem is established.

If K and L are any complexes we shall use $s_L K$ to stand for the sub-division of K which consists of starring every simplex in

$$K - L \quad (= K - K . L)$$

in order of decreasing dimensionality, and $s_L^2 K$ will mean $s_L(s_L K)$. We conclude this section with an existence lemma.

LEMMA 4. *If $N(L, K)$ is a normal neighbourhood of L $(L \subset K)$, then $N = N(L, s_L K)$ is a contractible neighbourhood of L. Moreover, $L . N_A$ is a single closed simplex, where A is any open simplex in $N - L$.*

Let $A_1, \ldots, A_p$ be the open simplexes in $K - L$, arranged so that† $\dim(A_\lambda) \leqslant \dim(A_\mu)$ if $\lambda < \mu$, and let a_λ be the internal vertex of $s_L A_\lambda$. By an easy extension of a known theorem‡, $s_L K$ contains the open simplex $A = B a_{\lambda_0} \ldots a_{\lambda_r}$, where $B \in L$ and $\lambda_0 < \ldots < \lambda_r$, if, and only if,

$$B \in \dot{A}_{\lambda_0}, \quad A_{\lambda_i} \in \dot{A}_{\lambda_{i+1}} \quad (i = 0, \ldots, r-1),$$

and any simplex in $s_L K$ is of this form (possibly with $B = 1$ or $r = -1$, *i.e.* $a_{\lambda_0} \ldots a_{\lambda_r} = 1$). Therefore N is a normal neighbourhood of L. Any open simplex in N whose closure does not meet L is of the form

$$A^* = a_{\lambda_0} \ldots a_{\lambda_r} \quad (\lambda_0 < \ldots < \lambda_r)$$

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† $\dim(P)$ stands for the dimensionality of P.

‡ H. Seifert und W. Threlfall, *Lehrbuch der Topologie* (Berlin, 1934), 230, Theorem II.

and it follows that

$$L . N_{A^*} = L . \dot{A}_{\lambda_0}.$$

Since any simplex in K is contained in L if all its vertices are in L it follows that the closure, and hence the boundary, of any simplex in $K-L$ meets L, if at all, in a single closed simplex. Therefore, $L . N_{A^*}$ is a single closed simplex and N is a contractible neighbourhood of L.

Let $A = A^* B$ be any open simplex in $N-L$ whose closure meets L, where $A^* = a_{\lambda_0} \dots a_{\lambda_r}$ and $B \epsilon L$ $(B \neq 1)$. To say that $B_1 \epsilon N_A$, where B_1 is any open simplex having no vertex in common with A, is to say that $AB_1 = A^* BB_1 \epsilon N$, or that $BB_1 \epsilon N_{A^*}$. If $B_1 \epsilon L$ it follows that $BB_1 \epsilon L$, since all its vertices are in L, and hence that $B_1 \epsilon (L . N_{A^*})_B$. Conversely, if $B_1 \epsilon (L . N_{A^*})_B$ it follows that $B_1 \epsilon L . N_A$. Therefore $L . N_A = (L . N_{A^*})_B$, and since $L . N_{A^*}$ is a single closed simplex, so is $L . N_A$.

If L is any sub-complex of a given complex K, it follows from the first argument in the proof of Lemma 4 that $N(L, s_L K)$ is a normal neighbourhood of L, and we have the corollary:

COROLLARY. *$N = N(L, s_L^2 K)$ is a contractible neighbourhood of L. Moreover $L . N_A$ is a single closed simplex, where A is any open simplex in $N-L$.*

4. *Sub-division.*

We shall need some theorems concerning sub-division and its relation to formal deformation. By a stellar sub-division we shall mean the resultant of a sequence of elementary sub-divisions (A, a), and a stellar sub-division will always be represented by the letter σ. In dealing simultaneously with two or more complexes $K_1, K_2, \dots$, it is always to be understood that no vertex introduced by a sub-division σK_1 belongs to any of the others. Thus σ may be regarded as operating simultaneously on all the complexes, with the convention that (A, a) leaves K unaltered if A does not belong to K. We admit the identical transformation as an elementary sub-division, which we denote by 1, and by $\sigma = 1$ in K we shall mean that σ is a product of elementary sub-divisions which leave K unaltered.

THEOREM 4. *If K contracts into L, then σK contracts into σL, where σ is any stellar sub-division of K.*

Using a double induction, first on the number of elementary sub-divisions in σ, and then on the number of elementary contractions in the transformation $K \rightarrow L$, we see that it is sufficient to consider the case where σ is a single elementary sub-division (A, a), and $K \rightarrow L$ a single elementary contraction. Let

$$K = L + bB,$$

where bB is a closed simplex meeting L in $b\dot{B}$. The theorem is then obvious unless $A \subset bB$ and, by Lemma 1, it is enough to prove that σbB contracts into $\sigma b\dot{B}$.

First let $A \subset B$. Then

$$\sigma bB = b(\sigma B), \quad \sigma b\dot{B} = b(\sigma\dot{B})$$

and σbB contracts into $\sigma b\dot{B}$, by Lemma 2. If $A \not\subset B$, let $A = bB_1$ and $B = B_1 B_2$. Then

$$\begin{aligned} \sigma bB &= a(b\dot{B}_1 + B_1)\, B_2 \\ &= ab\dot{B}_1\, B_2 + aB, \\ \sigma b\dot{B} &= \sigma b(B_1\, \dot{B}_2 + \dot{B}_1\, B_2) \\ &= a(b\dot{B}_1 + \dot{B}_1)\, \dot{B}_2 + b\dot{B}_1\, B_2. \end{aligned}$$

Removing the open simplexes B and aB from σbB, we are left with

$$\begin{aligned} ab\dot{B}_1\, B_2 + a\dot{B} &= ab\dot{B}_1\, B_2 + a(B_1\, \dot{B}_2 + \dot{B}_1\, B_2) \\ &= ab\dot{B}_1\, B_2 + aB_1\, \dot{B}_2, \end{aligned}$$

since $a\dot{B}_1\, B_2 \subset ab\dot{B}_1\, B_2$. Clearly $ab\dot{B}_1\, B_2$ and $aB_1\, \dot{B}_2$ meet in $a\dot{B}_1\, \dot{B}_2$, which is contained in $(a\dot{B}_1\, B_2)^{\cdot}$. It follows from Lemmas 2 and 1, with b taking the place of the simplex A in Lemma 2, that $ab\dot{B}_1\, B_2 + aB_1\, \dot{B}_2$ contracts into

$$\begin{aligned} b(a\dot{B}_1\, B_2)^{\cdot} + aB_1\, \dot{B}_2 &= b\dot{B}_1(a\dot{B}_2 + B_2) + aB_1\, \dot{B}_2 \\ &= a(b\dot{B}_1 + \dot{B}_1)\, \dot{B}_2 + b\dot{B}_1\, B_2 \\ &= \sigma b\dot{B}, \end{aligned}$$

and the theorem is established.

Let $P_1, \ldots, P_n$ be sub-complexes of a complex K and let Q_i be a sub-complex of P_i $(i = 1, \ldots, n)$ with the following property: If σ_1 is any stellar sub-division of P_i, some stellar sub-division $\sigma_2 \sigma_1 P_i$ contracts into $\sigma_2 \sigma_1 Q_i$. Then we have the corollary to Theorem 4:

COROLLARY. *There is a sub-division σK such that σP_i contracts into σQ_i for each value of i.*

The corollary is trivial if $n = 1$, and we shall prove it by induction on n. Assume that $\sigma_1 P_\lambda$ contracts into $\sigma_1 Q_\lambda$ $(\lambda = 1, \ldots, n-1)$, where σ_1 is some stellar sub-division of K. By hypothesis, there is a sub-division $\sigma_2 \sigma_1 P_n$ which contracts into $\sigma_2 \sigma_1 Q_n$. By Theorem 4, $\sigma_2 \sigma_1 P_\lambda$ contracts into $\sigma_2 \sigma_1 Q_\lambda$ $(\lambda = 1, \ldots, n-1)$, whence σP_i contracts into σQ_i $(i = 1, \ldots, n)$, where $\sigma = \sigma_2 \sigma_1$.

THEOREM 5. *If $K_q = D(K_0)$, there is a complex which contracts both into K_0 and into a stellar sub-division of K_q.*

Let
$$K_\lambda = C_\lambda E_\lambda \ldots C_1 E_1(K_0) \quad (\lambda = 1, \ldots, q),$$
where $E_1, \ldots, E_q$ are expansions and $C_1, \ldots, C_q$ contractions†. If σ is any stellar sub-division it follows from Theorem 4 that σK_λ expands into $\sigma E_{\lambda+1}(K_\lambda)$ $(\lambda < q)$ and that the latter contracts into $\sigma K_{\lambda+1}$, and from induction on $q-\lambda$ that
$$\sigma K_q = C_q' E_q' \ldots C'_{\lambda+1} E'_{\lambda+1}(\sigma K_\lambda),$$
where $E'_{\lambda+1}, \ldots, E_q'$ are expansions and $C'_{\lambda+1}, \ldots, C_q'$ contractions. Therefore the theorem will follow from induction on q if we can show that, when $q > 1$, K_0 expands into some complex which contracts into a stellar sub-division of K_2 [if $q = 1$ the complex $E_1(K_0)$ satisfies the required conditions].

Let
$$K_{01} = E_1(K_0), \quad K_{12} = E_2(K_1) = E_2 C_1 E_1(K_0),$$
let $K'_{12} = s_{K_1} K_{12}$, the new vertices introduced by s_{K_1} being, as usual, different from any of those in K_{01}, and let $e_1, \ldots, e_r$ be the elementary expansions of which E_2 is the resultant (e_λ applied before $e_{\lambda+1}$). If e_λ is the transformation $K^* \to K^* + aA$, let e_λ' stand for the transformation
$$s_{K_1} K^* \to s_{K_1}(K^* + aA),$$
which is an expansion by Theorem 4, and let $E_2' = e_r' \ldots e_1'$. If $C_2 = c_s \ldots c_1$, where $c_1, \ldots, c_s$ are elementary contractions, let c_λ' be similarly defined and let $C_2' = c_s' \ldots c_1'$. Then
$$s_{K_1} K_2 = C_2' E_2' C_1 E_1(K_0).$$

I say that the expansion E_2' is interchangeable with the contraction C_1. For, in general, let c be any elementary contraction $K \to L = K - aA - A$ and let e be an expansion $L \to L + bB + B$ $(b\dot{B} \subset L,\ B \bar{\epsilon} L)$. Then e and c are interchangeable if neither of the open simplexes aA and A coincides with either† bB or B. For, if B is neither aA nor A, the transformation $K \rightarrow K + bB + B$ is an elementary expansion, since $b\dot{B} \subset L \subset K$ and $B \bar{\in} K$. The only simplex in $(bB)^{\cdot}$ which is not in K is B and, if $B \neq A$, it follows that aA is the only simplex in $K + bB + B$ with A on its boundary. Therefore the transformation

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† Either E_1 or C_q, or both, may be the identity.

$$\begin{aligned} K+bB+B \rightarrow (K+bB+B)&-aA-A \\ &= (K-aA-A)+bB+B \\ &= L+bB+B \end{aligned}$$

is an elementary contraction. That is to say $ce = ec$. If C is any contraction of a given complex K, and E is an expansion of $C(K)$, it follows from an inductive argument that $EC = CE$ provided that none of the simplexes removed by C is restored by E. This condition is satisfied by C_1 and E_2'. For any open simplex A which is added by E_2' belongs to $K_{12}' - K_1$ and so contains at least one vertex introduced by s_{K_1}. According to our rule this vertex, and therefore A, does not belong to K_{01}. Therefore A is not one of the simplexes removed by C_1. Therefore $E_2' \, C_1 = C_1 \, E_2'$, and

$$\begin{aligned} s_{K_1} K_2 &= C_2' \, E_2' \, C_1 \, E_1(K_0) \\ &= C_2' \, C_1 \, E_2' \, E_1(K_0). \end{aligned}$$

Therefore the complex

$$K_{02} = E_2' \, E_1(K_0)$$

contracts both into K_0 and into $s_{K_1} K_2$, and the theorem is established.

We now give two definitions.

1. By the *order* of a deformation D will be meant the maximum order of the elementary deformations in D.

2. If $L \subset K_0 . K_q$ and if no simplex of L is removed by any of the elementary contractions in a deformation $K_q = D(K_0)$, we shall describe D as *relative* to L, and shall write

$$K_q = D(K_0) \quad (\text{rel. } L).$$

Two addenda follow from the proof of Theorem 5:

ADDENDUM 1. *If K_0 and K_q are at most n-dimensional and if the order of D does not exceed m, where $m \geqslant n$, there is a complex of at most m dimensions which contracts both into K_0 and into σK_q.*

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† Actually $ec = ce$ if $A \neq B$.

ADDENDUM 2. *If $K_q = D(K_0)$ (rel. L) there is a complex which contracts both into K_0 and into σK_q, where $\sigma = 1$ in L.*

For each simplex added by E_λ' is contained in a sub-division of some simplex added by E_λ ($\lambda = 2, \ldots, q$) and its dimensionality does not exceed m. Therefore neither the order of E_λ' ($\lambda = 2, \ldots, q$) nor the dimensionality of K_{02} exceeds m and the first addendum follows from induction on q. The second addendum follows from induction on q and the fact that $L \subset K_1$, since no simplex of L is removed during the contraction C_1.

Notice that the complex which contracts both into K_0 and into σK_q is uniquely determined by the construction used in proving Theorem 5, except for the actual vertices introduced by the sub-division σ.

THEOREM 6. *If K is any complex, there is a sub-division σK such that σE^m contracts into σE^{m-1} ($m > 0$), where E^m is any m-element*[†] *in K and E^{m-1} any ($m-1$)-element in $\dot{E}^m$, the sub-division σ being independent of m, of $E^m \subset K$ and of $E^{m-1} \subset \dot{E}^m$.*

If K is 0-dimensional there is nothing to prove and, assuming the theorem for a complex of at most n dimensions ($n \geqslant 0$), we shall prove it by induction on n. First notice that each element σE^m is collapsible, where $E^m \subset K$ and σK is any sub-division which satisfies the conditions of the theorem. For, if $m > 0$, σE^m contracts into σA^{m-1}, where A^{m-1} is any ($m-1$)-simplex in $\dot{E}^m$. From the corollary to Lemma 2 and Theorem 4 it follows that σA^{m-1} and hence σE^m are collapsible.

There are in K, which we now take to be ($n+1$)-dimensional, only a finite number of elements. Therefore the theorem will follow from the corollary to Theorem 4 if we can prove that some sub-division σE^m contracts into σE^{m-1} ($0 < m \leqslant n+1$), where E^m is a given m-element and E^{m-1} a given ($m-1$)-element in $\dot{E}^m$. This follows from the hypothesis of the induction unless $m = n+1$, which we assume to be the case.

Let C^m ($m = n+1$) be a rectilinear, geometrical representation of the m-element $ab\dot{A}^n$, where A^n is an n-simplex. Some partition $\pi_0 \dot{C}^m$ is the image of a stellar sub-division $\sigma_0 \dot{E}^m$ in an isomorphic transformation f_0, such that[‡] $f_0(\sigma_0 E^{m-1}) = \pi_0 a\dot{A}^n$. The sub-divisions $\pi_0 \dot{C}^m$ and $\sigma_0 \dot{E}^m$ and the transformation f_0 can be extended to sub-divisions $\pi_1 C^m$ and $\sigma_1 E^m$ and an isomorphism $f_1(\sigma_1 E^m) = \pi_1 C^m$, with $\sigma_1 = \sigma_0$ and $\pi_1 = \pi_0$ on $\dot{E}^m$ and

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[†] Following Alexander and Newman, we use the terms m-element and ($m-1$)-sphere to mean complexes which are combinatorially equivalent to an m-simplex and to its boundary.

[‡] J. H. C. Whitehead, *Proc. Cambridge Phil. Soc.*, 31 (1935), 69–75.

$\dot{C}^m$, and† $f_1 = f_0$ on $\sigma_0 \dot{E}^m$. Therefore we lose no generality in assuming that E^m was initially isomorphic to $\pi_1 C^m$ with E^{m-1} corresponding to $\pi_1 a\dot{A}^n$ and, assuming this to have been the case, we take E^m and E^{m-1} to be the geometrical complexes $\pi_1 C^m$ and $\pi_1 a\dot{A}^n$.

Let $p_0 = a, p_1, \dots, p_r = b$ be the points on the rectilinear segment ab, arranged in this order, such that all the vertices of E^m lie in the (geometrical) n-elements $p_i\dot{A}^n$ $(i = 0, \dots, r)$. By a construction used elsewhere‡, we can find a stellar sub-division σE^m which is a partition of the rectilinear m-element $(p_0p_1 + p_1p_2 + \dots + p_{r-1}p_r)\dot{A}^n$ and is such that all its vertices are in the n-elements $p_i\dot{A}^n$ $(i = 0, \dots, r)$. To economize our symbols, let us assume that E^m satisfied this condition initially, let E_i^n be the complex covering $p_i\dot{A}^n$, and let E_i^m be the complex covering the m-element

$$p_{i-1}p_i\dot{A}^n \quad (i = 1, \dots, r).$$

If $r > 1$, assume that some stellar sub-division $\sigma_{r-1}(E_1^m + \dots + E_{r-1}^m)$ contracts into $\sigma_{r-1}E^{m-1}$. If we can show that some stellar sub-division $\sigma^*\sigma_{r-1}E_r^m$ contracts into $\sigma^*\sigma_{r-1}E_{r-1}^n$, the theorem will be established directly, if $r = 1$ (taking $\sigma_{r-1} = 1$); and it will follow from Lemma 1, Theorem 4 and induction on r, if $r > 1$. The element $\sigma_{r-1}E_r^m$ has no internal vertices and, again simplifying our notation, it remains to prove the following: if E^m is a partition of $p_0p_1\dot{A}^n$ with no internal vertices, some stellar sub-division σE^m contracts into σE_0^n, where E_i^n is the sub-complex of E^m covering $p_i\dot{A}^n$ $(i = 0, 1)$.

Since E^m is a partition of $p_0p_1\dot{A}^n$ and since it has no internal vertices, the vertices of any simplex in E^m lie in $(p_0 + p_1)A^{n-1}$, where A^{n-1} is some closed simplex in $\dot{A}^n$. Since p_0A^{n-1} and p_1A^{n-1} are flat, every internal simplex in E^m is of the form B_0B_1, where $B_i \subset E_i^n$ $(i = 0, 1;$ possibly $B_0 . B_1 \neq 1$, being in $\dot{A}^n)$. Therefore E^m is a normal simplicial neighbourhood both of E_0^n and E_1^n.

Now let q be an inner point of the segment p_0p_1 and apply the sub-division $s_L E^m$, where $L = \dot{E}^m = E_0^n + E_1^n$, placing each new vertex on the locus $q\dot{A}^n$. Since each simplex of E^m is contained in one of the closed simplexes $p_0p_1A^{n-1}$ $(A^{n-1} \subset \dot{A}^n)$, and since the locus qA^{n-1} is flat, it follows that $q\dot{A}^n$ is covered by a sub-complex F^n of s_LE^m. Let

$$F^m = N(E_0^n, s_L E^m)$$

be the sub-complex of $s_L E^m$ covering the closure of the region between

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† Alexander, *loc. cit.*, Theorem 13.2, and Whitehead, *loc. cit.*, Theorem 2.

‡ Whitehead, *loc. cit.*, Theorem 1.

E_0^n and F^n. Clearly F^m is the same as it would be were $L = E_0^n$, instead of $E_0^n + E_1^n$, and, by Lemma 4 and Theorem 2, F^m contracts into E_0^n. It follows from an argument used by Newman† that $F^n . (s_L E^m)_B$ is an element, where B is any internal simplex of E_1^n. By the hypothesis of the original induction, and our preliminary observation, there is a stellar sub-division $\sigma_1 F^n$ such that each of the elements $\sigma_1\{F^n . (s_L E^m)_B\}$, with B inside E_1^n, is collapsible. Therefore σE^m, with $\sigma = \sigma_1 s_L$, is a contractible neighbourhood of $\sigma_1 F^m$ and contracts into $\sigma_1 F^m$, by Theorem 2. By what we have just proved and Theorem 4 it then contracts into $\sigma_1 E_0^n = \sigma E_0^n$ and the proof is complete.

5. *Geometrical deformation.*

If K_0 is any complex and

$$(5.1) \qquad K_1 = K_0 + E^n,$$

where E^n is an n-element which meets K_0 in an $(n-1)$-element on $\dot{E}^n$, we shall describe the transformation $K_0 \rightarrow K_1$ as a *geometrical expansion* and $K_1 \rightarrow K_0$ as a *geometrical contraction*. We shall also say that a complex expands and contracts geometrically into any general sub-division‡ of itself. Finally, any sequence of geometrical expansions (contractions) will also be called a geometrical expansion (contraction). When a contrast is unnecessary, or when it is obvious from the context which kind is meant, we shall refer to either a formal or a geometrical expansion (contraction) simply as an expansion (contraction).

THEOREM 7. *If K contracts geometrically into L some stellar sub-division σK contracts formally into σL.*

As explained in § 14 below, a general sub-division of any sub-complex of K may be extended to the whole of K. It follows from a straightforward inductive argument that some general sub-division γK contracts geometrically into γL without further sub-division. That is to say $\gamma K \rightarrow \gamma L$ is the resultant of transformations of the form $K_1 \rightarrow K_0$, where K_0 and K_1 are related by (5.1). Clearly the same is true of $\gamma^* \gamma K$ and $\gamma^* \gamma L$, where γ^* is any general sub-division of γK, and there is a sub-division γ^* such that $\gamma^* \gamma K$ is a stellar sub-division† $\sigma_1 K$. It follows from Theorem 6 that there is a further stellar sub-division σ_2, such that σK contracts formally into σL, where $\sigma = \sigma_2 \sigma_1$, and the theorem is established.

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† *Journal London Math. Soc.*, 2 (1926), 56–64, Lemma 2. The double sub-division is not needed for Newman's argument provided that the $(n-1)$-element in question has a normal neighbourhood initially (*cf.* Lemma 10 below).

‡ See Newman, *Journal London Math. Soc.* (*loc. cit.*), also §14 below. We denote a general sub-division by the symbol γ.

COROLLARY. *If K contracts geometrically into L any sub-division γK contracts geometrically into γL.*

For some stellar sub-division σK contracts formally into σL. Clearly any sub-division $\gamma_1 \sigma K$ contracts geometrically into $\gamma_1 \sigma L$, and the corollary follows from the fact that γ and σ have a common sub-division $\gamma_2 \gamma = \gamma_1 \sigma$.

A complex which contracts geometrically into a single vertex may be described as *geometrically collapsible* and it follows from the corollary to Theorem 7 that the property of being geometrically collapsible is a combinatorial invariant. It also follows from Theorem 7 that geometrical expansions and contractions are formal deformations.

6. *Maps and homotopy.*

Let f be a simplicial map of a complex K in a complex L, where $K \,.\, L = 1$, meaning a transformation of vertices such that, if $A = a_0 \ldots a_n$ is any simplex in K, then $f(a_0), \ldots, f(a_n)$, which need not be distinct, are the vertices of a simplex $f(A)$ in L. As usual, we shall refer to f as a map of K on, as distinguished from in, L only if each simplex in L is the image of one or more simplexes in K. We shall describe f as (1-1) if no two simplexes in K have the same image, even if f is not a map on L, and a (1-1) simplicial map of K on L will be called an *isomorphism*.

We now define what we call the *mapping cylinder* $C_f(K)$, of a map f. For convenience we represent K and L as rectilinear, geometrical complexes, and we take f to be the semi-linear map determined by the given transformation of vertices. Let K_{01} be the simplicial complex derived from the topological product $K \times \langle 0, 1 \rangle$, by starring all the cells $A \times \langle 0, 1 \rangle$ $(A \subset K)$, leaving $K \times 0$ and $K \times 1$ untouched. Then we define $C_f(K)$ as the simplicial complex obtained from K_{01} by identifying each simplex $A \times 0$, of $K \times 0$, with the corresponding simplex A, in K, and each simplex $A_1 = A \times 1$ in $K \times 1$ with the simplex $f(A)$ in L. If c is the centre of the star in K_{01} covering $A \times \langle 0, 1 \rangle$ the simplex cA_1 is thus transformed into $cf(A)$, which may be of lower dimensionality than cA, and $C_f(K)$ is the image of K_{01} in a simplicial map ϕ, which is an isomorphism if, and only if, K is (1–1). If K^* is any sub-complex of K, and if K^*_{01} is the sub-complex of K_{01} covering $K^* \times \langle 0, 1 \rangle$, then $\phi(K^*_{01})$ is obviously the mapping cylinder $C_f(K^*)$ of the map $f(K^*) \subset L$. If A is any closed simplex in K and c the centre of the star covering $A \times \langle 0, 1\rangle$ an easy inductive argument shows that†

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† See the addendum to Lemma 16 in §14.

$$C_f(A) = c[A + f(A) + C_f(\dot{A})].$$

We now prove two theorems which are analogous to Lemmas 2 and 3.

THEOREM 8. *If $f(K) \subset L$ is a simplicial map of K in L, and if K^* is any sub-complex of K (possibly $K^* = 1$), then $L + C_f(K)$ contracts formally into $L + C_f(K^*)$.*

If $K^* = K$ there is nothing to prove. Otherwise let A be the closure of any principal open simplex in $K - K^*$. Assuming that $C_f(A)$ contracts into $f(A) + C_f(\dot{A})$, we deduce the theorem from Lemma 1 and induction on the number of simplexes in $K - K^*$. If $\dim(A) = 0$ it is obvious that $C_f(A)$ contracts into $f(A) + C_f(\dot{A})$ $[= f(A)$ with the convention $C_f(1) = 1]$, and if $\dim(A) > 0$ we assume this to be true of any k-simplex for $k < \dim(A)$. Then, taking $K = A$, $L = f(A)$, and $K^* = 1$, we deduce by our first argument that $f(A) + C_f(\dot{A})$ contracts into $f(A)$, and is therefore collapsible, $f(A)$ being a single closed simplex. If c is the centre of the star $C_f(A)$ we first remove the open simplexes A and cA from $C_f(A)$, leaving

$$c[f(A) + C_f(\dot{A})],$$

which contracts into $f(A) + C_f(\dot{A})$ by what we have already proved and the corollary to Lemma 3. Therefore, the fact that $C_f(A)$ contracts into $f(A) + C_f(\dot{A})$, and hence the theorem, follow from a second induction on $\dim(A)$.

Let E_0^n be an n-element which is completely inside an n-element E^n, meaning that $E_0^n \subset E^n - \dot{E}^n$. As a corollary to Theorem 8 we have:

COROLLARY. *$Cl(E^n - E_0^n)$ contracts geometrically both into $\dot{E}^n$ and into $\dot{E}_0^n$.*

For some sub-division σE^n may be represented as a partition of a geometrical simplex Δ^n in such a way that σE_0^n covers a "concentric" simplex Δ_0^n, contained in‡ Δ^n. The closure of the region between Δ^n and Δ_0^n is the geometrical mapping cylinder $C_t(\dot{\Delta}_0^n)$, where t is the projection of $\dot{\Delta}_0^n$ on $\dot{\Delta}^n$ from their common centre, and contracts geometrically into

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† This property may be taken as the basis of an inductive definition of $C_f(K)$. As a matter of convention we take $f(1) = C_f(1) = 1$.

‡ Newman, *Journal London Math. Soc.*, 2 (1927), 64, Theorem 3.

$\dot{\Delta}^n$, by Theorem 8, and similarly into $\dot{\Delta}_0^n$. Therefore, $Cl(E^n - E_0^n)$ contracts geometrically both into $\dot{E}^n$ and into $\dot{E}_0^n$.

THEOREM 9. *If* K *contracts formally into* K^* *and if* $f(K) \subset L$ *is a simplicial map of* K *in* L, *then* $L + C_f(K)$ *contracts formally into*

$$K + L + C_f(K^*).$$

If $K^* = K$ there is nothing to prove. Otherwise let $K = K_1 + aA$ ($a\dot{A} \subset K_1$, $A \not\subset K_1$), where $K \to K_1$ is the first step in a contraction $K \to K^*$. The theorem will follow from Lemma 1 and induction on the number of simplexes in $K - K^*$ if we can show that $C_f(aA)$ contracts into

$$P = aA + f(aA) + C_f(a\dot{A}).$$

By Lemma 2, $C_f(aA)$ contracts into cP, where c is the centre of $C_f(aA)$. Clearly P contracts into $f(aA) + C_f(a\dot{A})$, which contracts into $f(aA)$, by Theorem 8, and is therefore collapsible. Therefore P is collapsible, cP contracts into P, by the corollary to Lemma 3, and the theorem is established.

The purpose of our next theorem is to establish a certain relation between homotopic maps $f_0(K_0) \subset L$ and $f_1(K_1) \subset L$, the complexes K_0 and K_1 being combinatorially equivalent. We shall say that two simplicial maps $f_0(K_0) \subset L$ and $f_1(K_1) \subset L$ are *equivalent* if, and only if, K_1 is the image of K_0 in an isomorphism t, such that $f_0 = f_1 t$ [*i.e.* $f_0(A_0) = f_1(A_1)$, where A_0 is any simplex in K_0 and $A_1 = t(A_0)$]. If we represent K_0, K_1 and L as geometrical complexes, the maps f_0 and f_1 will be described as homotopic† in L if, and only if, first, supposing that $K_0 . K_1 = 1$, there is an isomorphism $t(\pi_0 K_0) = \pi_1 K_1$, where π_i is a partition of K_i ($i = 0, 1$), and a map‡ of $C_t(\pi_0 K_0)$ in L which, regarded as a transformation of points, coincides with f_i in $\pi_i K_i$; secondly, if K_1 meets K_0 the maps f_0 and f_1 will be described as homotopic if $f_0(K_0)$ is homotopic to $f_2(K_2)$, where the latter is equivalent to $f_1(K_1)$ and $K_0 . K_2 = 1$. This definition obviously includes the ordinary definition of homotopy in case $K_0 = K_1$.

Let K, K_0 and K_1 be combinatorially equivalent complexes and let $K_0 . K_1 = 1$. By a *simple cylinder joining* K_0 *to* K_1 we shall mean a (simplicial) complex containing the complexes K_0 and K_1 (*i.e.* K_0 and K_1 themselves, not merely sub-divisions of K_0 and K_1), some sub-division of which is isomorphic to a simplicial sub-division of $K \times \langle 0, 1 \rangle$ in a transformation which maps a sub-division of $K \times i$ ($i = 0, 1$) on a sub-division of K_i.

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† *Cf.* Hurewicz, *loc. cit.* (2nd paper), 524.

‡ By a map of a geometrical complex we shall always mean a continuous transformation.

THEOREM 10. *If two simplicial maps* $f_0(K_0) \subset L$ *and* $f_1(K_1) \subset L$ *are homotopic in* L *there is a simple cylinder* P, *joining* K_0 *to* K_1, *which contracts formally both into* K_0 *and into* K_1, *and a simplicial map* $f(P) \subset L$ *which coincides with* f_i *in* K_i $(i = 0, 1)$, *the map* f *being simplicial with respect to the complex* P, *not merely with respect to a sub-division of* P.

Let K_0, K_1 and L be represented as geometrical complexes. Since f_0 and f_1 are homotopic there is a map $\phi(C) \subset L$, which coincides, as a transformation of points, with f_i in $\pi_i K_i$ $(i = 0, 1)$, where $C = C_t(\pi_0 K_0)$ and t is an isomorphism of a partition $\pi_0 K_0$ on a partition $\pi_1 K_1$. The map ϕ may be approximated, in the usual way, by a simplicial map $\phi'(\sigma C) \subset L$, where σ is a stellar sub-division of C. Since $\phi = f_i$ in $\pi_i K_i$ and f_i is simplicial we may assume that $\phi'(a)$ is a vertex of $f_i(A)$, where a is any vertex of $\pi_i A$ and $A \subset K_i$. By Theorem 6 there is a further sub-division $\sigma_1 \sigma C = \sigma' C$ such that each of the elements $\pi_i' A$ is collapsible, where $A \subset K_i$ and $\pi_i' = \sigma_1 \sigma \pi_i$. By Theorem 8 the complex C contracts formally into each of $\pi_0 K_0$ and $\pi_1 K_1$, and by Theorem 4 the complex $\sigma' C$ contracts formally into each of $\pi_0' K_0$ and $\pi_1' K_1$. The map ϕ' may be replaced by a simplicial map $\phi''(\sigma' C) \subset L$, which transforms each vertex in $\pi_i' A$ $(A \subset K_i)$ into a vertex of $f_i(A)$.

We now form the topological product $K_i \times \langle 0, 1 \rangle$, taking $p \times 0 = p$, where p is any point in K_i. Let T_i be the polyhedral complex covering $K_i \times \langle 0, 1 \rangle$, which consists of the open simplexes $A \times 1$ $(A \in K_i)$, the open cells $A \times \langle 0, 1 \rangle$, and the simplexes in $\pi_i' K_i$. Let T_i^* be the simplicial complex derived from T_i by starring all the cells $A \times \langle 0, 1 \rangle$, leaving the complexes $K_i \times 1$ and $\pi_i' K_i$ untouched, and let

$$P_1 = T_0^* + \sigma' C + T_1^*.$$

Then P_1 is obviously a simple cylinder joining $K_0 \times 1$ to $K_1 \times 1$. It follows from Theorem 8, with trivial modifications, that T_i^* contracts formally into $K_i \times 1$, and also into $\pi_i' K_i$, since each of the elements $\pi_i' A$ is collapsible $(A_i \subset K)$. Therefore P_1 contracts, first into $T_0^* + \sigma' C$, then into T_0^*, since $\sigma' C$ contracts formally into $\pi_0' K_0$, and finally into $K_0 \times 1$. Similarly it contracts formally into $K_1 \times 1$.

We now extend the map $\phi''(\sigma' C)$ to a map $\phi''(P_1) \subset L$ by taking $\phi''(A \times 1) = f_i(A)$ and $\phi''(c)$ to be any vertex of $f_i(A)$, where $A \subset K_i$ and c is the vertex of T_i^* which is inside $A \times \langle 0, 1 \rangle$. The map so defined is simplicial since each vertex in $\pi_i' A$ is transformed by ϕ'' into a vertex of $f_i(A)$. Finally we replace P_1 by an isomorphic complex $P = \psi(P_1)$, where $\psi(A \times 1) = A$ $(A \subset K_i)$. Then P is a simple cylinder joining K_0 to K_1, it contracts formally both into K_0 and into K_1, and $\phi''\psi^{-1}(P) \subset L$ is a simplicial map which coincides with f_i in K_i. Thus the theorem is established.

A map $f(S^n) \subset L$, where S^n is an n-sphere, will be called an n-spherical *map* in L, or simply a *spherical map* if the dimensionality is irrelevant or obvious from the context. If $f(\dot{E}^n)$ is a simplicial, spherical map in L, where E^n $(n > 0)$ is an n-element which does not meet $L + C_f(\dot{E}^n)$ except in $\dot{E}^n$, we shall describe

$$\mathscr{E}^n = E^n + C_f(\dot{E}^n)$$

as a *simple membrane*[†] bounded by the spherical map $f(\dot{E}^n)$. Notice that $\mathscr{E}^n$ is an n-sphere if $f(\dot{E}^n)$ is a single point, and that any n-sphere which meets L in a single point may be regarded as a simple membrane bounded by such a map. In dealing simultaneously with a set of simple membranes, of the same or different dimensionalities, and with any number of complexes, it is always to be understood that none of the membranes has an inner point in common with any of the others or with any of the complexes.

If t is an isomorphic map of $\dot{E}^n$ on an $(n-1)$-sphere which does not meet E^n, it can be proved without difficulty that $E^n + C_t(\dot{E}^n)$ is an n-element. From this and from the definition of a mapping cylinder it follows that $\mathscr{E}^n = E^n + C_f(\dot{E}^n)$ is an n-cell[‡] bounded by a map which is equivalent[§] to f. For, if t is an isomorphism of $\dot{E}^n$ on an $(n-1)$-sphere S^{n-1}, we have $\mathscr{E}^n = \phi(E_0^n)$, where $E_0^n = E^n + C_t(\dot{E}^n)$ and $\phi(A) = A$ if $A \subset E^n$, $\phi(A_1) = ft^{-1}(A_1)$ and $\phi(c_1) = c$, where $A_1 \subset S^{n-1}$ and c_1 and c are the vertices inside the cells $C_t(A)$ and $C_f(A)$ respectively. Since the vertices inside different cells $C_f(A)$ and $C_f(B)$ are distinct the map ϕ has no folds, which means that no two n-simplexes in E_0^n have the same image in $\mathscr{E}^n$. We shall denote the boundary of a simple membrane, or of any cell Γ, by $F(\Gamma)$, remembering that $F(\Gamma)$ is a spherical map rather than a complex.

Let $\Gamma_i^n = f_i(E_i^n) \subset L$ $(i = 1, 2)$ be n-cells bounded by equivalent spherical maps $f_i(\dot{E}_i^n)$, and, replacing $f_2(E_2^n)$ by an equivalent map if necessary, let $\dot{E}_2^n = \dot{E}_1^n$ and $f_2 = f_1$ in $\dot{E}_1^n$. Let σE_2^n be an internal sub-division of E_2^n (*i.e.* $\sigma = 1$ in $\dot{E}_2^n$) which has no internal simplexes in common with E_1^n. Then $E_1^n + E_2^n = \phi(S^n)$, where $S^n = E_1^n + \sigma E_2^n$ and ϕ is a simplicial map such that $\phi(A) = A$ if $A \subset E_1^n$. We shall write

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[†] *Cf.* N. Aronszajn, *Akad. Wet. Amsterdam*, 40 (1937), 69–69.

[‡] By a cell we shall always mean a map of an element, which (*i.e.* the map) may be singular or non-singular. It will be obvious from the context when a simple membrane is to be regarded as no more than a complex (which will usually be the case) and when it is to be regarded as a cell.

[§] In speaking of the boundary of a cell we shall often refer to a map f when, strictly speaking, we mean a map which is equivalent to f.

$$\Gamma_1^n + \Gamma_2^n = f(E_1^n + E_2^n) = f\phi(S^n),$$

where $f = f_i$ in E_i^n. Thus $\Gamma_1^n + \Gamma_2^n$ is a spherical map which is determined up to the choice of the sub-division σ, and what we shall later call a homotopic deformation rel. E_1^n (see § 8).

THEOREM 11. *If* $\mathcal{E}_0$ *and* $\mathcal{E}_1$ *are simple membranes bounded by homotopic maps in a complex* L, *then*

$$L + \mathcal{E}_1 = D(L + \mathcal{E}_0) \quad (\text{rel. } L).$$

Let $$\mathcal{E}_i = E_i + C_{f_i}(\dot{E}_i) \quad (i = 0, 1),$$

where $F(\mathcal{E}_i) = f_i(\dot{E}_i)$, E_0 and E_1 being n-elements. We first dispose of two trivial cases. If $\dot{E}_0 = \dot{E}_1$ and $f_0 = f_1$, we may transform E_0 into E_1, and hence $L + \mathcal{E}_0$ into $L + \mathcal{E}_1$, by internal combinatorial transformations. The result is a formal deformation of $L + \mathcal{E}_0$, by Theorem 1, and it is obviously relative to L. Secondly, by elementary sub-divisions of order zero applied to the internal vertices of $\mathcal{E}_0$, which again are formal deformations relative to L, we may replace $\mathcal{E}_0$ by a simple membrane having no internal simplex in common with $\mathcal{E}_1$. Thus we may suppose, first that $E_i = a_i \dot{E}_i$ $(i = 0, 1)$, and secondly that our universal condition relative to $\mathcal{E}_0 . \mathcal{E}_1$ is satisfied.

By Theorem 10 there is a complex P, which contracts formally both into $\dot{E}_0$ and into $\dot{E}_1$, and a simplicial map $f(P) \subset L$, which coincides with f_i in $\dot{E}_i$. We may take $P . L = 1$ and $C_{f_i}(\dot{E}_i) = C_f(\dot{E}_i)$, and it follows from Theorem 9 that $L + C_f(P)$ contracts into $L + P + C_f(\dot{E}_i)$, and hence into $L + C_f(\dot{E}_i)$. Let

$$K = L + C_f(P) + b(E_0 + P + E_1),$$

where b is a new vertex. By Lemma 2 the complex $b(E_0 + P + E_1)$ contracts into $b(E_0 + P)$. Since P contracts into $\dot{E}_0$ and $E = a_0 \dot{E}_0$, it follows that $E_0 + P$ is collapsible, and by the corollary to Lemma 3 that $b(E_0 + P)$ contracts into $E_0 + P$. Therefore K contracts into

$$L + C_f(P) + E_0$$

and hence into $L + \mathcal{E}_0$, since $C_f(P)$ contracts into $C_f(\dot{E}_0)$. Similarly K contracts into $L + \mathcal{E}_1$, and it follows that

$$L + \mathcal{E}_1 = D(L + \mathcal{E}_0),$$

where D is the resultant of the expansion $L+\mathcal{E}_0 \to K$, followed by the contraction $K \to L+\mathcal{E}_1$. Clearly D is relative to L, and the theorem is established.

The cylinder P is n-dimensional, $\dot{E}^n$ being $(n-1)$-dimensional. Therefore $K-L$ is $(n+1)$-dimensional and we have the addendum:

ADDENDUM. *The order of the deformation D in Theorem* 10 *need not exceed* dim $(\mathcal{E}_i)+1$.

LEMMA 5. *If $\mathcal{E}^n$ is a simple membrane bounded by a spherical map in K, the complexes K and $K+\mathcal{E}^n$ have the same $(n-1)$-group. If $F(\mathcal{E}^n)$ is homotopic to a point in K they have the same n-group.*

Let $\mathcal{E}^n = E^n + C_f(\dot{E}^n)$, where $F(\mathcal{E}^n) = f(\dot{E}^n)$, and let A^n be an open n-simplex in E^n, none of whose vertices lie in $\dot{E}^n$. Then $K+\mathcal{E}^n$ has the same $(n-1)$-group as $K+\mathcal{E}^n-A^n$. By the corollary to Theorem 8, $K+\mathcal{E}^n-A^n$ contracts into $K+C_f(\dot{E}^n)$ and by Theorem 8 itself the latter contracts into K. Therefore $K+\mathcal{E}^n$ and K have the same $(n-1)$-group.

If $f(\dot{E}^n)$ is homotopic to a point in K it follows from Theorem 11 that

$$K+\mathcal{E}^n = D(K+S^n),$$

where S^n is a simple membrane bounded by a single point. That is to say, S^n is an n-sphere, which we may take to be $\dot{A}^{n+1}$, where A^{n+1} is a closed $(n+1)$-simplex meeting K in a single vertex. Then $K+\dot{A}^{n+1}$, and therefore $K+\mathcal{E}^n$, has the same n-group as $K+A^{n+1}$, which contracts into K. Therefore K and $K+\mathcal{E}^n$ have the same n-group, and the lemma is established.

Let $\mathcal{E}^n = E^n + C_f(\dot{E}^n)$ be a simple membrane bounded by a spherical map $f(\dot{E}^n)$ in a complex K, and let K contain a principal open $(n-1)$-simplex B^{n-1}, which is covered in the map f by one, and only one, open simplex A^{n-1}, in $\dot{E}^n$.

LEMMA 6. *$K+\mathcal{E}^n$ contracts into $K-B^{n-1}$.*

We first take away from $K+\mathcal{E}^n$ the open simplex B^{n-1} and the interior of the cell $C_f(A^{n-1})$. The resulting complex, namely

$$K-B^{n-1}+E^n+C_f(\dot{E}^n-A^{n-1}),$$

contracts into

$$K-B^{n-1}+C_f(\dot{E}^n-A^{n-1}),$$

since E^n meets the latter in the $(n-1)$-element $\dot{E}^n-A^{n-1}$. The lemma now follows from Theorem 8.

7. *m-Groups.*

THEOREM 12. *Two connected complexes have the same 2-group if, and only if, they have the same fundamental group.*

Two connected complexes with the same 2-group obviously have the same fundamental group. For neither a formal deformation nor the addition or removal of an open m-simplex ($m > 2$) alters the fundamental group.

Conversely, let K and L be two connected complexes with the same fundamental group G. After removing the simplexes of higher dimensionality, if any, we may assume that K and L are at most 2-dimensional. After a familiar process of shrinking segments into a point we may further assume that K consists of oriented circuits $a_1, \ldots, a_p$, with a common point (no two meeting anywhere else), together with certain simple membranes $\mathcal{E}_1^2, \ldots, \mathcal{E}_k^2$, bounded by circuits which are represented in the usual way as products

$$R = a_{i_1}^{m_1} \ldots a_{i_s}^{m_s}. \tag{7.1}$$

Then K determines a system of generators and relations

$$a_1, \ldots, a_p; \quad R_1 = 1, \ldots, R_k = 1, \tag{7.2}$$

for the group G, where R_λ is the product of the form (7.1) corresponding to $F(\mathcal{E}_\lambda^2)$. The complex L may be treated similarly so as to determine a system of generators and relations

$$b_1, \ldots, b_q; \quad S_1 = 1, \ldots, S_l = 1. \tag{7.3}$$

If the two systems (7.2) and (7.3) are identical it is obvious from Theorem 11 that $L = D(K)$.

In general, the system (7.2) can be transformed into the system (7.3) by a finite sequence of transformations of the two following types†, and their inverses:

(1) adding a new generator a_0, together with a relation of the form

$$a_0^{-1} W(a) = 1,$$

where W is a product of the existing generators and their inverses;

(2) adding a new relation $R_0 = 1$, which is a consequence of the existing relations.

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† See K. Reidemeister, *Einführung in die kombinatorische Topologie* (Brunswick, 1932), 46–48.

Let $\mathcal{E}_0^2$ be a simple membrane corresponding to the relation

$$a_0^{-1} W(a) = 1$$

(a_0 being a new circuit) in the first case, and to the relation $R_0 = 1$ in the second case. In either case

$$K_1 = K + \mathcal{E}_0^2$$

is a complex which determines the new system of generators and relations. In the first case $\mathcal{E}_0^2$ may obviously be chosen so as to satisfy the conditions of Lemma 6, with $n = 2$ and B^1 in the circuit a_0. Clearly K expands into $K + a_0 - B^1$ and hence, by Lemma 6, into K_1. In the second case the boundary of $\mathcal{E}_0^2$ is homotopic to a point in K since the relation $R_0 = 1$ is redundant. Therefore K and K_1 have the same 2-group, by Lemma 5. Equally, if a generator a_0 and a relation $a_0^{-1} W = 1$, or a redundant relation $R_0 = 1$, are removed, K and K_1 have the same 2-group, where

$$K = K_1 + \mathcal{E}_0^2.$$

Therefore there is a sequence of complexes $K_0 = K, K_1, \ldots, K_r = L$, all of which have the same 2-group, and the theorem is established.

We now introduce a new kind of elementary transformation which we shall call a *special filling of order m*, and its inverse which we call a *special perforation of order m*. A special filling is a transformation of the form

$$K \rightarrow K + A^m,$$

where $\dot{A}^m$, but not A^m, belongs to K, and $\dot{A}^m$ is homotopic to a point in K. If L is derived from K by a special filling or perforation of order m it follows from Lemma 5 that K and L have the same m-group.

Let p^m be a perforation given by

$$K \rightarrow K_1 = (K - A^m) + \dot{A}^m,$$

where A^m is a principal closed simplex in K, and let σ be a stellar subdivision of K.

LEMMA 7. *The transformation $\sigma K \rightarrow \sigma K_1$ is the resultant of a perforation p_0^m followed by a formal contraction. If p^m is special so is p_0^m.*

As in Theorem 4 we may take σ to be a single elementary sub-division (A, a), where $A \subset A^m$. Let $A = bB_1$ and $A^m = bB$, where $B = B_1 B_2$. Then

$$\sigma A^m = a(b\dot{B}_1 + B_1) B_2$$

$$= ab\dot{B}_1 B_2 + aB,$$

and we remove the interior of the simplex aB by a perforation $\dot{p}_0^m$. If $(bB)^{\cdot}$ is homotopic to a point in K_1 so obviously is $(aB)^{\cdot}$ in $(\sigma K - aB) + (aB)^{\cdot}$. Therefore p_0^m is special if p^m is special. It follows from the second part of the proof of Theorem 4 that $(\sigma K - aB) + (aB)^{\cdot}$ contracts into σK_1, and the lemma is established.

LEMMA 8. *If K and L are two complexes with the same n-group, each of which is at most n-dimensional, then K is transformable into L by special fillings and perforations of order n and elementary deformations whose orders do not exceed n.*

We first replace any elementary expansion

$$K \to K + bB^{m-1} \quad (m > n,\ b\dot{B}^{m-1} \subset K),$$

whose order exceeds n, by the filling

$$K \to K + B^{m-1}$$

followed by the filling

$$K + B^{m-1} \to K + bB^{m-1}.$$

Since $\dot{B}^{m-1} = (b\dot{B}^{m-1})^{\cdot}$ it is homotopic to a point in K. Therefore the first of these is a special filling and is permissible even if $m = n+1$. Similarly we replace any elementary contraction of order m $(m > n)$ by a perforation of order m followed by a special perforation of order $m-1$. Therefore we may suppose that K is transformed into L by elementary deformations whose orders do not exceed n and fillings and perforations whose orders exceed $n-1$, those of order n, if any, being special. Let k be the maximum order of the fillings. If $k = n$ there is nothing more to be said. For K is at most n-dimensional and if no simplexes of higher dimensionality are introduced none can be taken away. If $k > n$, let A^k be one of the open simplexes introduced by a filling f^k, of order k. Since $k > n$, A^k is subsequently removed. Let p^k be the first† perforation after f^k which removes A^k. Then p^k commutes with each of the elementary transformations between f^k and p^k. For none of the latter add or remove a simplex belonging to the closure of A^k, since A^k is present throughout, and, since k is the maximum order, no simplex is introduced having A^k on its boundary. Therefore p^k, which is the inverse of f^k, may be applied immediately after f^k and both may be omitted from the sequence. The lemma now follows from induction on the number of elementary transformations in the passage from K to L.

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† It may happen that A^k is inserted and removed more than once.

By a *cluster* of simplexes (spheres) attached to a complex K will be meant a set of simplexes (spheres) with a vertex of K in common, which do not meet K or each other anywhere else.

THEOREM 13. *If K and L are connected complexes of at most n dimensions with the same n-group, then*

$$L+\sum_{\mu=1}^{s}\dot{B}_{\mu}^{n+1}=D\Big(K+\sum_{\lambda=1}^{r}\dot{A}_{\lambda}^{n+1}\Big),$$

where A_1^{n+1}, ..., A_r^{n+1} and B_1^{n+1}, ..., B_s^{n+1} are clusters of $(n+1)$-simplexes attached to K and L respectively†.

We may suppose that $K \rightarrow L$ by a transformation of the kind described in Lemma 8. It follows from Lemma 7 and an argument similar to the proof of Theorem 5 that, after a suitable sub-division, the elementary transformations in $K \rightarrow L$ may be arranged so that every filling and expansion precedes every perforation and contraction. Further, if e^k is an elementary expansion of order $k \leqslant n$ which follows immediately after a filling f^n, it is obvious that

$$e^k f^n = f^n e^k.$$

Similarly

$$p^n c^k = c^k p^n,$$

where c^k is an elementary contraction of order $k \leqslant n$ and p^n is a perforation of order n. Therefore we may exhibit the transformation $K \rightarrow L$ in the form

$$L = Cp_s{}^n \dots p_1{}^n f_r{}^n \dots f_1{}^n\, E(K),$$

where E is an expansion, $f_\lambda{}^n$ and $p_\lambda{}^n$ are special fillings and perforations of order n, and C is a contraction. That is to say

$$L_0 = p_s{}^n \dots p_1{}^n f_r{}^n \dots f_1{}^n\,(K_0),$$

where $K_0 = E(K)$, $L_0 = C^{-1}(L)$.

Let $A_\lambda{}^n$ be the open simplex added by $f_\lambda{}^n$, let $B_\mu{}^n$ be the open simplex removed by $p_\mu{}^n$, and let

$$K_\lambda = f_\lambda{}^n \dots f_1{}^n (K_0).$$

Since $\dot{A}^n_{\lambda+1}$ is homotopic to a point in K_λ it is obvious that any spherical map in K_λ which is homotopic to a point in $K_\lambda + A^n_{\lambda+1}$ is homotopic to a point in K_λ. It follows from induction on λ that any spherical map in K_0 which

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† If $R^n(P)$ stands for the n-th connectivity of P, it is obvious that

$$s-r = R^n(L)-R^n(K).$$

is homotopic to a point in K_r is homotopic to a point in K_0. Therefore each of $\dot{A}_1{}^n, \ldots, \dot{A}_r{}^n$ is homotopic to a point in K_0 and it follows from Theorem 11, as in the proof of Lemma 5, that

$$K_r = D_1\Big(K_0 + \sum_{\lambda=1}^{r} \dot{A}_\lambda^{n+1}\Big), \tag{7.5}$$

where $A_1^{n+1}, \ldots, A_r^{n+1}$ is a cluster of $(n+1)$-simplexes attached to K_0. Similarly

$$K_r = D_2\Big(L_0 + \sum_{\mu=1}^{s} \dot{B}_\mu^{n+1}\Big), \tag{7.6}$$

where $B_1^{n+1}, \ldots, B_s^{n+1}$ is a cluster of $(n+1)$-simplexes attached to L_0. Since K_0 and L_0 contract into K and L respectively we may assume that the vertices $A_\lambda^{n+1} . K_0$ and $B_\mu^{n+1} . L_0$ belong to K and L. Then

$$K_0 + \sum_{\lambda=1}^{r} \dot{A}_\lambda^{n+1} \quad \text{and} \quad L_0 + \sum_{\mu=1}^{s} \dot{B}_\mu^{n+1}$$

contract into

$$K + \sum_{\lambda=1}^{r} \dot{A}_\lambda^{n+1} \quad \text{and} \quad L + \sum_{\mu=1}^{s} \dot{B}_\mu^{n+1},$$

and the theorem follows from (7.5) and (7.6).

From the addendum to Theorem 11 we have the addendum:

ADDENDUM. *The order of the deformation D in Theorem* 13 *need not exceed $n+1$.*

From the first addendum to Theorem 5 we have the corollary:

COROLLARY. *If K and L are two complexes of at most n dimensions with the same n-group there is a complex of at most $n+1$ dimensions which contracts into*

$$K + \sum_{\lambda=1}^{r} \dot{A}_\lambda^{n+1}$$

and also into some sub-division of

$$L + \sum_{\mu=1}^{s} \dot{B}_\lambda^{n+1}.$$

8. *Retracts by deformation.*

Let $f_0(p)$ $(p \in P)$ be a map of a topological space P in a space Q and let P^* be any sub-space of P. We shall describe a deformation of f_0 into a map f_1, given by $f_t(p) = f(p, t)$ $(0 \leqslant t \leqslant 1)$, as *relative to* P^* if $f_t(p^*) = f_0(p^*)$ for every point p^* in P^* and every t in $\langle 0, 1 \rangle$.

We recall that a sub-space of a topological space, and in particular a sub-complex L of a geometrical complex K, is called a retract by deformation† of K if there is a deformation

$$p_t = \phi_t(p) \quad (p \in K,\ 0 \leqslant t \leqslant 1),$$

such that $p_0 = p$, $p_1 \in L$ and $p_1 = p_0$ if $p_0 \in L$. If the deformation ϕ_t is relative to L we shall describe L as a retract by deformation *relative to itself*. If K contracts into L it is obvious that L is a retract by deformation relative to itself. If

$$p = f(q) \quad (q \in Q)$$

is a map of a topological space Q, in K, the map

$$p_1 = \phi_1 f(q)$$

is uniquely determined up to homotopy in L. For if

$$p_t = \psi_t(p) \quad (0 \leqslant t \leqslant 1;\quad p_0 = p)$$

is any deformation of K into L the map $\psi_1 f$ is homotopic to f and hence to $\phi_1 f$ in K. Therefore the two maps are homotopic in L since the latter is a retract of K. If K_1 and K_2 are complexes such that $K_2 = D(K_1)$, there is a complex K^* which contracts into K_1 and into a stellar sub-division σK_2, and, except for the choice of the new vertices, K^* is uniquely determined by the deformation D. If $f_1(Q)$ is a map of Q in K_1 it follows that D determines a unique homotopy class of maps $[f_2] = D[f_1]$ in K_2, given by

$$D[f_1] = [\sigma^{-1}(\psi f_1)],$$

where ψ is the final result of a deformation of K^* into σK_2, and $\sigma^{-1}(\psi f_1)$ is a map obtained from ψf_1 by a canonical displacement‡ of the vertices of σK_2 into the vertices of K_2. Let $\mathcal{E}_{i1}, \ldots, \mathcal{E}_{ip}$ be simple membranes bounded by (simplicial) spherical maps $f_{i1}(S_1), \ldots, f_{ip}(S_p)$ in K_i $(i = 1, 2)$, where $f_{2\lambda} \in D[f_{1\lambda}]$. Further let D be relative to L, where $L \subset K_1 . K_2$.

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† *Cf.* K. Borsuk, *Fundamenta Math.*, 21 (1933), 91–98.

‡ The purpose of the sub-division σ in Theorem 5 is to eliminate unwanted intersections. Therefore we cannot, in general, take K_2 and σK_2 to be complexes covering the same (polyhedral) point-set. However, each vertex of σK_2 is internal to just one element σA, where $A \subset K_2$, and a canonical displacement is a simplicial map in which every vertex inside σA corresponds to a vertex of A, for each $A \subset K_2$. If follows from a well-known argument that any two such maps of σK_2 on K_2 are homotopic in K_2.

THEOREM 14. *Under these conditions*

$$K_2+\sum_{\lambda=1}^{p}\mathcal{E}_{2\lambda}=D^*\left(K_1+\sum_{\lambda=1}^{p}\mathcal{E}_{1\lambda}\right)\quad(\text{rel. } L).$$

First assume that D is the resultant of an expansion $K_1\rightarrow K^*$, followed by a contraction $K^*\rightarrow K_2$. Then $F(\mathcal{E}_{1\lambda})$ is homotopic in K^* to $F(\mathcal{E}_{2\lambda})$ and it follows from Theorem 11 that

$$K^*+\sum_{\lambda=1}^{p}\mathcal{E}_{2\lambda}=D_1\left(K^*+\sum_{\lambda=1}^{p}\mathcal{E}_{1\lambda}\right),$$

where the deformation D_1 is relative to K^* and hence to L. Since $F(\mathcal{E}_{i\lambda})\subset K_i$, we have

$$K^*+\sum_{\lambda=1}^{p}\mathcal{E}_{i\lambda}=E_i\left(K_i+\sum_{\lambda=1}^{p}\mathcal{E}_{i\lambda}\right)\quad(i=1,2),$$

where E_i is an expansion, and the special case of the theorem follows.

In general the complex K^* contracts both into K_1 and into some stellar sub-division $K_1{}^*=\sigma K_2$, where $\sigma=1$ in L, by the second addendum to Theorem 5. So, if we begin again with $K_1{}^*$, the proof will be complete if we can prove the theorem in case $D=\sigma^{-1}$, where $\sigma=1$ in L. Using induction on the number of elementary sub-divisions in σ, we may take σ to be a single elementary sub-division $K_2=AP+Q\rightarrow a\dot{A}P+Q=K_1{}^*$. There is then a complex which contracts both into $K_1{}^*$ and into K_2, namely $aAP+Q$. Moreover, if a_0 is a vertex of A, the canonical displacement $a\rightarrow a_0$ may be realized in $aAP+Q$, by shrinking the edge aa_0 into the vertex a_0. Therefore a homotopic deformation of $aAP+Q$ into K_2 determines the given transformation of classes of maps in $K_1{}^*$ into classes of maps in K_2. Since $\sigma=1$ in L, it follows that $A\not\subset L$, whence $L\subset K_1{}^*.K_2$, and the theorem follows from what we have already proved.

Under the conditions of Theorem 14 we shall write

$$\mathcal{E}_{2\lambda}=D^*(\mathcal{E}_{1\lambda}).$$

From the addendum to Theorem 11 we have the addendum to Theorem 14:

ADDENDUM. *If D is of order m and if the membranes $\mathcal{E}_{1\lambda}$ are at most n-dimensional, the order of D^* need not exceed* max $(m, n+1)$.

We now quote for reference a lemma which is essentially a restatement of a familiar result. Let $p_t=f_t(p)$ $(0\leqslant t\leqslant 1;\ p_0=p)$ be a deformation of a connected complex K into itself.

LEMMA 9. *If p_0 is any point in K the circuits $f_1(C)$ constitute a geometrical basis for $\pi_1(K, p_1)$, the fundamental group of K with p_1 as a base point, where $\{C\}$ is the set of circuits beginning and ending at p_0.*

If s is the segment $p = p_t$ described by p_0 in the deformation, the singular circuits of the form $s + C + s$ (orientations ignored) constitute a geometrical basis for $\pi_1(K, p_1)$. Such a circuit is homotopic, rel. p_1, to $f_1(C)$.

Let L be a sub-complex of K. We specify four sets of conditions and shall show that each of them implies all the others. The first is

R. *L is a retract by deformation of K.*

R_L. *L is a retract by deformation of K relative to itself.*

A. *Any r-cell in K ($r = 0, 1, \ldots$) with its boundary in L is homotopic, relative to its boundary, to an r-cell in L.*

Our final set, which we shall denote by B, contains three conditions B_1, B_2, and B_3, namely

B_1. *If p is any point in L the circuits in L beginning and ending at p constitute a geometrical basis for $\pi_1(K, p)$.*

B_2. *Any spherical map in K is homotopic to a spherical map in L.*

B_3. *Any spherical map in L which bounds a cell in K bounds a cell in L.*

THEOREM 15. *Each of the conditions R, R_L, A and B implies all the others.*

It is obvious that R_L implies each of the others. It follows from an argument used by Hurewicz†, in establishing this result when L is a single point, that A implies R_L, and therefore R and B. In the presence of Lemma 9 it is obvious that R implies B. Therefore the theorem will follow if we can show that B implies A.

Let pq be a segment in K whose end points, p and q, lie in L. According to B_3, there is a segment s joining p to q in L. According to B_1 the circuit $pq + s$ is homotopic, rel. p, to a circuit C, in L. Then the singular segment $s + C$ is homotopic to pq, rel. $(p + q)$, and the condition A is established in case $r = 1$.

Let Σ_0^n ($n > 0$) be any n-spherical map in K containing a point p_0, in L. According to B_2, Σ_0^n is homotopic to a map Σ_1^n in L. Let $p_0 p_1$ be the

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† Hurewicz, *Proc. Akad. Amsterdam* (*loc. cit.*), 2nd paper, § 6.

segment described by p_0 during a deformation of Σ_0^n into Σ_1^n. Then Σ_0^n is homotopic, rel. p_0, to a spherical map of the form† $p_0 p_1 + \Sigma_1^n$. By what we have just proved, $p_0 p_1$ is homotopic, rel. $(p_0 + p_1)$, to a segment s in L. Therefore Σ_0^n is homotopic, rel. p_0, to a spherical map in L, namely $s + \Sigma_1^n$.

Finally, let Γ_1^n be any n-cell in K whose boundary lies in L. According to B_3, $F(\Gamma_1^n)$ bounds a cell Γ_2^n in L. By what we have just proved the spherical map $\Gamma_1^n + \Gamma_2^n$ is homotopic, rel. p_0, to a spherical map Σ^n in L, where p_0 is any point in Γ_2^n. As in the case $n = 1$ it follows that Γ_1^n is homotopic, rel. $F(\Gamma_1^n)$, to the cell $\Gamma_1^n + \Sigma^n$, which lies in L, and the proof is complete.

THEOREM 16. *If L is a retract by deformation of K, then K and L have the same m-group for all values of m.*

Let K^p $(p > 0)$ be the complex consisting of L together with all the simplexes in $K - L$ whose dimensionalities do not exceed p. Thus $K^p = K$ if K is n-dimensional and $p \geqslant n$. For a given value of p let us assume that

$$(8.1) \qquad L_p = D_p(K^p) \quad (\text{rel. } L),$$

where

$$(8.2) \qquad L_p = L + \sum_{i=1}^{r} \mathcal{E}_i^{p-1} + \sum_{\lambda=1}^{s} \mathcal{E}_\lambda^{p},$$

$\mathcal{E}_1^{p-1}, \ldots, \mathcal{E}_r^{p-1}$ being simple membranes whose boundaries are in L, and $\mathcal{E}_1^{p}, \ldots, \mathcal{E}_s^{p}$ being simple membranes whose boundaries are in

$$(8.3) \qquad L + \sum_{i=1}^{r} \mathcal{E}_i^{p-1}.$$

By Theorem 14 there is a deformation of K^{p+1}, which we also denote by D_p, such that

$$(8.4) \qquad D_p(K^{p+1}) = L_p + \sum_{\rho=1}^{t} \mathcal{E}_\rho^{p+1} \quad (\text{rel. } L)$$
$$= L_p^*,$$

where‡ $\mathcal{E}_\rho^{p+1} = D_p(A_\rho^{p+1})$ and $F(\mathcal{E}_\rho^{p+1}) \subset L_p$, $A_1^{p+1}, \ldots, A_t^{p+1}$ being the $(p+1)$-simplexes in K.

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† *Cf.* §10 below.

‡ If $p \geqslant n$ we have $t = 0$, the corresponding sets of cells being empty, and $L_p^* = L_p$.

Since L is a retract by deformation of K, any simplicial $(p-1)$-cell in K^{p+1}, say $\tilde{\Gamma}^{p-1}$, whose boundary lies in L, is homotopic in K, rel. $F(\tilde{\Gamma}^{p-1})$, to a $(p-1)$-cell Γ^{p-1}, in L. The p-dimensional deformation cell may be deformed into K^{p+1}, holding Γ^{p-1} and $\tilde{\Gamma}^{p-1}$ fixed, and it follows that $\tilde{\Gamma}^{p-1}$ is homotopic to Γ^{p-1}, rel. $F(\tilde{\Gamma}^{p-1})$, in K^{p+1}. This property is obviously invariant under a formal deformation, rel. L, of K^{p+1}. Therefore $\mathscr{E}_i^{p-1}$ is homotopic, rel. $F(\mathscr{E}_i^{p-1})$, in $L_p{}^*$ to a $(p-1)$-cell Γ_i^{p-1}, in L. Let ${}_i\mathscr{E}^p$ be a simple membrane bounded by the spherical map $\Gamma_i^{p-1}+\mathscr{E}_i^{p-1}$. Then $F({}_i\mathscr{E}^p)$ is homotopic to a point in $L_p{}^*$ and it follows from Theorem 11 that

$$L_p{}^*+\sum_{i=1}^{r}{}_i\mathscr{E}^p=D^*\Big(L_p{}^*+\sum_{i=1}^{r}B_i^{p+1}\Big)\quad(\text{rel. } L_p{}^*),$$

where $(B_1^{p+1}, \ldots, B_r^{p+1})$ is a cluster of $(p+1)$-simplexes attached to $L_p{}^*$. By Theorem 14 there is a deformation of

$$L_p{}^*+\sum_{i=1}^{r}B_i^{p+1}, \tag{8.5}$$

which we also denote by D^*, such that

$$D^*\Big(L_p{}^*+\sum_{i=1}^{r}B_i^{p+1}\Big)=L_p{}^*+\sum_{i=1}^{r}\Big(\mathscr{E}_{t+i}^{p+1}+{}_i\mathscr{E}^p\Big)\quad(\text{rel. } L_p{}^*), \tag{8.6}$$

where $\mathscr{E}_{t+i}^{p+1}=D^*(B_i^{p+1})$ and $F(\mathscr{E}_{t+i}^{p+1})\subset L_p{}^*+\sum_{i=1}^{r}{}_i\mathscr{E}^p.$

Let $$L_p{}^*+\sum_{i=1}^{r}(\mathscr{E}_{t+i}^{p+1}+{}_i\mathscr{E}^p)=L_p^{**}.$$

Clearly $\mathscr{E}_i^{p-1}$ is a sub-complex of ${}_i\mathscr{E}^p$. Therefore $\mathscr{E}_i^{p-1}+{}_i\mathscr{E}^p={}_i\mathscr{E}^p$ and

$$L_p^{**}=L+\sum_{i=1}^{r}{}_i\mathscr{E}^p+\sum_{\lambda=1}^{s}\mathscr{E}_\lambda{}^p+\sum_{\rho=1}^{t+r}\mathscr{E}_\rho^{p+1}. \tag{8.7}$$

From (8.4), from the fact that $L_p{}^*$ expands into (8.5) and from (8.6) we have

$$L_p^{**}=D_0{}^*(K^{p+1})\quad(\text{rel. } L).$$

We now show that L_p^{**} is deformable, rel. L, into a complex L_{p+1}, given by an equation of the form (8.2) with p replaced by $p+1$. As explained in § 6, the cell $\mathscr{E}_i^{p-1}$ is defined by a map without folds. Therefore no two $(p-1)$-simplexes in the original $(p-1)$-sphere of the map

$$\Gamma_i^{p+1}+\mathscr{E}_i^{p-1}=F({}_i\mathscr{E}^p)$$

correspond to the same $(p-1)$-simplex in $\mathcal{E}_i^{p-1}$, and it follows from Lemma 6 that

$$L+\sum_{i=1}^{r} {}_i\mathcal{E}^p$$

contracts into

$$L+\sum_{i=1}^{r} (\mathcal{E}_i^{p-1}-B^{p-1}),$$

where B_i^{p-1} is an open $(p-1)$-simplex in $\mathcal{E}_i^{p-1}$, and so contracts into L†. It follows from Theorem 14 that there is a deformation of

$$L+\sum_{i=1}^{r} {}_i\mathcal{E}^p+\sum_{\lambda=1}^{s} \mathcal{E}_\lambda^p,$$

which we denote by D_1^*, such that

$$D_1^*\left(L+\sum_{i=1}^{r} {}_i\mathcal{E}^p+\sum_{\lambda=1}^{s} \mathcal{E}_\lambda^p\right)=L+\sum_{\lambda=1}^{s} \tilde{\mathcal{E}}_\lambda^p \quad (\text{rel. } L),$$

where

$$\tilde{\mathcal{E}}_\lambda^p=D_1^*(\mathcal{E}_\lambda^p) \quad \text{and} \quad F(\tilde{\mathcal{E}}_\lambda^p)\subset L.$$

Therefore it follows from (8.7) and yet another appeal to Theorem 14 that there is a deformation of L_p^{**}, which we also denote by D_1^*, such that

$$D_1^*(L_p^{**})=L+\sum_{\lambda=1}^{s} \tilde{\mathcal{E}}_\lambda^p+\sum_{\rho=1}^{t+r} \tilde{\mathcal{E}}_\rho^{p+1} \quad (\text{rel. } L),$$

where

$$\tilde{\mathcal{E}}_\rho^{p+1}=D_1^*(\mathcal{E}_\rho^{p+1}) \quad \text{and} \quad F(\tilde{\mathcal{E}}_\rho^{p+1})\subset L+\sum_{\lambda=1}^{s} \mathcal{E}_\lambda^p.$$

Therefore

$$L_{p+1}=D_{p+1}(K^{p+1}) \quad (\text{rel. } L), \tag{8.8}$$

where $D_{p+1}=D_1^* D_0^*$ and

$$L_{p+1}=L+\sum_{\lambda=1}^{s} \tilde{\mathcal{E}}_\lambda^p+\sum_{\rho=1}^{t+r} \tilde{\mathcal{E}}_\rho^{p+1}. \tag{8.9}$$

The equations (8.8) and (8.9) are similar to (8.1) and (8.2) with p replaced by $p+1$. Equations of the form (8.1) and (8.2) are obviously satisfied when $p=1$, taking the set $\mathcal{E}_1^0, \ldots, \mathcal{E}_r^0$ to be empty, and it follows by induction that they are satisfied for all values of p‡. By Lemma 5,

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† *Cf.* the proof of Lemma 5.

‡ Notice that the step from $p=1$ to $p=2$ is achieved by (8.4), since L_1^* is of the form L_2.

L_{m+2} and (8.3), with $p=m+2$, have the same $(m+1)$-group and (8.3) and L have the same m-group. Therefore K^{m+2} and L have the same m-group. But K and K^{m+2} obviously have the same m-group and the theorem is established.

From the addendum to Theorem 14 and induction on p we have the addendum to Theorem 16:

ADDENDUM. *The order of the deformation* D_p *in* (8.1) *need not exceed* $p+1$.

9. *Homotopy types.* Two topological spaces, P and Q, are said to belong to the same homotopy type if there is a map $f(P)$ in Q and a map $g(Q)$ in P such that the maps gf and fg, of P and Q into themselves, are each homotopic to the identity.

THEOREM 17. *Two complexes are of the same homotopy type if, and only if, they have the same m-group for each value of m.*

We first show that two given complexes K_1 and K_2, of at most n dimensions, belong to the same homotopy type if they have the same $(n+1)$-group. Let K_1 and K_2 have the same $(n+1)$-group. By Theorem 13

$$K_2+\Sigma_2=D(K_1+\Sigma_1),$$

where Σ_i is a cluster of $(n+1)$-spheres attached to K_i at a vertex a_i $(i=1, 2)$. Two complexes with the same nucleus are obviously of the same homotopy type, and it follows that there are maps $f_i(K_i+\Sigma_i)$ in $K_j+\Sigma_j$ $(i=1, 2; j=i+1 \bmod 2)$ such that each of $f_2 f_1$ and $f_1 f_2$ is homotopic to the identity. We may take f_i to be semi-linear, in which case, since K_i is at most n-dimensional, $f_i(K_i)$ does not cover the whole of Σ_j. If part of $f_i(K_i)$ lies in Σ_j, it may therefore be deformed into a_j, holding the rest of the map fixed. Therefore f_i and f_j may be deformed into maps f_i^* and f_j^* such that $f_i^*(K_i)\subset K_j$ and $f_j^* f_i^*(K_i)$ is homotopic to the identity in $K_i+\Sigma_i$. But K_i is obviously a retract (not by deformation) of $K_i+\Sigma_i$ and it follows that $f_j^* f_i^*(K_i)$ is homotopic to the identity in K_i. Therefore K_1 and K_2 are of the same homotopy type.

Conversely, let K and L be two complexes of the same homotopy type, which we may assume to be connected, and let $f(K)$ and $g(L)$ be maps of the kind described above. We may suppose that $K.L=1$ and, after a suitable sub-division of K, that the map f is simplicial. By Theorem 8 the mapping cylinder $C_f(K)$ contracts into L. Therefore the theorem will follow from Theorem 16 if we can show that K is a retract by deformation of $C_f(K)$, and hence from Theorem 15 if we can show that K satisfies the conditions B.

Let P be any complex and let $\phi(P) \subset K$ be a map of P in K. When we compare $C_f(K)$ with $K \times \langle 0, 1 \rangle$, it follows from the definition of $C_f(K)$, in § 6, that $\phi(P)$ is homotopic in $C_f(K)$ to the map $f\phi(P) \subset L$. Conversely, any map of the form $f\phi(P) \subset L$, where $\phi(P) \subset K$, is homotopic in $C_f(K)$ to $\phi(P)$. If P is a circuit and p is any point in P, the singular circuit $s+f\phi(P)+s$ is obviously homotopic, rel. p, to $\phi(P)$, where s is the segment $C_f\{\phi(p)\}$.

Let q_0 be any point in L, let $p_0 = g(q_0)$ and let $q_1 = f(p_0) = fg(q_0)$. By Lemma 9 the set of circuits $fg(\Sigma)$ constitutes a geometrical basis for $\pi_1(L, q_1)$, where Σ is any circuit in L beginning and ending at q_0. Since $C_f(K)$ contracts into L, the circuits $fg(\Sigma)$ also constitute a geometrical basis for $\pi_1\{C_f(K), q_1\}$. Therefore the circuits of the form $s+fg(\Sigma)+s$ constitute a geometrical basis for $\pi_1\{C_f(K), p_0\}$, where $s = C_f(p_0)$. But such a circuit is homotopic, rel. p_0, to the circuit $g(\Sigma)$. Therefore the circuits in K which contain p_0 constitute a geometrical basis for $\pi_1\{C_f(K), p_0\}$, and B_1 is satisfied.

Since L is a retract by deformation of $C_f(K)$, any spherical map in $C_f(K)$ is homotopic to a spherical map Σ, in L, and hence to $fg(\Sigma)$, since fg is homotopic to the identity. But $fg(\Sigma)$ is homotopic in $C_f(K)$ to $g(\Sigma) \subset K$ and it follows that B_2 is satisfied.

Finally, let Σ be any spherical map in K which bounds a cell Γ, in $C_f(K)$. Comparing $C_f(K)$ with $K \times \langle 0, 1 \rangle$, we see that L is a retract of $C_f(K)$ by a deformation f_t $(0 \leqslant t \leqslant 1)$ such that $f_1 = f$ in K. Therefore $f(\Sigma) \subset L$ bounds the cell $f_1(\Gamma) \subset L$ and $gf(\Sigma)$ bounds a cell in K, namely $gf_1(\Gamma)$. But gf is homotopic to the identity. Therefore Σ bounds a cell in K and B_3 is satisfied. Therefore K is a retract by deformation of $C_f(K)$ and the theorem is established.

10. *Certain questions: a ring.* In this section we ask certain questions and introduce a ring which is in many ways analogous to Reidemeister's homotopy ring†. The first question is:

Q. 1. *If two complexes of at most n dimensions have the same n-group and the same connectivities, have they the same nucleus?*

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† K. Reidemeister, *Abhand. Math. Sem. Hamburg*, 10 (1934), 211–215; *Journal für Math.*, 173 (1935), 164–173, and other papers.

If K and L, being at most n dimensional, have the same n-group it follows from Theorem 13 that

$$L+\sum_{\mu=1}^{s}\dot{B}_{\mu}^{n+1}=D\left(K+\sum_{\lambda=1}^{r}\dot{A}_{\lambda}^{n+1}\right),$$

where $A_1^{n+1}, \ldots, A_r^{n+1}$ and $B_1^{n+1}, \ldots, B_s^{n+1}$ are clusters of $(n+1)$-simplexes attached to K and L. Then $r=s$ if $R^n(K)=R^n(L)$, where $R^n(P)$ stands for the n-th connectivity of P, and Q. 1 raises the question:

Q. 2. *If* $K_2=D(K_1)$, $K_2+L_2=D_0(K_1+L_1)$ *and if* $K_i.L_i$ *is geometrically collapsible* $(i=1, 2)$, *have* L_1 *and* L_2 *the same nucleus?*

An affirmative answer to Q. 2 carries with it an affirmative answer to Q. 1. We shall see, in § 12, that Q. 2 is equivalent to the apparently narrower question:

Q. 3. *If* M^n, $M_1{}^n$, *and* $M_2{}^n$ *are bounded n-dimensional manifolds*†, *with connected boundaries, such that* M^n *meets* $M_i{}^n$ *in an* $(n-1)$*-element on the boundary of both* $(i=1, 2)$, *and if the manifolds* $M^n+M_1{}^n$ *and* $M^n+M_2{}^n$ *are combinatorially equivalent, have* $M_1{}^n$ *and* $M_2{}^n$ *the same nucleus?*

We now leave these questions for the moment and turn to the ring. Let an n-spherical map in a connected‡ (geometrical) complex K be taken as a map $f(C_1{}^n)$ of a hyper-cube $C_1{}^n$, such that $f(\dot{C}_1{}^n)$ is a constant§ p_1, and let $C_1{}^n$ be given by $-t_1\leqslant x_i\leqslant t_1$ $(i=1, \ldots, n)$ in Cartesian space. Let p_0 be any point in K and let s be any oriented segment in K, beginning at p_0 and joining it to p_1, which is given by‖

$$p_t=\phi(t) \quad (t_1\leqslant t\leqslant t_0; \quad p_{t_0}=p_0, \quad p_{t_1}=p_1).$$

With the map $f(C_1{}^n)$ we associate the map ${}_sf(C_0{}^n)$, such that ${}_sf=f$ in $C_1{}^n$ and ${}_sf(\dot{C}_t{}^n)=p_t$ $(t_1\leqslant t\leqslant t_0)$, where $C_t{}^n$ is given by

$$-t\leqslant x_i\leqslant t \quad (C_{t_0}^n=C_0{}^n, \quad C_{t_1}^n=C_1{}^n).$$

We shall denote the map ${}_sf(C_0{}^n)$ by

$$s+f(C_1{}^n)-s$$

if $n=1$, and by

$$s+f(C_1{}^n)$$

---

† Here, as in § 12, a manifold is a complex M^n such that $M_a{}^n$ is an $(n-1)$-sphere or $(n-1)$ element according as a is inside or on the boundary of M^n.

‡ In any discussion involving the homotopy groups of a complex it is always to be understood that the latter is connected.

§ *Cf.* Hurewicz, 2nd paper (*loc. cit.*), § 2.

‖ Notice that the parameter t decreases as p_t describes s in the positive direction.

if $n > 1$. It is easy to verify that the transformation $f \rightarrow {}_s f$ determines an isomorphism of $\pi_n(K, p_1)$ on $\pi_n(K, p_0)$, where $\pi_n(P, p)$ $(p \in P)$ stands for the geometrical representation of the n-th homotopy group of a space P having p as its base point. If s_1 is a segment joining p_0 to p_1 which is homotopic, rel. $(p_0 + p_1)$, to s, it is obvious that ${}_{s_1}f(C_0^n)$ is homotopic, rel. $(\dot{C}_0^n + \dot{C}_1^n)$, to ${}_s f(C_0^n)$. Therefore the transformations $f \rightarrow {}_s f$ and $f \rightarrow {}_{s_1} f$ determine the same isomorphism $\pi_n(K, p_1) \rightarrow \pi_n(K, p_0)$.

If $p_0 = p_1$, the isomorphisms just described are automorphisms of the group $\pi_n(K, p_1)$. To each element g of the fundamental group $\pi_1(K, p_1)$ corresponds an automorphism $\psi_n(g)$, given by

$$f(C_1^n) \rightarrow c + f(C_1^n) - c$$

if $n = 1$, and by

$$f(C_1^n) \rightarrow c + f(C_1^n)$$

if $n > 1$, where c is an oriented circuit representing the element g. The transformation $g \rightarrow \psi_n(g)$ is obviously a homomorphism of $\pi_1(K, p_1)$ in the group of automorphisms of $\pi_n(K, p_1)$, which, in case $n = 1$, is the familiar homomorphism of $\pi_1(K, p_1)$ on its group of inner automorphisms. This homomorphism is invariant, relative to a change of base point, in the same sense that a tensor (not to be confused with its components in any one coordinate system) is invariant under a transformation of coordinates. For if $g' \rightarrow \psi'_n(g')$ is the homomorphism defined as above with p_0 as a base point, and if T_k is the isomorphism of $\pi_k(K, p_1)$ on $\pi_k(K, p_0)$ $(k = 1, 2, \ldots)$ determined by a segment joining p_0 to p_1, it may be verified either formally or geometrically that the transformation law of $\psi_n(g)$ is

$$\psi_n'(g') = T_n \psi_n(g) T_n^{-1},$$

where $g' = T_1(g)$. In particular, if $p_0 = p_1$ and $T_k = \psi_k(g_0)$, we have

$$\begin{aligned} \psi_n'(g') &= \psi_n(g_0)\psi_n(g)\psi_n^{-1}(g_0) \\ &= \psi_n(g_0 g g_0^{-1}) \\ &= \psi_n(g'). \end{aligned}$$

With the group of automorphisms of the form $\psi_n(g)$ is associated a ring[†] $\mathfrak{R}_n = \mathfrak{R}_n(K, p)$, consisting of homomorphisms of $\pi_n = \pi_n(K, p)$ into itself $(n > 1)$. If we rewrite $\psi_n(g)\alpha$ as $g\alpha$, the elements of the ring are transformations of the form[†]

[†] *Cf.* B. L. van der Waerden, *Moderne Algebra*, 1 (Berlin, 1930), 133.

$$\alpha \to r\alpha = \Sigma_i a_i g_i \alpha,$$

where $\alpha \in \pi_n$, $g_i \in \pi_1$, and $a_1, a_2, \ldots$ are rational integers, all but a finite number of which are zero. The addition and multiplication $(r+r')\alpha$ and $rr'\alpha$ are defined in the usual way. If $\alpha_1, \ldots, \alpha_k$ are fixed elements in π_n, the set of all elements of the form

$$r_1 \alpha_1 + \ldots + r_k \alpha_k \quad (r_i \in \mathfrak{R}_n),$$

is a sub-group of π_n, which we shall denote by $r(\alpha_1, \ldots, \alpha_k)$.

Let $K^* = K + \mathcal{E}_1^{n+1} + \ldots + \mathcal{E}_k^{n+1}$ $(n > 1)$, where $\mathcal{E}_i^{n+1}$ is an oriented simple membrane bounded by a (simplicial) map representing the element α_i in π_n, and let $\pi_n^* = \pi_n(K^*, p)$. If α is any element in π_n, represented by an oriented map $f(S^n)$ in K, and if α^* is the element in π_n^* which is represented by the same map $f(S^n)$, the transformation $\alpha \to \alpha^* = \phi(\alpha)$ is obviously a homomorphism of π_n in π^*. It is also obvious that any map $f^*(S^n)$ in K^* is homotopic, rel. q, to a map in K, where $q \in S^n$ and $f^*(q) = p$. Therefore ϕ is a homomorphism of π_n on π_n^*.

THEOREM 18. *The kernel of the homomorphism* ϕ *is*[‡] $r(\alpha_1, \ldots, \alpha_k)$.

If $\alpha \in r(\alpha_1, \ldots, \alpha_k)$, it is obvious that $\phi(\alpha) = 0$. Conversely, let $\phi(\alpha) = 0$, where α is a given element in π_n, let $f(\dot{E}^{n+1})$ be an oriented map in K representing α and $\phi(\alpha)$, and let $f(E^{n+1})$ be an oriented $(n+1)$-cell in K^* bounded by $f(\dot{E}^{n+1})$.

Let $\mathcal{E}_i^{n+1} = E_i^{n+1} + C_{f_i}(\dot{E}_i^{n+1})$ and let A_i^n be any closed n-simplex in $\dot{E}_i^{n+1}$ whose image $f_i(A_i^n)$ contains p, which we take to be a vertex of K. Let B_i^{n+1} be any principal open simplex in $C_i = C_{f_i}(A_i^n)$ of which p is a vertex. I say that $C_i - B_i^{n+1}$ contracts into $\dot{C}_i = A_i^n + f_i(A_i^n) + C_{f_i}(\dot{A}_i^n)$. For, if $f_i(A_i^n)$ is n-dimensional and $B_i^{n+1} = c_i f_i(A_i^n)$, where c_i is the centre of C_i, this follows from an argument used in proving Theorem 8. Otherwise $B_i^{n+1} = c_i B_i^n$, where B_i^n is a principal open simplex of $C_{f_i}(A_i^{n-1})$ $(A_i^{n-1} \subset \dot{A}_i^n)$. In this case the assertion follows from induction on n, Lemma 3, and an argument used in proving Theorem 9. Therefore $C_i - B_i^{n+1}$ contracts into $\dot{C}_i$ and, by an argument used in proving Lemma 6, $\mathcal{E}_i^{n+1} - B_i^{n+1}$ contracts into $F(\mathscr{E}_i^{n+1})$. Let the map $f(E^{n+1})$ be simplicial, $f(\dot{E}^{n+1})$ being the spherical map defining α, and let $A_{i1}^{n+1}, \ldots, A_{ir_i}^{n+1}$ be the open simplexes in E^{n+1} which fall on B_i^{n+1}. Let B_i^{n+1} be oriented so that $\dot{B}_i^{n+1}$ is homotopic in $\mathscr{E}_i^{n+1} - B_i^{n+1}$, with regard to sense, to the oriented map $F(\mathscr{E}_i^{n+1})$, representing α_i, and let $A_{i\lambda}^{n+1}$ be oriented so as to cover B_i^{n+1} positively.

[†] The (commutative) homotopy groups π_n $(n > 1)$ will always be written additively and the fundamental group with multiplication.

[‡] That is to say, $\phi^{-1}(0) = r(\alpha_1, \ldots, \alpha_k)$.

If the set of simplexes $A_{i\lambda}^{n+1}$ is empty, $f(E^{n+1})$ is deformable into a cell in K bounded by $f(\dot{E}^{n+1})$, and it follows that $\alpha = 0$. We suppose, therefore, that the set of simplexes $A_{i\lambda}^{n+1}$ is not empty. After a suitable subdivision† of E^{n+1} we may suppose that no two of the simplexes $A_{i\lambda}^{n+1}$ meet $\dot{E}^{n+1}$ or have a vertex in common. This being so, let q and $q_{i\lambda}$ be points in $\dot{E}^{n+1}$ and $\dot{A}_{i\lambda}^{n+1}$ such that $f(q) = f(q_{i\lambda}) = p$. Let $t_{i\lambda}$ be an oriented segment in E^{n+1}, beginning with q and joining it to $q_{i\lambda}$, such that $t_{i\lambda}$ does not meet $Cl(A_{i\lambda}^{n+1})$ except in $q_{i\lambda}$ and does not meet $t_{j\mu} + Cl(A_{j\mu}^{n+1})$ except in q $(t_{j\mu} \neq t_{i\lambda})$. Then

$$E^{n+1} - \sum_{i,\lambda} (t_{i\lambda} + \epsilon_{i\lambda} A_{i\lambda}^{n+1}) \quad (\epsilon_{i\lambda} = \pm 1)$$

may be regarded as a singular cell bounded by the oriented singular n-sphere

$$\dot{E}^{n+1} - \sum_{i,\lambda} (t_{i\lambda} + \epsilon_{i\lambda} \dot{A}_{i\lambda}^{n+1}).$$

Therefore $f(\dot{E}^{n+1})$ is homotopic, rel. q, in

$$K_0^* = K^* - \sum_{i=1}^{k} B_i^{n+1},$$

to

$$\sum_{i,\lambda} (s_{i\lambda} + \epsilon_{i\lambda} \dot{B}_i^{n+1}),$$

where $s_{i\lambda} = f(t_{i\lambda})$, and hence to

$$\sum_{i,\lambda} \{s_{i\lambda} + \epsilon_{i\lambda} F(\mathscr{E}_i^{n+1})\},$$

since $\mathscr{E}_i^{n+1} - B_i^{n+1}$ contracts into $F(\mathscr{E}_i^{n+1})$. Since $\mathscr{E}_i^{n+1} - B_i^{n+1}$ contracts into $F(\mathscr{E}_i^{n+1})$, the complex K_0^* contracts into K. Therefore each of the circuits $s_{i\lambda}$ is homotopic in K_0^*, rel. q, to a circuit $c_{i\lambda}$ in K, and $f(\dot{E}^{n+1})$ is homotopic in K, rel. q, to

$$\sum_{i,\lambda} \{c_{i\lambda} + \epsilon_{i\lambda} F(\mathscr{E}_{i\lambda}^{n+1})\}.$$

---

† We can modify E^{n+1} and the map f by applying a sub-division (A, a) to E^{n+1}, making the new vertex a correspond to $f(b)$, where b is a vertex of A. Repeating this process we can isolate the simplexes $A_{i\lambda}^{n+1}$ from $\dot{E}^{n+1}$ and from each other.

Therefore
$$a = \sum_{i,\lambda} \epsilon_{i\lambda} g_{i\lambda} a_i,$$
where $g_{i\lambda}$ is the element in $\pi_1(K, p)$ corresponding to the circuit $c_{i\lambda}$, whence $a \epsilon r(a_1, \ldots, a_k)$ and the theorem is established.

COROLLARY. *$\pi_n{}^*$ is isomorphic to the residue group $\pi_n - r(a_1, \ldots, a_k)$.*

Let $\mathcal{E}_1^{n+1}, \ldots, \mathcal{E}_k^{n+1}$ be simple membranes bounded by oriented spherical maps in K, which do not necessarily contain p, and let
$$K^* = K + \sum_{i=1}^{k} \mathcal{E}_i^{n+1}.$$
Let s_i be an oriented segment beginning with p and joining it to some point in $F(\mathcal{E}_i^{n+1})$, and let a_i be the element in $\pi_n(K, p)$ corresponding to the map $s_i + F(\mathcal{E}_i^{n+1})$. Then the group $r(a_1, \ldots, a_k)$ is independent of the particular segments s_i. For, if s_i is replaced by $\bar{s}_i$, the map $\bar{s}_i + F(\mathcal{E}_i^{n+1})$ is homotopic to
$$(\bar{s}_i - s_i + s_i) + F(\mathcal{E}_i^{n+1}) = (\bar{s}_i - s_i) + \{s_i + F(\mathcal{E}_i^{n+1})\},$$
and the corresponding element $\bar{a}_i$ is given by $\bar{a}_i = g_i a_i$, where g_i is the element in $\pi_1(K, p)$ corresponding to the circuit $\bar{s}_i - s_i$. It follows that
$$r(\bar{a}_1, \ldots, \bar{a}_k) = r(a_1, \ldots, a_k) = r(f_1, \ldots, f_k), \quad \text{say}.$$
We can deform $\mathcal{E}_i^{n+1}$ into a simple membrane bounded by the map $s_i + f_i$, without altering any of the groups concerned, and Theorem 18, with its corollary, may be restated in terms of the group $r(f_1, \ldots, f_k)$.

Now let $f_i = f_i(S^n)$ and $\bar{f}_i = \bar{f}_i(S^n)$ $(i = 1, \ldots, k)$ be oriented maps in K and let a_i and $\bar{a}_i$ be the elements in π_n corresponding to the maps $s_i + f_i$ and $\bar{s}_i + \bar{f}_i$, where s_i and $\bar{s}_i$ are segments joining p to points of f_i and $\bar{f}_i$. Also let $\mathcal{E}_i^{n+1}$ and $\bar{\mathcal{E}}_i^{n+1}$ be simple membranes bounded by f_i and $\bar{f}_i$, and let

$$K^* = K + \sum_{i=1}^{k} \mathcal{E}_i^{n+1}, \quad \bar{K}^* = K + \sum_{i=1}^{k} \bar{\mathcal{E}}_i^{n+1}. \tag{10.1}$$

Then it follows from Theorem 18 that each map $\bar{f}_i$ is homotopic to a point in K^*, and f_i is homotopic to a point in $\bar{K}^*$, if, and only if, $\bar{a}_i \epsilon r(a_1, \ldots, a_k)$ and $a_i \epsilon r(\bar{a}_1, \ldots, \bar{a}_k)$. That is to say, if, and only if, there are elements r_{ij} and $\bar{r}_{ij}$ in $\mathfrak{R}_n$ such that

$$\left\{\begin{aligned} \bar{a}_i &= \sum_{j=1}^{k} r_{ij} a_j, \\ a_i &= \sum_{j=1}^{k} \bar{r}_{ij} \bar{a}_j. \end{aligned}\right. \tag{10.2}$$

If these conditions are satisfied, K^* and $\bar{K}^*$ have the same $(n+1)$-group. For, by Lemma 5, they both have the same $(n+1)$-group as

$$K^*+\bar{K}^*=K+\sum_{i=1}^{k}(\mathcal{E}_i^{n+1}+\bar{\mathcal{E}}_i^{n+1}).$$

On the other hand, if two complexes of at most $(n+1)$-dimensions have the same $(n+1)$-group and if their $(n+1)$-st connectivities are the same, it follows from arguments used in proving Theorem 13 that one is transformable into the other by formal deformations and a transformation of the form† $K^*\to\bar{K}^*$, where K^* and $\bar{K}^*$ are given by (10.1), subject to the conditions (10.2). Therefore, if $n>1$, the question Q. 1 is equivalent to the following:

Q. 4. *Have* K^* *and* $\bar{K}^*$ *the same nucleus if the conditions* (10.2) *are satisfied,* K *being at most* $(n+1)$*-dimensional?*

We conclude this section with an example of complexes K^* and $\bar{K}^*$ satisfying the conditions (10.2), though there seems to be no easy method for finding out whether or no they have the same nucleus. Let P be a 2-dimensional complex such that $\pi_1(P)$ is a cyclic group of order five. Let A^{n+1} $(n>1)$ be an oriented $(n+1)$-simplex meeting P in the single point p, and let $K=P+\dot{A}^{n+1}$. Let α be the element in $\pi_n(K, p)$ which is represented by $\dot{A}^{n+1}$ and let g be a generator of $\pi_1(K, p)$. Let $\bar{\mathcal{E}}^{n+1}$ be a simple membrane bounded by a representative of the element

$$\bar{\alpha}=(1-g-g^4)\,\alpha.$$

It may be verified that $(1-g^2-g^3)\,(1-g-g^4)=1$ as a consequence‡ of $g^5=1$, whence

$$\alpha=(1-g^2-g^3)\,\bar{\alpha}.$$

If $$K^*=K+A^{n+1} \quad\text{and}\quad \bar{K}^*=K+\bar{\mathcal{E}}^{n+1},$$

it follows that K^* and $\bar{K}^*$ satisfy the conditions of Q. 4. Since

$$K^* \quad (=K+A^{n+1}=P+A^{n+1})$$

contracts into P, the question is: "Have $\bar{K}^*$ and P the same nucleus?"

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† Replace n by $n+1$ in Theorem 13, take the complex $K_0-\sum_{\lambda=1}^{r}B_\lambda^{n+1}$ of Theorem 13 for K in (10.1), the simplexes B_λ^{n+1} and A_λ^{n+1} for the membranes $\mathcal{E}_i^{n+1}$ and $\bar{\mathcal{E}}_i^{n+1}$ and the complexes K_0 and L_0 of Theorem 13 for K^* and $\bar{K}^*$ in (10.1).

‡ I am indebted to Prof. L. J. Mordell for this example.

11. *A special class of groups.* In this section we show that two complexes have the same nucleus if they are of the same homotopy type, provided that their fundamental groups satisfy a certain condition. This condition will be stated in terms of the integral ring $\mathfrak{R}(G)$ of a group G, whose elements are linear forms

$$r = \Sigma_i a_i g_i,$$

where $g_i \epsilon G$ and $a_1, a_2, \ldots$ are rational integers, almost all of which are zero. If $G = \pi_1(K, p)$ and if $\psi_n(r)$ stands for the transformation $\alpha \rightarrow r\alpha$ in $\mathfrak{R}_n(K, p)$ $(n > 1)$, the transformation $r \rightarrow \psi_n(r)$ is obviously a homomorphism of $\mathfrak{R}(\pi_1)$ on $\mathfrak{R}_n = \mathfrak{R}_n(K, p)$. If $\alpha_i \epsilon \pi_n(K, p)$, an element

$$r_1 \alpha_1 + \ldots + r_k \alpha_k$$

in the group $r(\alpha_1, \ldots, \alpha_k)$ may be regarded as a linear form in† $\alpha_1, \ldots, \alpha_k$ with coefficients in $\mathfrak{R}(\pi_1)$, and the elements $\alpha_1, \ldots, \alpha_k$ will be described as *linearly independent* if

$$r_1 \alpha_1 + \ldots + r_k \alpha_k = 0$$

implies $r_1 = \ldots = r_k = 0$. Notice that $\alpha_1, \ldots, \alpha_k$ are linearly independent with coefficients in $\mathfrak{R}(\pi_1)$ if, and only if, the elements $g\alpha_i$ $(g \epsilon \pi_1)$ are linearly independent with integral coefficients; that is to say if $r(\alpha_1, \ldots, \alpha_k)$ is freely generated, with commutative addition, by the elements $g\alpha_i$. If there is a linearly independent set of elements in π_n the homomorphism $r \rightarrow \psi_n(r)$ is an isomorphism. For, if $\alpha_1, \ldots, \alpha_k$ is a linearly independent set, $r_1 \alpha_1 \neq 0$ unless $r_1 = 0$.

Let $$K^* = K + \sum_{i=1}^{k} \mathcal{E}_i^n \quad (n > 1),$$

where $\mathcal{E}_1^n, \ldots, \mathcal{E}_k^n$ are simple membranes such that $F(\mathcal{E}_i^n) = F(\Gamma_i^n)$, where Γ_i^n is an n-cell in K. Join $\Gamma_1^n, \ldots, \Gamma_k^n$ to a base point p in K and let α_i be the element in $\pi_n^* = \pi_n(K^*, p)$, corresponding to the spherical map $\mathcal{E}_i^n + \Gamma_i^n$, oriented either way. As a complement to Theorem 18 we have

THEOREM‡ 19. $\pi_n^* = \pi_n + r(\alpha_1, \ldots, \alpha_k)$,

and the elements $\alpha_1, \ldots, \alpha_k$ *are linearly independent.*

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† *Cf.* Reidemeister, *Abhand. Math. Sem. Hamburg* (*loc. cit.*).

‡ If P is a retract of P^* (not necessarily by deformation) $\pi_n = \pi_n(P, p)$ is isomorphic to a sub-group of $\pi_n^* = \pi_n(P^*, p)$, corresponding elements being represented by the same map in P. For simplicity of statement we shall identify each element in π_n with the corresponding element in π_n^*. Here K is a retract of K^*. For, since $F(\mathcal{E}_i^n) = F(\Gamma_i^n)$, the polyhedron $\mathcal{E}_i^n$ may be mapped on Γ_i^n so that each point in $F(\Gamma_i^n)$ corresponds to itself. Actually, as will appear in the proof, no generality is lost in taking Γ_i^n to be a single point.

We have to show, first that any element a^* in π_n^* can be expressed as a sum of the form $r_1 a_1 + \ldots + r_k a_k + a$, where $a \epsilon \pi_n$, secondly that

$$r_1 a_1 + \ldots + r_k a_k + a = 0$$

implies $r_1 a_1 + \ldots + r_k a_k = a = 0$, and thirdly that $r_1 a_1 + \ldots + r_k a_k = 0$ implies $r_1 = \ldots = r_k = 0$.

A formal deformation of K^*, rel. K, obviously does not alter

$$\pi_n, \ \pi_n^*, \ r(a_1, \ldots, a_k)$$

or the relations between them. Therefore we may suppose $\mathcal{E}_i^n$ to be an n-sphere which meets K in the single vertex p. Then K is obviously a retract of K^*, and the projection $\psi(K^*) = K$, such that $\psi(p^*) = p^*$ if $p^* \epsilon K$ and $\psi(p^*) = p$ if $p^* \epsilon \mathcal{E}_i^n$, determines a homomorphism ϕ, of π_n^* on π_n, such that $\phi(a^*) = a^*$ if $a^* \epsilon \pi_n$ and $\phi(a^*) = 0$ if $a^* \epsilon r(a_1, \ldots, a_k)$. Therefore $\phi(r_1 a_1 + \ldots + r_k a_k + a) = a$, and $r_1 a_1 + \ldots + r_k a_k + a = 0$ implies $a = 0$, and hence $r_1 a_1 + \ldots + r_k a_k = 0$.

To prove that any element a^* in π_n^* is a sum of the form

$$r_1 a_1 + \ldots + r_k a_k + a,$$

let $\mathcal{E}_i^n = \dot{E}_i^{n+1}$, where E_i^{n+1} does not meet K or E_j^{n+1} except in p $(j \neq i)$, and let

$$K^{**} = K^* + \sum_{i=1}^{k} E_i^{n+1}.$$

Then the projection $\psi(K^*) = K$ may be realized in K^{**} by shrinking each of $E_1^{n+1}, \ldots, E_k^{n+1}$ into the point p. It follows first that $\pi_n^{**} = \pi_n$, where $\pi_n^{**} = \pi_n(K^{**}, p)$, and secondly that the homomorphism $\phi(\pi_n^*) = \pi_n^{**}$ of Theorem 18, with the complexes K and K^* of Theorem 18 replaced by K^* and K^{**}, is the same as the homomorphism $\phi(\pi_n^*) = \pi_n$ of the last paragraph. Since $\phi^2 = \phi$ we have

$$\phi\{a^* - \phi(a^*)\} = \phi(a^*) - \phi(a^*) = 0,$$

whence $a^* - \phi(a^*) \epsilon r(a_1, \ldots, a_k)$, by Theorem 18. Since $\phi(a^*) \epsilon \pi_n$, it follows that $a^* = r_1 a_1 + \ldots + r_k a_k + a$, where a^* is any element in π_n^* and $a \epsilon \pi_n$. Therefore $\pi_n^* = \pi_n + r(a_1, \ldots, a_k)$.

If $\pi_1(K) = 1$, the independence of $a_1, \ldots, a_k$ follows from one of Hurewicz's theorems†, or directly from the corresponding theorem concerning homology groups. For a spherical map representing the element $m_1 a_1 + \ldots + m_k a_k$, where $m_1, \ldots, m_k$ are integers, is homologous to

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† *Loc. cit.* (3rd paper), p. 120, Theorem 1.

$$m_1 \mathcal{E}_1^n + \ldots + m_k \mathcal{E}_k^n ;$$

and, if $m_1 a_1 + \ldots + m_k a_k = 0$, it follows that $m_1 = \ldots = m_k = 0$. For this argument it is unnecessary that K and k should be finite, or that $\mathcal{E}_1^n$, $\mathcal{E}_2^n$, ..., should all meet K in the same point p. Therefore, when $\pi_1(K)$ is arbitrary, the linear independence of $a_1, \ldots, a_k$ with coefficients in $\mathfrak{R}(\pi_1)$ follows by a standard type of argument from the isomorphism between π_n^* and $\pi_n(\tilde{K}^*)$, where $\tilde{K}^*$ is the general covering complex of K^*. Thus the proof is complete.

We now state a certain condition on a group G. A square matrix $\|r_{ij}\|$ $(i, j = 1, \ldots, k)$, whose elements belong to $\mathfrak{R}(G)$, will be said to have a *left inverse*, namely $\|r_{ij}^*\|$, if

$$\sum_{t=1}^{k} r_{it}^* r_{tj} = \delta_{ij}.$$

Our condition on G is that any square matrix with a left inverse can be transformed into the empty matrix, having no rows and columns, by a finite sequence of operations which consist either of:

(1) multiplying each element in a row or column by $\pm g$, where $g \in G$;

(2) interchanging two rows or columns;

(3) adding a "left multiple" of one row to another, the multiplier being any element in $\mathfrak{R}(G)$ (*i.e.* $\rho_i \to \rho_i + \lambda \rho_j$, where $j \neq i$, $\lambda \in \mathfrak{R}(G)$ and ρ stands for the t-th row);

(4) adding a row $(r_{00}, \ldots, r_{0k})$ and a column $(r_{00}, \ldots, r_{k0})$ such that $r_{00} = 1$, $r_{10} = \ldots = r_{k0} = 0$;

or of

(5) removing such a row and column[†].

The group consisting of the unit element alone satisfies this condition. For then $\mathfrak{R}(G)$ is isomorphic to the ring of rational integers and a matrix

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[†] Notice that these operations allow us to add a right multiple of one column to another. For we can add an extra row and column such that

$$r_{00} = 1, \quad r_{0j} = -1, \quad r_{0i} = \lambda \quad \text{and} \quad r_{0t} = 0 \quad [t \neq 0,\ i, \text{ or } j;\ \lambda \in \mathfrak{R}(G)],$$

and then add $r_{sj} \rho_0$ to ρ_s for each $s = 1, \ldots, k$. We can then remove the row ρ_0 and the column containing r_{0j} and move the column $r'_{s0} = r_{sj}$ back to the j-th place. The final result is to replace r_{si} by $r_{si} + r_{sj} \lambda$ $(s = 1, \ldots, k)$, leaving the other columns unaltered.

with an inverse is unimodular. It is proved elsewhere[†] that the condition is satisfied by at least two other groups, namely the cyclic groups of order two and three.

THEOREM 20. *If* $\pi_1(K)$ *satisfies the above condition and if* L *is a retract by deformation of* K, *then*

$$L = D(K) \quad (\text{rel. } L).$$

Let K be n-dimensional and, after an expansion if necessary, let $n > 2$. With the notation used in proving Theorem 16,

$$L_n = L + \sum_{i=1}^{k} \mathscr{E}_i^{n-1} + \sum_{\lambda=1}^{l} \mathscr{E}_\lambda{}^n = D_n(K) \quad (\text{rel. } L),$$

and L is a retract by deformation of L_n. Since L is a retract by deformation of L_n, the connectivities of L_n, calculated mod L in the sense of Lefschetz[‡], are all zero. If $l \neq k$, some cycle (mod L), composed either of the cycles (mod L) $\mathscr{E}_i^{n-1}$ or of the chains $\mathscr{E}_\lambda{}^n$, would fail to bound mod L. Therefore $l = k$. Since L is a retract of L_n, the map $F(\mathscr{E}_i^{n-1})$ bounds a cell Γ_i^{n-1} in L. Let $f_i(S^{n-1})$ be the spherical map $\mathscr{E}_i^{n-1} + \Gamma_i^{n-1}$, and let s_i be a segment in L joining some point p to a point $f_i(q)$ in $f_i(S^{n-1})$, where $q \in S^{n-1}$. Let

$$L^* = L + \sum_{i=1}^{k} \mathscr{E}_i^{n-1},$$

and let a_i be the element in $\pi_{n-1}^* = \pi_{n-1}(L^*, p)$ corresponding to the map $s_i + f_i(S^{n-1})$, oriented either way. Join p to a point in $F(\mathscr{E}_i{}^n)$ by a segment t_i in L and let $a_i{}^*$ be the element in π_{n-1}^* corresponding to the map $t_i + F(\mathscr{E}_i{}^n)$. By Theorem 19, $\pi_{n-1}^* = \pi_{n-1} + r(a_1, \ldots, a_k)$, whence

$$a_i{}^* = \sum_{j=1}^{k} r_{ij}\, a_j + \beta_i{}^* \quad [\beta_i{}^* \in \pi_{n-1} = \pi_{n-1}(L, p)],$$

which we write as

$$a_i{}^* = \sum_{j=1}^{k} r_{ij}\, a_j \quad (\text{mod } \pi_{n-1}). \tag{11.1}$$

[†] See a forthcoming paper by G. Higman.

If G is Abelian, the determinant $|r_{ij}|$ can be calculated in the ordinary way, and the elementary transformations can only alter $|r_{ij}|$ by a factor $\pm g$ $(g \in G)$. Therefore the above condition is not satisfied if $\mathfrak{R}(G)$ has a unit ϵ, other than $\pm g$, as one sees by taking $k = 1$ and $r_{11} = \epsilon$ (*e.g.* $r_{11} = 1 - g - g^4$, where $g^5 = 1$).

[‡] *Topology* (New York, 1930), p. 17.

Since L is a retract by deformation of L_n each map $f_i(S^{n-1})$ is homotopic in L_n, rel. q, to a map $\bar{f}_i(S^{n-1})$ in L. If β_i is the element in π_{n-1} corresponding to the map $s_i + \bar{f}_i(S^{n-1})$, it follows from Theorem 18 that

$$\alpha_i - \beta_i = \sum_{=1}^{k} r^*_{ij} \alpha_j^*,$$

i.e. that

$$\alpha_i = \sum_{j=1}^{k} r^*_{ij} \alpha_j^* \pmod{\pi_{n-1}}. \tag{11.2}$$

From (11.1) and (11.2) we have

$$\Sigma_j (\Sigma_t r^*_{it} r_{tj} - \delta_{ij}) \alpha_j = 0 \pmod{\pi_{n-1}}. \tag{11.3}$$

Since $\pi^*_{n-1} = \pi_{n-1} + r(\alpha_1, \ldots, \alpha_k)$, the left-hand side of (11.3) is zero absolutely, and since $\alpha_1, \ldots, \alpha_k$ are linearly independent, we have

$$\sum_{t=1}^{k} r^*_{it} r_{tj} = \delta_{ij}.$$

Therefore the matrix $\|r_{ij}\|$ has a left inverse and by our condition on π_1 it can be reduced to nothing by the operations described above.

If the matrix $\|r_{ij}\|$ has no rows or columns, the sets of membranes $\mathcal{E}^{n-1}_1, \ldots, \mathcal{E}^{n-1}_k$ and $\mathcal{E}_1{}^n, \ldots, \mathcal{E}_k{}^n$ being empty, we have $L_n = L$. Therefore it is enough to show that each of the five kinds of elementary transformation $\|r_{ij}\| \to \|r'_{ij}\|$ can be copied by a formal deformation of L_n, rel. L, which transforms it into a complex $L_n{}'$ with the matrix $\|r'_{ij}\|$. We take them in order.

(1) The i-th row (column) is multiplied by -1 if we change the orientation of $\mathcal{E}_i{}^n$ ($\mathcal{E}^{n-1}_i$) and by g if we replace t_i (s_i) by $c + t_i$ ($c + s_i$), where c is a circuit corresponding to g.

(2) To interchange two rows (columns) we merely re-order $\mathcal{E}_i{}^n(\mathcal{E}^{n-1}_i)$.

(3) For the transformation $\rho_i \to \rho_i + \lambda \rho_j$ $(j \neq i)$, let $K_i = L_n - I(\mathcal{E}_i{}^n)$, where $I(\mathcal{E}_i{}^n)$ is the interior of $\mathcal{E}_i{}^n$, and let $\bar{\mathcal{E}}_i{}^n$ be a simple membrane whose boundary is a map representing the element $\alpha_i{}^* + \lambda \alpha_j{}^*$. Then $F(\mathcal{E}_i{}^n)$ is homotopic to $F(\bar{\mathcal{E}}_i{}^n)$ in K_i, since $\mathcal{E}_j{}^n \subset K_i$, and it follows from Theorem 11 that

$$K_i + \bar{\mathcal{E}}_i{}^n = D(L_n) \quad (\text{rel. } L).$$

Then $L_n{}' = K_i + \bar{\mathcal{E}}_i{}^n$ is a complex with the required matrix $\|r'_{ij}\|$.

(4, 5) Since (4) is the inverse of (5), it is sufficient to consider a transformation of type 5. If $r_{i1} = \delta_{i1}$ $(i = 1, \ldots, k)$, it follows from Theorem 19 that the map $F(\mathcal{E}_i^n)$ is homotopic to one which covers an open simplex B^{n-1}, in $\mathcal{E}_1^{n-1}$, just once if $i = 1$ and not at all if $i = 2, \ldots, k$. Therefore we may assume that the maps $F(\mathcal{E}_i^n)$ have this property, in which case L_n contracts into $L_n - I(\mathcal{E}_1^n) - B^{n-1}$, by Lemma 6, and then into

$$L_n' = L_n - I(\mathcal{E}_1^n) - I(\mathcal{E}_1^{n-1}),$$

as in the proof of Lemma 5. Then L_n' has the required matrix

$$\|r_{\lambda\mu}\| \quad (\lambda, \mu = 2, \ldots, k).$$

Thus each type of algebraic transformation can be copied geometrically and the theorem is established.

From the addenda to Theorems 16 and 11 we have the addendum:

ADDENDUM. *The order of the deformation D in Theorem 20 need not exceed $n+1$ $(n > 2)$.*

From Theorem 20, with its addendum, and the proof of Theorem 17, we have

THEOREM 21. *If two complexes K and K^* are of the same homotopy type and if $\pi_1(K)$ satisfies the above condition, then*

$$K^* = D(K).$$

If K and K^* are at most n-dimensional†, *the order of the deformation D need not exceed $n+2$.*

12. *Regular neighbourhoods in manifolds.* Let M be an n-dimensional manifold in the sense of Alexander and Newman. That is to say, the complement of any vertex in M is an $(n-1)$-sphere or an $(n-1)$-element according as the vertex in question is inside M or in $\dot{M}$. We recall that this implies the more general condition: the complement of any k-simplex in M is an $(n-k-1)$-sphere or an $(n-k-1)$-element according as the simplex in question is inside M or in $\dot{M}$. We also recall the relation‡ $(M_A)^\cdot = \dot{M}_A$, both sides being zero unless $A \in \dot{M}$.

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† Here we need not require $n > 2$. For the relevant dimensionality in Theorem 17 is $\dim \{C_f(K)\} = \dim(K)+1$, and, if $n = 0$ or 1, Theorem 21 may be verified directly.

‡ For let $B \in M_A$. Then $(M_A)_B = M_{AB}$ and $B \in (M_A)^\cdot$ if, and only if, $(M_A)_B$ is an element, and $B \in \dot{M}_A$ if, and only if, $AB \in \dot{M}$, *i.e.* if M_{AB} is an element.

If M is bounded, a transformation of the form

$$M \to M + E^n,$$

where E^n is an n-element which meets M in an $(n-1)$-element on the boundary of both, or the resultant of a finite sequence of such transformations, will be called a *regular expansion* of M. The inverse of a regular expansion will be called a *regular contraction*. As a matter of convention it is to be understood that M expands and contracts regularly into itself. If $M \to M^*$ is a regular expansion or contraction, M^* is combinatorially equivalent† to M.

Let Σ be any set of simplexes, not necessarily closed, in a complex K. By $O(\Sigma, K)$ and $N(\Sigma, K)$ $[= Cl\, O(\Sigma, K)]$ we shall mean respectively the set of all open simplexes in K whose closures contain one or more simplexes in Σ, other than 1 in case $1 \in \Sigma$, and the set of all closed simplexes which contain one or more simplexes in Σ, other than 1. Clearly $O(\Sigma, K)$ is open in the sense that $K - O(\Sigma, K)$ is a complex. Notice also that $A \in K_B$ if $O(A, K) . K_B \neq 0$, A being any open simplex in K. For K_B is a complex, and, if A is not in K_B, it is not on the boundary of any simplex in K_B. The distinction between closed and open simplexes is important when the symbols $O(A, K)$ and $N(A, K)$ are used. For, if A is an open simplex, $O(A, K)$ is its open star, while, if A is closed, $O(A, K)$ is the sum of the open stars of all the vertices of A.

Let M be a bounded n-dimensional manifold and let Σ be a set of open simplexes in $\dot{M}$ such that:

(1) if $A \in \Sigma$, all the internal simplexes of M_A are inside M (expressed formally: if $A \in \Sigma$ and $B \in \dot{M} . M_A$, then $AB \in \dot{M}$),

(2) if B is inside M and $A_1 + A_2 \subset \Sigma . M_B$, then $A_1 A_2 \in M_B$.

LEMMA 10. *Under these conditions, the transformation*

$$M \to M - O(\Sigma, M)$$

is a regular contraction.

If $\Sigma = 0$ or 1, there is nothing to prove. Otherwise let $A \neq 1$ be any open simplex in Σ. Then it follows easily enough from the first condition that the boundary of $N(A, M)$ does not meet $\dot{M}$ except in the $(n-1)$-element $N(A, \dot{M})$, and hence that the transformation

$$M \to M^* = M - O(A, M)$$

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† Newman, *Proc. Akad. Amsterdam*, 29 (1926), 635, Theorem 8a, or Alexander (*loc. cit.*), 317, Theorem 14.3.

is a regular contraction. Clearly $M - O(\Sigma, M) = M^* - O(\Sigma^*, M^*)$, where $\Sigma^* = \Sigma - A$, and, if we can show that the above conditions are also satisfied by M^* and Σ^*, the lemma will follow by induction on the number of simplexes in Σ.

Consider the second condition first. If B is inside M^*, the complement $M_B{}^*$ is a sphere and so cannot be a proper sub-set of M_B. Therefore

$$M_B = M_B{}^*,$$

and, if $A_1 + A_2 \subset \Sigma^* . M_B{}^* \subset \Sigma . M_B$, it follows from the second condition on M and Σ that $A_1 A_2 \epsilon M_B$. Since $M_B = M_B{}^*$, the second condition is satisfied by M^* and Σ^*. To verify the first condition let $B \epsilon \dot{M}^* . M^*_{A^*}$, where $A^* \epsilon \Sigma^*$, and first let $B \epsilon \dot{M}$. Then $B \epsilon \dot{M} . M_{A^*}$, since $M^* \subset M$, and it follows from the first condition on M and Σ that $A^* B \epsilon \dot{M}$, whence $A^* B \epsilon \dot{M}^*$. If $B \epsilon \dot{M}^* - \dot{M}$, then M_B is a sphere and $M_B{}^*$ is an element. Therefore $M_B{}^*$ is a proper sub-set of M_B, and it follows that

$$O(A, M) . M_B \neq 0,$$

i.e. that $A \epsilon M_B$. But $A^* \epsilon M_B$, since $B \epsilon M^*_{A^*} \subset M_{A^*}$, and it follows that $A + A^* \subset \Sigma . M_B$. Since $B \epsilon M - \dot{M}$, it follows from the second condition on M and Σ that $AA^* \epsilon M_B$, or that $AA^* B \epsilon M$. Therefore $A \epsilon M_{A^*B}$ and $M^*_{A^*B}$ is a proper sub-set of M_{A^*B}. It is therefore an element, rather than a sphere, and $A^* B \epsilon \dot{M}^*$. Therefore both conditions are invariant under the transformation $M \rightarrow M^*$ and the lemma is established.

Let M_1 be an n-dimensional manifold such that $M . M_1$ consists of one or more unbounded $(n-1)$-dimensional manifolds in $\dot{M} + \dot{M}_1$ (in particular $M . M_1 = \dot{M} = \dot{M}_1$ if $\dot{M}$ and $\dot{M}_1$ are connected). Then the following corollary follows immediately from the proof of the lemma:

COROLLARY 1. *If $\Sigma \epsilon \dot{M} . \dot{M}_1$ and M and Σ satisfy the above conditions, the transformation $M_1 \rightarrow M_1 + N(\Sigma, M)$ is a regular expansion.*

It follows from Lemma 4 that $s_{\dot{M}} M$ and $\dot{M}$ satisfy the conditions of Lemma 10 provided that no simplex inside M has all its vertices in $\dot{M}$. Therefore† $s_{\dot{M}'} M'$ and $\dot{M}'$ satisfy them. If K is any complex and $L \subset K$, the sub-division $K \rightarrow K'$ is the resultant of $s_L K$ followed by a regular sub-division of L. Also it is obvious that if $M_1 \rightarrow M_2$ is a regular contraction (expansion), so is $\sigma M_1 \rightarrow \sigma M_2$, where σ is any sub-division of M_1 (of M_2).

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† Throughout this section K' and K'' will stand for the first and second derived complexes of a given complex K. In this section it is to be understood that the vertices of K are unaltered by a sub-division of the form $s_L K$.

Finally, if M and Σ satisfy the conditions of Lemma 10, so do M and any sub-set of Σ. Therefore we have a second corollary to Lemma 10:

COROLLARY 2. *The transformation* $M'' \rightarrow M'' - O(\Sigma'', M'')$ *is a regular contraction, where* Σ *is any set of simplexes in* $\dot{M}$. *If* M *and* M_1 *satisfy the conditions of Corollary* 1 *and* $\Sigma \in \dot{M} . \dot{M}_1$, *the transformation*

$$M_1'' \rightarrow M_1'' + N(\Sigma'', M'')$$

is a regular expansion.

We now come to the main purpose of this section. Let K be a sub-complex of an n-dimensional manifold M. By a *regular neighbourhood* of K in M we shall mean a sub-complex $U(K, M)$, of M, such that:

(1) $U(K, M)$ *is an* n*-dimensional manifold,*

(2) $U(K, M)$ *contracts geometrically into* K.

It follows from a well-known theorem on sub-division that the first of these conditions, and from the corollary to Theorem 7 that the second, are invariant under a general sub-division of M. The two main theorems are:

THEOREM 22_n. *If* $K \subset M$, *where* M *is an* n*-dimensional manifold,* $N(K, s_K^2 M)$ *is a regular neighbourhood of* K $(K \neq 1)$.

THEOREM 23_n. *Any two regular neighbourhoods of* K *in* M *are combinatorially equivalent,* M *being any* n*-dimensional manifold.*

These theorems are trivial if $n = 0$ and we prove them together by induction on n, assuming Theorems 22_{n-1} and 23_{n-1}. First notice the corollary to Theorem 23_n.

COROLLARY 1_n. *If* K *is geometrically collapsible* $U(K, M)$ *is an* n*-element.*

For if K contracts geometrically into L then $U(K, M)$ is also a regular neighbourhood of L. Taking L to be a single vertex, $N(L, M)$ is an n-element and, being a star, contracts into L. It is, therefore, a regular neighbourhood of L. By Theorem 23_n, $U(K, M)$ is combinatorially equivalent to $N(L, M)$ and is therefore an n-element.

Let $K \subset M$ be any complex other than 1 such that:

(12.1) (*a*) *no simplex in* $M - K$ *has all its vertices in* K,

(*b*) *if* $A \in M - K$, *then* $K . M_A$ *is a single closed simplex* (*possibly* 1).

If B is any open simplex in M, the complexes $K . M_B$ and M_B also satisfy these conditions. For (12.1*a*) is obviously satisfied, and if $A \in M_B - K$, then $AB \epsilon M-K$ and $K.M_{AB}$ is a single closed simplex. Therefore (12.1*b*) is satisfied, since $M_{AB} = (M_B)_A$ and $(K.M_B).(M_B)_A = K.(M_B)_A$. I say that, under the conditions (12.1), $N = N(K, M)$ is a regular neighbourhood of K. This will be proved by induction on n, being trivial if $n = 0$. By Theorem 2, N contracts into K, and it remains to prove that N_b is an $(n-1)$-sphere or an $(n-1)$-element, where b is any vertex in N. This is certainly the case if $b \subset K$. For then $N_b = M_b$. If $b \not\subset K$ a closed simplex Ab meets K if, and only if, A meets K. Therefore $N_b = N(B, M_b)$, where B is the closed simplex $K.M_b$. By the hypothesis of the induction, $N(B, M_b)$ is a regular neighbourhood of B, and is therefore an $(n-1)$-element by Theorem 23_{n-1}, Corollary 1_{n-1}. Therefore N is a manifold, and hence a regular neighbourhood of K.

By the corollary to Lemma 4 the conditions (12.1) are satisfied by K and $s_K^2 M$, where K is any sub-complex of M, and Theorem 22_n follows from Theorem 23_{n-1}.

We shall need two observations for the proof of Theorem 23_n. First, if M is an unbounded n-dimensional manifold and K is any sub-complex of M, then $N(K', M')$ is the aggregate of closed cells in the dual cell-structure (*Zellteilung*) which are dual to the simplexes in K. Therefore $N(K', M')$ consists of the closed n-cells which are dual to the vertices $a_1, \ldots, a_m$ of K. Therefore $N(K', M') = N(a_1 + \ldots + a_m, M')$. If we regard $N(K', M')$ as the cell-complex consisting of these dual cells, its $(n-1)$-cells are the duals of the edges in M which have an extremity in K. Such an $(n-1)$-cell is inside $N(K', M')$ if both extremities of the dual edge are in K and on the boundary (mod 2) if only one extremity is in K. If L is any sub-complex of M which does not meet K, it follows that $N(K', M')$ meets $N(L', M')$, if at all, in the aggregate of closed $(n-1)$-cells which are dual to the edges of M with one end in K and the other in L.

Secondly, no simplex in $M'-K'$ has all its vertices in K'. Therefore K' and $s_{K'} M'$ satisfy the conditions (12.1), by Lemma 4, and $N(K', s_{K'} M')$ is a regular neighbourhood of K'. As we have already remarked, the sub-division $M \rightarrow M''$ is the resultant of $s_{K'} M'$, followed by a regular sub-division of K'. Therefore $N(K'', M'')$ is a regular neighbourhood of K'' in M''.

Theorem 23_n will now follow without difficulty from the following lemma. Let M be an n-dimensional, unbounded manifold and K_p a sub-complex of M which contracts formally into K_0.

LEMMA 11. *The manifold* $N(K_0'', M'')$ *expands regularly into*

$$N(K_p'', M''),$$

Let $K_0, \ldots, K_p$ be a sequence of complexes such that the transformation $K_i \rightarrow K_{i+1}$ is an elementary expansion ($i = 0, \ldots, p-1$). Let

$$K_1 = K_0 + A,$$

where $A = aB$, $a\dot{B} \subset K$, $B \not\subset K$, and let x and y be the vertices of M' which are internal to A' and B' respectively. Then the vertices of $K_1{}'$ are the vertices of $K_0{}'$ together with x and y. Therefore, according to the first observation,

$$\begin{aligned} N(K_1{}'', M'') &= N(K_0{}'' + x + y, M'') \\ &= N(K_0{}'', M'') + N(x, M'') + N(y, M''). \end{aligned}$$

We shall prove that $N(K_0{}'', M'')$ expands regularly into

$$N(K_0{}'', M'') + N(x, M'')$$

and that the latter expands regularly into $N(K_1{}'', M'')$. The lemma will then follow by induction on p.

Since M' and $K_0{}'$ satisfy (12.1a), any edge xb, in M', lies in $K_1{}'$, if $b \in K_1{}'$. Since A is a principal simplex of K_1 it follows, if $b \in K_1{}'$, that $b \in \dot{A}'$ and $xb \in A'$. If $b \in K_0{}'$, it also follows that $b \in (a\dot{B})'$. Therefore $N(K_0{}'', M'') \,.\, N(x, M'') = E^{n-1}$, say, is the aggregate of closed cells in the dual of M' which are dual to $xb_1, \ldots, xb_k$, where $b_1, \ldots, b_k$ are the vertices in $(a\dot{B})'$. By a familiar property of regular sub-division, E^{n-1} is isomorphic to

$$N[\{(a\dot{B})'\}', (M_x{}')']. \tag{12.2}$$

The complexes M' and $(a\dot{B})'$, and *a fortiori* $M_x{}'$ and $(a\dot{B})'$, satisfy (12.1a). It follows from Lemma 4 and the proof of Theorem 22_{n-1} that

$$N\{(a\dot{B})', \; s_{(a\dot{B})'} M_x{}'\}$$

is a regular neighbourhood of $(a\dot{B})'$. Therefore (12.2) is a regular neighbourhood of $(a\dot{B})''$ and, by the corollary to Theorem 23_{n-1}, (12.2) and E^{n-1} are $(n-1)$-elements. By our first observation E^{n-1} is in the boundary of $N(K_0{}'', M'')$ and of $N(x, M'')$, and, since the latter is an n-element, $N(K_0{}'', M'')$ expands regularly into

$$N(K_0{}'', M'') + N(x, M'') = N(K_0{}'' + x, M'').$$

By our first observation

$$N(K_0{}'' + x, M'') = N[\{K_0{}' + x(a\dot{B})'\}', M''],$$

and it follows from the same argument as before, with a replaced by x and x by y, that $N(K_0''+x, M'')$ expands regularly into

$$N(K_0''+x, M'')+N(y, M'')=N(K_1'', M''),$$

and the lemma is established.

Now let $U_1=U_1(K, M)$ and $U_2=U_2(K, M)$ be two regular neighbourhoods of K in M. Since the conditions for a regular neighbourhood are intrinsic to $U(K, M)$ we may replace M, in case it is bounded, by an unbounded manifold $M+M^*$, where $M^*.M=\dot{M}=\dot{M}^*$. So, without loss of generality, we assume that $\dot{M}=0$. Since U_1 contracts geometrically into K there is, by Theorem 7, a stellar sub-division $\sigma_1 M$ such that $\sigma_1 U_1$ contracts formally into $\sigma_1 K$. By the corollary to Theorem 7, $\sigma_1 U_2$ contracts geometrically into $\sigma_1 K$, and by Theorem 7 itself there is a further stellar sub-division $\sigma_2\sigma_1 M$ such that $\sigma_2\sigma_1 U_2$ contracts formally into $\sigma_2\sigma_1 K$. By Theorem 4, $\sigma_2\sigma_1 U_1$ contracts formally into $\sigma_2\sigma_1 K$ and it follows that σU_i $(i=1, 2)$ contracts formally into σK, where $\sigma=\sigma_2\sigma_1$. So we may assume, after an initial sub-division, that each of the neighbourhoods U_1 and U_2 contracts formally into K, the manifold M being unbounded.

By the second corollary to Lemma 10, U_i'' expands regularly into $N(U_i'', M'')$. Since K expands formally into U_i, it follows from Lemma 11 that $N(K'', M'')$ expands regularly into $N(U_i'', M'')$. Therefore, with the sign of congruence denoting combinatorial equivalence,

$$U_i\equiv U_i''\equiv N(U_i'', M'')\equiv N(K'', M'').$$

Therefore $U_1\equiv U_2$ and the theorem is established.

Let K be a given sub-complex of a manifold M, let

$$U_i=U(\gamma_i K, \gamma_i M) \quad (i=1, 2)$$

be a regular neighbourhood of $\gamma_i K$ in a general sub-division $\gamma_i M$, and let $\gamma_1^*\gamma_1=\gamma_2^*\gamma_2=\gamma$ be a common sub-division of γ_1 and γ_2. Since the property of being a regular neighbourhood is invariant under sub-division, $\gamma_i^* U_i$ $(i=1, 2)$ is a regular neighbourhood of γK in γM. Therefore, $\gamma_1^* U_1\equiv\gamma_2^* U_2$ and hence $U_1\equiv U_2$. Therefore the simplicial space associated with a regular neighbourhood of $\gamma_0 K$ in $\gamma_0 M$, where γ_0 is a suitable sub-division of M (*e.g.* $\gamma_0 M=M''$), is uniquely determined by the given complexes M and K. We shall denote it by $\Sigma(K, M)$.

Let P be a (finite) polyhedron imbedded in an n-dimensional, polyhedral manifold† M. A regular neighbourhood of P in M may be defined geometrically as an n-dimensional, polyhedral manifold $U(P, M)$, contained in M and containing P, which contracts geometrically into the latter. This is obviously equivalent to the combinatorial definition stated in terms of a rectilinear triangulation of M with sub-complexes covering P and $U(P, M)$. Therefore Theorems 22 and 23 may be restated geometrically in the form of the corollary:

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† *I.e.* any rectilinear triangulation of M satisfies the combinatorial condition for a manifold.

COROLLARY 3. *Any polyhedron P, in a polyhedral manifold M, has a regular polyhedral neighbourhood in M, and any two such neighbourhoods are semi-linearly homeomorphic.*

Let K be a symbolic complex, let Σ be the simplicial space determined by K and let P be a polyhedron covered by a rectilinear, simplicial complex which is isomorphic to K. In addition to the symbolic complexes which are combinatorially equivalent to K, we shall admit as a representative of Σ any polyhedron which is semi-linearly homeomorphic to P. If P is imbedded in a polyhedral manifold M we shall use $\Sigma(P, M)$ to stand for the simplicial space determined by a regular neighbourhood of P in M. So long as P and the regular neighbourhoods are finite, M may obviously be infinite. In particular M may be Euclidean space.

THEOREM 24. *If P^n and Q^n are semi-linearly homeomorphic, n-dimensional polyhedra in Euclidean p-space R^p, then $\Sigma(P^n, R^p) = \Sigma(Q^n, R^p)$ provided $p \geqslant 2n+3$.*

Let $f(P^n) = Q^n$ be a semi-linear, topological map of P^n on Q^n and let Γ^{n+1} be the locus swept out by the linear segment $pf(p)$ as p describes P^n. Then Γ^{n+1} is an image, which may be singular, of the mapping cylinder $C_f(P^n)$, where we first assume that P^n does not meet Q^n. Then, since $p \geqslant 2n+3$, the singularities in Γ^{n+1} may be removed by simplicial subdivision and slight displacement of the vertices which do not lie on P^n or Q^n. The resulting complex C^{n+1} is a semi-linear topological image of $C_f(P^n)$ and, since f is (1-1), C^{n+1} contracts both into P^n and into Q^n. Therefore a regular neighbourhood of C^{n+1} in R^p is a regular neighbourhood both of P^n and of Q^n. Therefore $\Sigma(P^n, R^p) = \Sigma(Q^n, R^p)$ in case P^n and Q^n do not meet each other. If they do meet, we have

$$\Sigma(P^n, R^p) = \Sigma(P_1^n, R^p) = \Sigma(Q^n, R^p),$$

where P_1^n is any semi-linear topological image of P^n in R^p which does not meet P^n or Q^n, and the theorem is established.

Any n-dimensional (simplicial) complex K^n may be imbedded in R^p if $p \geqslant 2n+1$, and if $p \geqslant 2n+3$ it follows from Theorem 24 that $\Sigma(K^n, R^p)$ is independent of the way in which K^n is imbedded in R^p. It is also the same for combinatorially equivalent complexes. Therefore we have the corollary:

COROLLARY. *If* $p \geqslant 2n+3$, *the simplicial space* $\Sigma(K^n, R^p)$ *is a combinatorial invariant of* K^n.

Notice that $\Sigma(S^n, R^p)$ is the topological product $S^n \times E^{p-n}$, where S^n is an n-sphere, E^{p-n} a $(p-n)$-element and $p \geqslant 2n+3$. For it is obvious that $\Sigma(S^n, R^p) = \Sigma(S^n, S^p)$, where S^p is a p-sphere, and we can take

$$S^p = (E^{n+1} \times E^{p-n})^{\cdot}$$

$$= \dot{E}^{n+1} \times E^{p-n} + E^{n+1} \times \dot{E}^{p-n},$$

where $\dot{E}^{n+1} = S^n$, and $\dot{E}^{n+1} \times E^{p-n}$ is clearly a regular neighbourhood of $\dot{E}^{n+1}$.

THEOREM 25. *If* $L^m = D(K^n)$, *where* K^n *and* L^m *are complexes of* n *and* m *dimensions and* D *is a formal deformation of order* k, *then*

$$\Sigma(K^n, R^p) = \Sigma(L^m, R^p)$$

provided that $p \geqslant 2l+1$, *where* $l = \max(m+1, n+1, k)$.

By Theorem 5, Addendum 1, there is an l-dimensional complex K^*, which contracts formally into both K^n and a sub-division of L^m. Imbedding K^* in R^p, where $p \geqslant 2l+1$, it follows, as in Theorem 24, that

$$\Sigma(K^n, R^p) = \Sigma(K^*, R^p) = \Sigma(L^m, R^p),$$

and the theorem is established.

From Theorem 21 we have the corollary to Theorem 25:

COROLLARY 1. *If* $\pi_1(K^n)$ *satisfies the condition imposed in* §11 *and if* K^n *and* L^m *are of the same homotopy type, then* $\Sigma(K^n, R^p) = \Sigma(L^m, R^p)$ *provided that* $p \geqslant 2\max(m, n)+5$.

As a special case of Corollary 1 we have:

COROLLARY 2. *If* $\pi_1(K^n)$ *satisfies the condition imposed in* §11, *the simplicial space* $\Sigma(K^n, R^p)$ *is a topological invariant of* K^n *provided that*

$$p \geqslant 2n+5.$$

Combining Corollary 1 with Hurewicz's results we have:

COROLLARY 3. *If* $\pi_1(K^n) = 1$ *and all the homology groups* $\beta_i(K^n)$ *vanish* $(i = 2, \ldots, n)$, *then* $U(K^n, R^p)$ *is a* p*-element* $(p \geqslant 2n+5)$.

From a previous remark we have also:

COROLLARY 4. *If* $\pi_1(K^n) = 1$, $\beta_i(K^n) = \beta_i(S^n)$ $(i = 2, \dots, n)$, S^n *being an* n*-sphere, then* $U(K^n, R^p) = S^n \times E^{p-n}$ $(p \geqslant 2n+5)$.

We now return to the questions in § 10. If K and L are sub-complexes of a manifold H^n, which is either an n-sphere or an n-element, we shall say that K can be *freed isotopically* from L if there is an n-element in some sub-division σH^n, which contains σK and does not meet σL.

LEMMA 12. *If* $K, L \subset H^n$, $K \,.\, L = 1$ *and* $n > r+s+1$, *where* $r = \dim(K)$ *and* $s = \dim(L)$, *then* K *can be freed isotopically from* L.

If H^n is an n-sphere, it contains an n-element containing K and L. Therefore we may suppose that $H^n = E^n$ and, after a suitable sub-division, E^n may be represented as a simplicial covering of a rectilinear simplex Δ^n. Let p_0 be a point in Δ^n whose position is general with respect to K and L. Since K does not meet L and $r+s+1 < n$, the cone C, swept out by the segment $p_0 p$ as p varies over K, does not meet L. After a further sub-division we may suppose that C is covered by a sub-complex of E^n, which we also denote by C. Then $N = N(C'', E''^n)$ does not meet L''. But N is a regular neighbourhood of C'', and C'', being a sub-division of the star $p_0 K$, contracts into p_0. Therefore N is an n-element, by Theorem 23, Corollary 1, and the lemma is established.

If two bounded, connected, n-dimensional manifolds M_1 and M_2 meet in an $(n-1)$-element on the boundary of both, we shall describe the simplicial space associated with $M_1 + M_2$ as a *topological sum* of the simplicial spaces associated with M_1 and M_2. Let K_1 and K_2 be two sub-complexes of an n-dimensional manifold M, which meet in a single vertex b.

THEOREM 26. *If* K_{1b}, *the complement of* b *in* K_1, *can be freed isotopically from* K_{2b} *in* M_b, *the simplicial space* $\Sigma(K_1 + K_2, M)$ *is a topological sum of the simplicial spaces* $\Sigma(K_1, M)$ *and* $\Sigma(K_2, M)$.

Adding an n-element of the form bE^{n-1} to M, if necessary, we may suppose that b is inside M. After a suitable sub-division we also assume that M_b does not meet $\dot{M}$, also that $K_{1b} \subset E_1^{n-1} \subset M_b$, where E_1^{n-1} does not meet K_{2b}. Let $E_2^{n-1} = Cl(M_b - E_1^{n-1})$ and let

$$M_i = Cl(M - bE_j^{n-1}) \quad (i = 1,\ 2;\ j = i+1 \bmod 2).$$

Then $K_i \subset M_i$ and M_i is a manifold since M_b does not meet $\dot{M}$. Let

$$N_i = N(K_i'', M_i'').$$

Then N_i is a regular neighbourhood of K_i'' in M_i'' and therefore in M'', since the conditions for a regular neighbourhood are intrinsic. Therefore N_i is a triangulation of the simplicial space $\Sigma(K_i, M)$. Moreover, N_1+N_2 consists of all the closed simplexes in $M_1''+M_2''=M''$ which meet $K_1''+K_2''$. That is to say $N_1+N_2=N(K_1''+K_2'', M'')$, and N_1+N_2 is a triangulation of $\Sigma(K_1+K_2, M)$. The intersection $N_1 . N_2$ consists of the closed simplexes in $M_1'' . M_2''$ which meet $K_1'' . K_2''=b$. But

$$M_1 . M_2 = E+\{M-O(b, M)\},$$

where $E=b\dot{E}_1^{n-1}=b\dot{E}_2^{n-1}$. Therefore $N_1 . N_2$ is an $(n-1)$-element, namely $N(b, E'')$. Finally $E \subset \dot{M}_1 . \dot{M}_2$, whence $N_1 . N_2 \subset \dot{N}_1 . \dot{N}_2$, and the theorem is established.

In Theorem 26 the manifold M may be infinite, K_1 and K_2 being finite. In particular M may be a triangulation of R^n.

THEOREM 27. *The questions* Q. 2 *and* Q. 3 *of* § 10 *are equivalent.*

It is clear that an affirmative answer to Q. 2 carries with it an affirmative to Q. 3. It remains to show that an affirmative to Q. 3 implies an affirmative to Q. 2. Let $K_2=D(K_1)$, $K_2+L_2=D_0(K_1+L_1)$ and let $P_i=K_i . L_i$ be geometrically collapsible. After a suitable sub-division we may assume that P_i is formally collapsible and that $N(P_i, K_i)$, $N(P_i, L_i)$ and therefore $N(P_i, K_i+L_i)$ are contractible neighbourhoods of P_i. By Theorem 3 the operation of shrinking P_i into a point is a formal deformation of K_i, L_i and of K_i+L_i. Therefore we may take $K_i . L_i$ to be a single vertex b_i. This being so, let l be the maximum of m_i+1, n_i+1 and the orders of the deformations D and D_0, where $n_i=\dim(K_i)$ and $m_i=\dim(L_i)$. After a suitable sub-division, K_i+L_i may be imbedded in R^p, as a sub-complex of some triangulation M^p, of R^p, where $p \geqslant 2l+1$. Since $p-1>(m_i-1)+(n_i-1)+1$ it follows from Lemma 12 that K_{ib_i} may be separated isotopically from L_{ib_i} in $M_{b_i}^p$, and from Theorem 26 that $\Sigma(K_i+L_i, R^p)$ is a topological sum of $\Sigma(K_i, R^p)$ and $\Sigma(L_i, R^p)$, Since $p \geqslant 2l+1$, it follows from Theorem 25 that $\Sigma(K_1, R^p)=\Sigma(K_2, R^p)$ and that $\Sigma(K_1+L_1, R^p)=\Sigma(K_2+L_2, R^p)$. Moreover the boundaries of regular neighbourhoods $U(K_i, R^p)$ and $U(L_i, R^p)$ are connected. For these neighbourhoods contain no non-bounding $(p-1)$-cycles since $m_i, n_i<p-1$. If the answer to Q. 3 is "yes" it follows that any two regular neighbourhoods $U(L_1, R^p)$ and $U(L_1, R^p)$ have the same nucleus and, since $U(L_i, R^p)$ contracts into L_i, that L_1 and L_2 have the same nucleus. Therefore an affirmative answer to Q. 3 implies an affirmative answer to Q. 2, and the theorem is established.

13. *Newman's moves.* The object of this section is to prove two theorems which are similar to Theorems 4 and 6 with elementary expansions and reductions replaced by Newman's moves of Type 1 and 2. We recall that a move of type 1, applied to an n-dimensional manifold M_0, is a regular expansion of the form

$$M_0 \rightarrow M_1 = AB + M_0,$$

where AB is n-dimensional, $A\dot{B} \subset \dot{M}_0$, $A \not\subset M_0$. A move of Type 2 is the inverse of a move of type 1. For $M \rightarrow Cl(M-C)$ to be a move of type 2 it is necessary and sufficient that $C = AB \subset M$, where

1. $B = M_A$,

2. B is internal to M.

An n-element E^n will be described as *regularly collapsible* if $E^n \underset{2}{\rightarrow} A^n$, where A^n is a closed n-simplex and the symbol $M \underset{\alpha}{\rightarrow} M$ ($\alpha = 1$ or 2) means that the transformation indicated is a product of moves of type α. As a matter of convention $M \underset{\alpha}{\rightarrow} M$, so that a closed simplex is regularly collapsible.

LEMMA 13. *If* $M = AE + M^*$ *and if* $E \underset{2}{\rightarrow} E_q$, *then* $M \underset{2}{\rightarrow} AE_q + M^*$, *provided that every internal simplex in* E *is inside* M.

If $E_q = E$ there is nothing to prove. Otherwise let $E = CB + E_1$, where $E \rightarrow E_1$ is the first move in the transformation $E \underset{2}{\rightarrow} E_q$ and $B = E_C$, B being inside E. Then $B = M_{AC}$ and B is inside M. Therefore

$$M \rightarrow Cl(M - ABC) = AE_1 + M^*$$

is a move of type 2, and the lemma follows by induction on the number of moves in $E \underset{2}{\rightarrow} E_q$.

COROLLARY. *If* E *is regularly collapsible,* $M \underset{2}{\rightarrow} M^*$.

For if $E = B$, a single closed simplex which is inside M, the transformation $AB + M \rightarrow M^*$ is a move of type 2.

LEMMA 14. *If* $E \underset{2}{\rightarrow} E_q$, *where* E *is an element, then* $AE \underset{2}{\rightarrow} AE_q$, *where* A *is a closed simplex which does not meet* E.

For, with the notation used in proving Lemma 13, the simplex AB is inside AE and is the complement of C. Therefore $AE \rightarrow AE_1$ is a move of type 2 and the lemma follows from induction on the number of moves in $E \underset{2}{\rightarrow} E_q$.

COROLLARY. *If E is regularly collapsible so is AE.*

LEMMA[†] 15. *If A and B are (closed) simplexes which do not meet each other ($A \neq 1$, $B \neq 1$), $A\dot{B}$ is regularly collapsible.*

If B is 0-dimensional, $A\dot{B} = A$. Otherwise let $B = bB_1$ ($B_1 \neq 1$). Then

$$A\dot{B} = A(b\dot{B}_1 + B_1).$$

The simplex A is inside $A\dot{B}$ and is the complement of B_1 in $A\dot{B}$. Therefore the transformation $A\dot{B} \rightarrow Ab\dot{B}_1$ is a move of type 2. Writing $Ab = A_1$, and assuming that $A_1\dot{B}_1$ is regularly collapsible, we deduce the lemma by induction on $\dim(B)$.

THEOREM 28. *If $M \underset{2}{\rightarrow} M_1$, then*

$$\sigma M \underset{2}{\rightarrow} \sigma M_1,$$

where σ is any stellar sub-division of M.

This will follow by an inductive argument similar to the one used in proving Theorem 4 if we can prove it in case $M \underset{2}{\rightarrow} M_1$ is a single move and σ is an elementary sub-division (A, a).

Let $$M = CB + M_1,$$

where $B = M_C$ and B is inside M. As in Theorem 4 the result is obvious unless $A \subset BC$. So let $A = B_1C_1$, where $B = B_1B_2$ and $C = C_1C_2$ (possibly $B_1 = 1$ or $C_1 = 1$). Then

$$\begin{aligned} \sigma M &= a(C_1\dot{B}_1 + \dot{C}_1B_1)B_2C_2 + \sigma M_1 \\ &= Ca\dot{B}_1B_2 + aC_2B\dot{C}_1 + \sigma M_1, \end{aligned} \tag{13.1}$$

and $a\dot{B}_1B_2 = (\sigma M)_C$, since $C \not\subset M_1$ and hence $C \not\subset \sigma M_1$. The simplex BC_1 is inside M, being incident with B, and

$$\begin{aligned} \sigma(BC_1) &= \sigma(B_1B_2C_1) \\ &= \sigma(AB_2) \\ &= a\dot{A}B_2. \end{aligned}$$

Therefore aB_2 is inside $\sigma(BC_1)$ and hence inside σM. The internal simplexes of $a\dot{B}_1B_2$ are those incident with aB_2 and are therefore inside σM. By

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[†] *Cf.* Newman, *Proc. Akad. Amsterdam* (second paper, *loc. cit.*), 619, Theorem 20.

Lemma 15, $a\dot{B}_1 B_2$ is regularly collapsible and, by the corollary to Lemma[†] 13,

$$\sigma M \underset{2}{\rightarrow} aC_2 B\dot{C}_1 + \sigma M_1 = M^*, \tag{13.2}$$

say.

If $C_1 = 1$, we have $M^* = \sigma M_1$ and the proof is complete. So we assume that $C_1 \neq 1$. Then $B \subset \sigma M$ and B is inside σM, since it is inside M. Also $M_B^* = (\sigma M)_B$, since $B \not\subset Ca\dot{B}_1 B_2$. It follows that B, and therefore the internal simplexes of $B\dot{C}_1$, are inside M^*. Also $B_1 C \not\subset M_1$, since $C \not\subset M_1$, and

$$\begin{aligned} \sigma(B_1 C) &= \sigma(B_1 C_1 C_2) \\ &= \sigma(AC_2) \\ &= a\dot{A}C_2. \end{aligned}$$

Therefore aC_2 is inside $\sigma(B_1 C)$ and it follows that $aC_2 \not\subset \sigma M_1$. Therefore $M^*_{aC_2} = B\dot{C}_1$ and is regularly collapsible by Lemma 15. Since the internal simplexes of $B\dot{C}_1$ are inside M^* it follows from the corollary to Lemma 13 that $M^* \underset{2}{\rightarrow} \sigma M_1$. Therefore $\sigma M \underset{2}{\rightarrow} \sigma M_1$ and the theorem is established.

In order to state the next theorem we shall extend our notation by writing

$$E^n \underset{2}{\rightarrow} E^{n-1} \quad (n > 0)$$

if $E^{n-1} \subset \dot{E}^n$ and
$$M + E^n \underset{2}{\rightarrow} M,$$

where M is any n-dimensional manifold such that $M \,.\, E^n = \dot{M} \,.\, \dot{E}^n = E^{n-1}$.

THEOREM 29. *If K is any complex, there is a stellar sub-division σK such that $\sigma E^m \underset{2}{\rightarrow} \sigma E^{m-1}$ $(m > 0)$, where E^m is an arbitrary m-element in K, and E^{m-1} is an arbitrary $(m-1)$-element in $\dot{E}^m$, the sub-division σ being independent of any particular choice of m or of E^m and E^{m-1}.*

There are in K only a finite number of elements and the theorem will follow from Theorem[‡] 28 if we can show that there is a sub-division σE^m, such that $\sigma E^m \underset{2}{\rightarrow} \sigma E^{m-1}$, where E^m is a given m-element and E^{m-1} a given $(m-1)$-element in $\dot{E}^m$. Assuming that this is true when $0 < m \leqslant n$ we shall prove the following corollary for $0 \leqslant m \leqslant n$.

[†] Notice that the first term on the right-hand side of (13 . 1) is absent if $B_1 = 1$. In this case $M = M^*$.

[‡] *Cf.* the corollary to Theorem 4.

COROLLARY. *Some stellar sub-division of any m-element is regularly collapsable.*

The corollary is trivial if $m = 0$ and we therefore take $m > 0$. After a preliminary sub-division, if necessary, we may assume that the internal simplexes of $E_b{}^m$ are inside E^m, where E^m is a given m-element and b is a vertex in $\dot{E}^m$. Then

$$E_0{}^m = E^m - O(b,\, E^m)$$

is an m-element and it follows from the above hypothesis that there is a stellar sub-division σ_1 such that $\sigma_1 E_0{}^m \underset{2}{\rightarrow} \sigma_1 E_b{}^m$. Let us assume that some sub-division $\sigma_2 \sigma_1 E_b{}^m = \sigma E_b{}^m$ is regularly collapsible. Then

$$b(\sigma E_b{}^m) = \sigma(bE_b{}^m)$$

is regularly collapsible, by the corollary to Lemma 14, and by Theorem 28 $\sigma E^m \underset{2}{\rightarrow} \sigma(bE_b{}^m)$, where $\sigma = \sigma_2 \sigma_1$. Therefore σE^m is regularly collapsible and the corollary follows by induction on m for $m \leqslant n$.

Now let $m = n+1$ and let E^m be a partition of the geometrical simplex $p_0 p_1 \dot{A}^n$ $(0 \leqslant n = m-1)$ having no internal vertices, and let $E_i{}^n$ be the sub-complex of E^m covering $p_i \dot{A}^n$ $(i = 0,\, 1)$. I say that $\sigma E^m \underset{2}{\rightarrow} \sigma E_0{}^n$, where σ is some stellar sub-division of E^m. For let $L = \dot{E}^m = E_0{}^n + E_1{}^n$ and let F^n and F^m mean the same as in the proof of Theorem 6. If we assume Theorem 29 for n-dimensional complexes, it follows from the corollary that there is a stellar sub-division $\sigma_1 F^n$ such that $\sigma_1 E$ is regularly collapsible, where E is any element in F^n. As in the proof of Theorem 6, $F^n . (s_L E^m)_B$ is an element, where B is any internal open simplex of $E_1{}^n$. Therefore $\sigma\{F^n . (s_L E^m)_B\} = \sigma_1 F^n . (\sigma E^m)_B$ is regularly collapsible, where $\sigma = \sigma_1 s_L$. If we take the open simplexes inside $\sigma_1 E_1{}^n (= \sigma E_1{}^n)$ in order of decreasing dimensionality, it follows from the corollary to Lemma 13 that the open stars $O(B, \sigma E^m)\, (B \epsilon\, \sigma E_1{}^n - \sigma \dot{E}_1{}^n)$ may be removed successively by moves of type 2. Therefore $\sigma E^m \underset{2}{\rightarrow} \sigma_1 F^m$. Let M and M_1 be any m-dimensional manifolds such that $M . M_1 = \dot{M} . \dot{M}_1$ and let $E^n (n = m-1)$ be an $(m-1)$-element in $\dot{M} . \dot{M}_1$. If $M \rightarrow Cl(M - AB)$ is a move of type 2, where A is inside E^n and B is inside M, the transformation $M_1 \rightarrow M_1 + AB$ is a move of type 1. If we take $M = \sigma E^m$ and $E^n = \sigma E_1{}^n$, it follows that $M_1 \underset{1}{\rightarrow} M_1 + \sigma_1 F_1{}^m$, where $F_1{}^m = Cl(s_L E^m - F^m)$. Therefore

$$M_1 + \sigma_1 F_1{}^m \underset{2}{\rightarrow} M_1 \quad \textit{i.e.} \quad \sigma_1 F_1{}^m \underset{2}{\rightarrow} \sigma E_1{}^n.$$

But the construction of F^n and σ_1 is symmetrical between $E_0{}^n$ and $E_1{}^n$.

Therefore $\sigma_1 F^m \underset{2}{\rightarrow} \sigma E_0{}^n$, and since $\sigma E^m \underset{2}{\rightarrow} \sigma_1 F^m$ we have $\sigma E^m \underset{2}{\rightarrow} \sigma E_0{}^n$, as stated. The rest of the proof is the same as the proof of Theorem 6.

COROLLARY. *If M contracts regularly into M^* there is a stellar sub-division σ such that $\sigma M \underset{2}{\rightarrow} \sigma M^*$.*

14. *Infinite complexes.* Starting with a given aggregate of vertices, we now define a complex as in § 2, with the single condition that the dimensionality of each simplex is to be finite. For example, if we take the vertices to be real numbers, the totality of finite sets of real numbers is a permissible complex. In order that sub-divisions and expansions shall be applicable to such a complex, we allow ourselves to create new vertices if and when they are needed. More precisely, we assume that a given aggregate of vertices can be duplicated at any stage in an argument, and the duplicate set combined with the original into a single set. We rely on the axiom of choice but, except at one stage in the proof of Theorem 37, only so far as sub-sets of the given vertices are concerned. For, if the original aggregate of vertices is well-ordered, choice can be eliminated from the combinatorial constructions by lexicographical and other standard devices.

We proceed to extend the definitions of equivalence to infinite complexes†. For combinatorial equivalence we shall use the idea of general sub-division rather than elementary transformations. We recall that a complex γK is a *general sub-division* of K if the simplexes of γK are grouped into k-elements γA^k $(k = -1, 0, 1, \ldots;\ \gamma 1 = 1)$, which are in a (1-1), incidence-preserving correspondence $\gamma A^k \rightarrow A^k$ with the closed simplexes in K. We shall write $\gamma = 1$ if γ is the identical sub-division, given by $\gamma A = A$ for each closed simplex in K. It is always to be understood that no vertex in γK belongs to K, or to any complex which is being considered simultaneously with K, unless it is a 0-simplex A^0, in K, and $\gamma A^0 = A^0$. If L is any sub-complex of K, the cells in γK which correspond to the simplexes in L constitute a sub-division of L, which we denote by γL. Thus γ may be regarded as a transformation which operates on each sub-complex of K. Conversely, let γL be given, where L is a sub-complex of K. We define γK as follows. Let K^n be the complex consisting of

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† A little care is needed here, as the following example shows. Let K_i be a triangulation of the topological product $A^n \times \langle i, \infty \rangle$ $(i = 0, 1, \ldots)$, together with the rest of the cylinder $C = \dot{A}^n \times \langle 0, \infty \rangle$. Let c_i be the geometrical contraction $K_i \rightarrow K_{i+1}$. Then the infinite sequence of contractions $c_0, c_1, \ldots$ transforms K_0 into C, and therefore alters the homotopy group $\pi_{n-1}(K_0)$.

L together with all the simplexes of K whose dimensionalities do not exceed n. Assuming that $\gamma_n K^n$ has been defined and that $\gamma_n = \gamma$ in L (*i.e.* $\gamma_n A = \gamma A$ if $A \subset L$) and $\gamma_n A = A$ if $\gamma = 1$ in $A . L$, we define $\gamma_{n+1} K^{n+1}$ by the conditions $\gamma_{n+1} = \gamma_n$ in K^n, and

$$\gamma_{n+1} A^{n+1} = A^{n+1} \quad \text{if} \quad \gamma_n = 1 \quad \text{in} \quad \dot{A}^{n+1}$$

$$= a \gamma_n \dot{A}^{n+1} \quad \text{otherwise,}$$

where A^{n+1} is the closure of any open simplex in $K^{n+1} - L$ and a is a new vertex. If we begin with $\gamma_{-1} = \gamma$ in $K^{-1} = L$, the sub-division $\gamma_n K^n$ is thus defined inductively for all values of n. Clearly $\gamma_m = \gamma_n$ in K^n if $m > n$, and γK is defined by the condition

$$\gamma A = \gamma_n A \quad \text{if} \quad A \subset K^n,$$

this definition being unique except for the choice of the new vertices. A given set of complexes L, L^*, ..., may be combined to form a single complex K. If γL is given, it follows from what we have said that γ may be treated as an operator which is applicable to K, and therefore to any complex in the set. If γ is initially defined as a sub-division of L and if $\gamma = 1$ in LL^*, notice that $\gamma = 1$ in L^*.

A sub-division γK will be described as a partition† πK, or a stellar sub-division, if it is a partition, or a stellar sub-division, of each finite sub-complex of K. With the notation explained in the last paragraph, if a given sub-division γK (or γL) is a partition, or a stellar sub-division, it is obvious that γL (or γK) is also a partition, or a stellar sub-division, where L is any sub-complex of‡ K. If K is infinite, the sub-division $s_L K$ cannot be defined as a sequence of elementary sub-divisions since there are, in general, no simplexes of highest dimensionality with which to start. We define $s_L K$ inductively by the construction used in extending γL to γK, starting with K^0 and a sub-division $\gamma K^0 = s_L K^0$ ($s_L = 1$ in L). If $A^0 \subset K^0 - L$, it is to be a matter of choice whether or no $s_L A^0 = A^0$.

LEMMA 16. *Two sub-divisions $\gamma_1 K$ and $\gamma_2 K$ have a common sub-division* $\gamma_1^* \gamma_1 = \gamma_2^* \gamma_2$.

Let K^n be the *n-dimensional skeleton* (*Gerüst*) of K, that is the set of all simplexes whose dimensionalities do not exceed n. Assume that there are sub-divisions γ_1^n and γ_2^n such that

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† *Cf.* Whitehead, *loc. cit.*

‡ If σL was given as a sequence of elementary sub-divisions this would differ from the natural definition of σK. But we shall adhere to the single definition of γK, whether γ is a stellar sub-division or not.

$$(14.1) \qquad \begin{cases} \gamma_1^n \gamma_1 = \gamma_2^n \gamma_2 \quad \text{in} \quad K^n, \\ \gamma_i^n = \gamma_i^{n-p} \quad \text{in} \quad \gamma_i K^{n-p} \quad (i = 1, 2;\ p \geqslant 0). \end{cases}$$

Since $$\gamma_1^n \gamma_1 \dot{A}^{n+1} = \gamma_2^n \gamma_2 \dot{A}^{n+1},$$

where A^{n+1} is any $(n+1)$-simplex in K, there are sub-divisions γ_1^{n+1} and γ_2^{n+1} such that†

$$\gamma_1^{n+1} \gamma_1 A^{n+1} = \gamma_2^{n+1} \gamma_2 A^{n+1},$$

where $\gamma_i^{n+1} = \gamma_i^n$ in $\gamma_i \dot{A}^{n+1}$ and therefore in $\gamma_i K^n$. Extending γ_i^n in this way throughout all the $(n+1)$-simplexes in K we arrive at sub-divisions which satisfy (14.1) with n replaced by $n+1$. If we begin with $\gamma_i^{-1} = \gamma_i = 1$ in K^{-1}, it follows by induction on n that there are sub-divisions satisfying (14.1) for all values of n. The required sub-divisions γ_1^* and γ_2^* are given by

$$\gamma_i^* \gamma_i A = \gamma_i^n \gamma_i A \quad \text{if} \quad A \subset K^n,$$

and the lemma is established.

From the sharper results proved in my paper on sub-divisions we have the addendum:

ADDENDUM. *The sub-divisions γ_1^* and γ_2^* of Lemma* 16 *may be chosen so that a given one of them is a stellar sub-division and the other is a partition*‡. *If $\gamma_1 = \gamma_2 = \gamma$ in some sub-complex L, they may be chosen so that also $\gamma_1^* = \gamma_2^* = \gamma$ in L.*

We now define two complexes K_1 and K_2 as *combinatorially equivalent* if, and only if, they have a common sub-division $\gamma_1 K_1 = \gamma_2 K_2$. It follows from Lemma 16 that two complexes are combinatorially equivalent to each other if each is combinatorially equivalent to a third. Therefore the equivalence classes are mutually exclusive.

We now come to formal deformations and fillings and perforations. Our method, which we have already used in proving Lemma 16, is to replace "long" sequences of individual elementary transformations (*i.e.* sequences with high ordinal numbers) by countable sequences of

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† Alexander, *loc. cit.*, Theorem 13.2, and Whitehead, *loc. cit.*, Theorem 2.

‡ Taking $\gamma_2 = 1$ and $\gamma_1^* = \sigma$, $\gamma_2^* = \pi$, we have $\sigma\gamma_1 = \pi$, and taking $\gamma_1^* = \pi$, $\gamma_2^* = \sigma$, we have $\pi\gamma_1 = \sigma$.

"composite" transformations. Though each composite set may have any cardinal number, the individual transformations contained in it can all be applied at once, and there is no need to consider questions of order within the set.

We shall denote the elementary expansion $K \to K + aA + A$, where A and aA are open simplexes, by aA and it is to be understood that the transformation aA is defined in the abstract though it is not applicable to every complex. It is applicable to a complex K if, and only if, K contains $a\dot{A}$ but not A. When aA denotes an elementary expansion we shall allow A to be 0, the empty set of simplexes, in which case aA will be the identical transformation, operating on every complex and transforming it into itself. Two elementary expansions aA and bB will be described as *independent* if, and only if†,

$$(14.2) \qquad A \bar{\in} Cl(bB), \quad B \bar{\in} Cl(aA).$$

In particular the identity is independent of every elementary expansion. The conditions (14.2) are obviously equivalent to the conditions

(1) *aA and bB are both applicable to some one complex K [e.g. $Cl(a\dot{A}+b\dot{B})$],*

(2) *bB is applicable to $K+aA+A$;*

and, if these are satisfied, the transformation

$$K \to K + aA + bB + A + B$$

will be described as the composite expansion due to the simultaneous application of aA and bB. More generally, let $\{aA\}$ be any set of elementary expansions, finite or infinite, each of which is applicable to a given complex K and any two of which are independent. The transformation

$$K \to K + \Sigma,$$

where Σ is the totality of open simplexes aA and A, will be called the *composite expansion* due to the *simultaneous application* of the elementary expansions aA. A composite expansion $\{aA\}$ will be described as applicable to K if, and only if, each of the elementary expansions aA is applicable to K, and $\{aA\}$ and $\{bB\}$ will be described as independent if, and only if, each elementary expansion aA is independent of each bB. From this definition it follows that a necessary and sufficient condition for the independence of two composite expansions is that one of them shall be applicable to some complex both before and after the other. Any set of mutually independent composite expansions can obviously be combined into a single composite expansion, and any sub-set of the elementary expansions in a composite expansion may be applied simultaneously to form a composite expansion. Though we shall be working entirely with expansions, notice that composite contractions may be similarly defined; also composite deformations, consisting of expansions and contractions.

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† Notice that, in dealing with elementary expansions, the question is not so much "are two given transformations interchangeable?" as "can one be applied both before and after the other?"

Now let $\bar{e}_1$ be a composite expansion† of a given complex K_0 and let $K_{n+1} = \bar{e}_{n+1} K_n$, where $\bar{e}_1, \bar{e}_2, \ldots$ is an enumerable sequence of composite expansions. Let Σ_{ω_0} be the aggregate of simplexes added by all the elementary expansions in $\bar{e}_1, \bar{e}_2, \ldots$, and let

$$K_{\omega_0} = K_0 + \Sigma_{\omega_0}.$$

Proceeding by transfinite induction, let $K_p = K_0 + \Sigma_p$, where p is any ordinal number and Σ_p is the aggregate of simplexes added by a given transfinite sequence of composite expansions $\{\bar{e}_j\}$ $(j \prec p+1)$, such that $K_{i+1} = \bar{e}_{i+1} K_i = \bar{e}_{i+1}(K_0 + \Sigma_i)$ $(i \prec p)$.

THEOREM 30. *The transformation* $K_0 \rightarrow K_p$ *is the resultant of a countable sequence of composite expansions.*

If an elementary expansion in $\bar{e}_j$ is applicable to K_i, where $i+1 \prec j$, it may be transferred from $\bar{e}_j$ to $\bar{e}_{i+1}$. We do this whenever possible, so that each elementary transformation is applied at the first opportunity. Then the elementary expansions in $K_0 \rightarrow K_p$ form a countable set of composite expansions. For if not, there is an elementary expansion aA in the (ω_0+1)-th set. But the complex $Cl(aA)$, being finite, is in K_n for some finite value of n, and $A \bar{\epsilon} K_n$ since $A \bar{\epsilon} K_{\omega_0}$ and $K_n \subset K_{\omega_0}$. Therefore aA is applicable to K_n, contrary to the fact that it is applied as soon as possible.

It follows from this theorem that we need only consider countable sequences of composite expansions. The resultant of such a sequence will be called an *expansion*.

Geometrical expansions, defined as in § 5, may be treated in the same way as elementary expansions. A geometrical expansion Σ is the addition of a set of open simplexes $\Sigma = E^n - E^{n-1}$, where E_n is an n-element and

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† When we write $\bar{e}K$, it is to be understood that $\bar{e}$ is applicable to K.

E^{n-1} is an $(n-1)$-element in $\dot{E}^n$. We shall say that Σ is *applicable* to a complex K if, and only if,

$$(14.3) \qquad Cl(\Sigma)-\Sigma \subset K, \quad \Sigma . K = 0,$$

and two geometrical expansions Σ_1 and Σ_2 will be described as *independent* if, and only if,

$$(14.4) \qquad \Sigma_1 . Cl(\Sigma_2) = \Sigma_2 . Cl(\Sigma_1) = 0.$$

All that we have said about elementary expansions, including Theorem 30, obviously applies with minor alterations to geometrical expansions. A product of composite geometrical expansions will also be called a geometrical expansion and, when a contrast is necessary, a product of composite formal expansions will be called a formal expansion.

THEOREM[†] 31. *If K_0 expands geometrically into K, it expands formally into some stellar sub-division σK, where $\sigma = 1$ in K_0.*

Our proof depends on the lemma:

LEMMA 17. *There is an internal stellar sub-division σE^n $(n > 0)$ which contracts formally into E^{n-1}, where E^n is any n-element and E^{n-1} is an $(n-1)$-element in $\dot{E}^n$.*

The lemma is trivial if $n = 1$ and will be proved by induction on n. First notice that, when we assume the lemma for k-elements if $k < n$, some internal stellar sub-division of a k-element is collapsible. For some internal stellar sub-division contracts into a closed $(k-1)$-simplex on the boundary. Now let $\Sigma = \dot{E}^n - E^{n-1}$, let $\sigma_0 = s^2_{\dot{E}^n}$ and let

$$E_1{}^n = \sigma_0 E^n - O(\Sigma, \sigma_0 E^n).$$

As proved by Newman[‡], $\sigma_0 E^n$ is transformed into $E_1{}^n$ by a sequence of regular contractions of the form

$$A^{n-k-1} E^k + \tilde{E}^n \rightarrow \tilde{E}^n \quad (E^k \subset \dot{\tilde{E}}^n),$$

where E^k is a k-element $(k < n)$ whose internal simplexes are inside $A^{n-k-1} E^k + \tilde{E}^n$. By the corollary to the inductive hypothesis and Theorem 4, there is an internal stellar sub-division $\sigma_1 \sigma_0 E^n$ such that each of the elements $\sigma_1 E^k$ is collapsible. Then $\sigma_1 \sigma_0 = 1$ in $\dot{E}^n$, and $\sigma_1 \sigma_0 E^n$

[†] Notice that this theorem is sharper than the analogous Theorem 7. Because of Lemma 17, proved below, it is unnecessary to sub-divide K_0.

[‡] *Journal London Math. Soc.* (*loc. cit.*), 509, Lemma 2.

contracts into $\sigma_1 E_1{}^n$ by Lemma 3. Let $L = E^{n-1}$ and let

$$E_2{}^n = s_L{}^2 \sigma_1 E_1{}^n - O(E^{n-1}, s_L{}^2 \sigma_1 E_1{}^n)$$

$$= \sigma_2 E_1{}^n - O(E^{n-1}, \sigma_2 E_1{}^n),$$

where $\sigma_2 = s_L{}^2 \sigma_1$. Then $\sigma_2 = 1$ in $\dot{E}^n$ and $E_2{}^n$ does not meet $\dot{E}^n$. Since each internal simplex of $s_L \sigma_1 E_1{}^n$ is starred by s_L, it follows that E^{n-1} and $\sigma_2 E_1{}^n$ satisfy the first condition imposed on Σ and M in Lemma 10. By the corollary to Lemma 4 they also satisfy the second condition, and it follows from Lemma 10 that $E_2{}^n$ is an n-element. Let us assume for the moment that

$$N = N(E^{n-1}, \sigma_2 E_1{}^n)$$

meets $E_2{}^n$ in an $(n-1)$-element on $\dot{E}_2{}^n$. Then it follows from Theorem 6 that some stellar sub-division $\sigma_3 E_2{}^n$ contracts into $\sigma_3(N \,.\, E_2{}^n)$, and $\sigma_3 = 1$ in $\dot{E}^n$ since $E_2{}^n$ does not meet $\dot{E}^n$. By Theorem 2, N contracts into E^{n-1}, and, by Theorem 4, $\sigma_3 N$ also contracts into E^{n-1}. Therefore $\sigma_3 \sigma_2 E_1{}^n$ contracts first into $\sigma_3 N$ and then into E^{n-1}, and it follows that σE^n contracts into E^{n-1}, where $\sigma = \sigma_3 \sigma_2 \sigma_0$ and $\sigma = 1$ in $\dot{E}^n$.

It remains to prove that $N \,.\, E_2{}^n \subset \dot{E}_2{}^n$ and is an $(n-1)$-element. An open simplex A $(A \neq 1)$ in $\sigma_2 E_1{}^n$ belongs to $N \,.\, E_2{}^n$ if, and only if, $(\sigma_2 E_1{}^n)_A$ meets E^{n-1}, since $A \in N$, but $Cl(A)$ does not, since $A \in E_2{}^n$. If $A \in N \,.\, E_2{}^n$, it follows that some simplex in $(\sigma_2 E_1{}^n)_A$ is absent from E^n_{2A}. Therefore E^n_{2A} is a proper sub-set of $(\sigma_2 E_1{}^n)_A$ and so cannot be a sphere. Therefore $A \in \dot{E}_2{}^n$, whence $N \,.\, E_2{}^n \subset \dot{E}_2{}^n$. Moreover, N is an n-element, by Theorem 22 and Theorem 23, Corollary 1. If $A \in N \,.\, E_2{}^n$, then $Cl(A)$ does not meet E^{n-1} and it follows from an argument in the proof of Theorem 22 that N_A is an element. Therefore $N \,.\, E_2{}^n \subset \dot{N} - O(E^{n-1}, \dot{N})$. But $Cl(A) \,.\, E^{n-1} = 1$ if $A \in \dot{N} - O(E^{n-1}, \dot{N})$, whence $\dot{N} - O(E^{n-1}, \dot{N}) \subset E_2{}^n$. Therefore $N \,.\, E_2{}^n = \dot{N} - O(E^{n-1}, \dot{N})$. If $Cl(A)$ meets E^{n-1} $(A \in \sigma_2 E_1{}^n)$, we have $N_A = (\sigma_2 E_1{}^n)_A$, whence A is in both $\dot{N}$ and $\sigma_2 \dot{E}_1{}^n$ if it is in either. Therefore $O(E^{n-1}, \dot{N}) = O(E^{n-1}, \sigma_2 \dot{E}_1{}^n)$ and is the interior of $N(E^{n-1}, \sigma_2 \dot{E}_1{}^n)$, which is an $(n-1)$-element by Lemma 10, Corollary 2. Therefore $N \,.\, E_2{}^n$ is an $(n-1)$-element and the lemma is established.

The theorem now follows by an inductive argument similar to the one used in proving Lemma 16. Let $\bar{e}_1, \bar{e}_2, \ldots$ be a countable sequence of geometrical expansions which transform K_0 into K, and let $K_r = \bar{e}_r K_{r-1}$. Assume that K_0 expands formally into some stellar sub-division

$$\sigma_i K \quad (i = 0, \ldots, m),$$

where $\sigma_j = \sigma_i$ in K_i if $i < j$. If $K_m \rightarrow K_m + E^n$ is any one of the individual expansions in $\bar{e}_{m+1}$, where $E_n . K_m = \dot{E}^n . K_m = E^{n-1}$, it follows from Lemma 17 that $\sigma_m E^{n-1}$ expands formally into $\sigma^* \sigma_m E^n$, where σ^* is some internal stellar sub-division of $\sigma_m E^n$. Therefore $\sigma_m K_m$ expands formally into $\sigma_{m+1} K_{m+1}$, where σ_{m+1} is the resultant of σ_m followed by all the sub-divisions σ^* corresponding to the various elements added by $\bar{e}_{m+1}$. Since σ^* is internal to $\sigma_m E^n$ it leaves $\sigma_m E^{n-1}$, and hence $\sigma_m K_m$, unaltered. Therefore $\sigma_{m+1} = \sigma_m$ in K_m and it follows that $\sigma_{m+1} = \sigma_i$ in K_i if $i < m$. If we begin with $\sigma_0 = 1$, it follows inductively that there is a sub-division σ_m satisfying the above conditions for each value of m. Moreover, $\sigma_m = \sigma_0 = 1$ in K_0. The required sub-division σK is defined by the condition

$$\sigma A = \sigma_m A \quad \text{if} \quad A \subset K_m,$$

and the theorem is established.

If K_0 expands formally into K, it is obvious that γK_0 expands geometrically into γK, where γ is any sub-division. Therefore we have the corollary:

COROLLARY. *If K_0 expands formally into K and γ is any sub-division, then γK_0 expands formally into some stellar sub-division $\sigma \gamma K$, where $\sigma = 1$ in γK_0.*

We now define a *formal deformation* of an infinite complex as a transformation of the form

$$D = \gamma_2^{-1} E_2^{-1} E_1 \gamma_1,$$

where γ_1 and γ_2 are general sub-divisions and E_1 and E_2 are formal expansions. That is to say

$$K_1 = D(K_0)$$

if, and only if, there are sub-divisions γ_1 and γ_2 and formal expansions E_1 and E_2, such that

$$E_1(\gamma_1 K_0) = E_2(\gamma_2 K_1).$$

If $\gamma_1 = \gamma_2 = 1$ in L, where $L \subset K_0 . K_1$, we shall write

$$K_1 = D(K_0) \quad (\text{rel. } L).$$

THEOREM 32. *If $K_1 = D_1(K_0)$ (rel. L) and $K_2 = D_2(K_1)$ (rel. L), then $K_2 = D(K_0)$ (rel. L), where $L \subset K_0 . K_1 . K_2$.*

Let
$$E_1(\gamma_1 K_0) = E_2{}^*(\gamma_2 K_1) = K_{01} \quad (\gamma_1 = \gamma_2 = 1 \text{ in } L),$$
and
$$E_2(\gamma_2{}^* K_1) = E_3(\gamma_3 K_2) = K_{12} \quad (\gamma_2{}^* = \gamma_3 = 1 \text{ in } L).$$
By Lemma 16 and its addendum, we have $\sigma\gamma_2 K_1 = \gamma\gamma_2{}^* K_1 = K_1{}^*$, say, where σ is some stellar sub-division and γ a general sub-division and $\sigma = \gamma = 1$ in L. By Theorem 4, which is obviously true of infinite complexes, $\sigma\gamma_1 K_0$ and $K_1{}^*$ expand formally into σK_{01}. By the corollary to Theorem 31, $K_1{}^*$ expands formally into some stellar sub-division $\sigma_1 \gamma K_{12}$, where $\sigma_1 = 1$ in $K_1{}^*$, and hence in L. By Theorem 4, $\sigma_1 \sigma \gamma_1 K_0$ and $K_1{}^*$ expand formally into $\sigma_1 \sigma K_{01}$. Finally, $\sigma_1 \gamma\gamma_3 K_2$ expands formally into some stellar sub-division $\sigma_2 \sigma_1 \gamma K_{12}$ ($\sigma_2 = 1$ in $\sigma_1 \gamma\gamma_3 K_2$ and hence in L), $\sigma_2 \sigma_1 \sigma \gamma_1 K_0$ expands formally into $\sigma_2 \sigma_1 \sigma K_{01}$ and $\sigma_2 K_1{}^*$ expands formally both into $\sigma_2 \sigma_1 \sigma K_{01}$ and into $\sigma_2 \sigma_1 \gamma K_{12}$. Therefore we may assume that $\gamma_2{}^* = \gamma_2$, after preliminary sub-divisions if necessary.

Assuming that $\gamma_2{}^* = \gamma_2$, let $K'_{12} = s_P K_{12}$, where $P = \gamma_2 K_1$, the new vertices being, as usual, different from any of those in K_{01}. Then $(K_{01} - \gamma_2 K_1) . (K'_{12} - \gamma_2 K_1) = 0$, whence $A \mathrel{\bar{\epsilon}} K_{01}$ if $A \epsilon K'_{12} - \gamma_2 K_1$ and $A \mathrel{\bar{\epsilon}} K'_{12}$ if $A \epsilon K_{01} - \gamma_2 K_1$. Let $\gamma_3{}' = s_P \gamma_3$ and let $E_2{}'$ and $E_3{}'$ be the expansions $\gamma_2 K_1 \to K'_{12}$ and $\gamma_3{}' K_2 \to K'_{12}$, defined as in the proof of Theorem 5. Since $A \mathrel{\bar{\epsilon}} K_{01}$ if $A \epsilon K'_{12} - \gamma_2 K_1$ and $A \mathrel{\bar{\epsilon}} K'_{12}$ if $A \epsilon K_{01} - \gamma_2 K_1$, it follows from (14.2) that each elementary expansion in $E_2{}'$ is independent of every elementary expansion in $E_2{}^*$. Therefore $E_2{}'$ is applicable to $E_2{}^*(\gamma_2 K_1) = K_{01}$ and $E_2{}^*$ is applicable to $E_2{}'(\gamma_2 K_1) = K'_{12}$. But
$$E_2{}'(K_{01}) = E_2{}^*(K'_{12}) = K_{01} + K'_{12}.$$
Therefore
$$E_2{}' E_1(\gamma_1 K_0) = E_2{}^* E_3{}'(\gamma_3{}' K_2) \quad (\gamma_1 = \gamma_3{}' = 1 \text{ in } L),$$
and the theorem follows from Theorem 30.

As a special case of Theorem 32 we have the corollary:

COROLLARY. *If two sub-divisions of K_1 expand into K_{01} and into K_{12} respectively, then*
$$K_{12} = D(K_{01}).$$

The relation of equivalence under formal deformations is reflexive and symmetric by definition, and it follows from Theorem 32 that it is also transitive. Therefore the equivalence classes are mutually exclusive and with each of these classes we associate an abstract *infinite nucleus*, or simply a *nucleus*. It is an immediate consequence of the definition that the nucleus of a complex is a combinatorial invariant.

Fillings, perforations, and m-groups may be treated in the same way as expansions, contractions, and nuclei. Let us describe either an elementary expansion or a filling whose order exceeds some given m as an *elementary addition*. Then an addition Σ consists of adding a set of open simplexes Σ to a complex, the set containing one simplex in the case of a filling and two in the case of an expansion. The conditions (14.3) are necessary and sufficient for an elementary addition Σ to be applicable to a complex K, and two elementary additions Σ_1 and Σ_2 will be described as *independent* if, and only if, the conditions (14.4) are satisfied. As in the case of expansions, any set of mutually independent elementary additions may be combined into a *composite addition*. All that was said about expansions, up to and including Theorem 30, obviously applies, with minor alterations, to additions. As the analogue of the corollary to Theorem 31 we have:

THEOREM 33. *If K_0 is transformed into K by a countable sequence of composite additions, and if γ is a given sub-division, then γK_0 is transformed into some stellar sub-division $\sigma\gamma K$ by a countable sequence of composite additions ($\sigma = 1$ in γK_0).*

The proof is similar to the proof of Theorem 31, with Lemma 17 supported by the auxiliary lemma:

If E^n is a given n-element, there is an internal sub-division σE^n, such that the transformation $\dot{E}^n \rightarrow \sigma E^n$ is the resultant of a formal expansion followed by a filling of order n.

Let A^{n-1} be an open $(n-1)$-simplex in $\dot{E}^n$, let aA^{n-1} be the open simplex in E^n with A^{n-1} on its boundary and, after a suitable internal sub-division, let a be inside E^n. Let $E^{n-1} = \dot{E}^n - A^{n-1}$ and $E_1{}^n = E^n - aA^{n-1}$. By Lemma 17, E^{n-1} expands formally into some stellar sub-division $\sigma E_1{}^n$, where $\sigma = 1$ in $\dot{E}_1{}^n$ and therefore in $\dot{E}^n$. Since $\sigma = 1$ in $a\dot{A}^{n-1}$, the required transformation is the resultant of the expansion

$$\dot{E}^n = A^{n-1} + E^{n-1} \rightarrow A^{n-1} + \sigma E_1{}^n,$$

followed by the filling $A^{n-1} + \sigma E_1{}^n \rightarrow A^{n-1} + \sigma E_1{}^n + aA^{n-1} = \sigma E^n$, and the lemma is established.

The theorem now follows from the proof of Theorem 31 with trivial modifications.

Two complexes K_0 and K_1 will be said to have the same m-group if, and only if, there are sub-divisions γ_1 and γ_2 and additions T_1 and T_2 (containing no fillings whose orders do not exceed m) such that

$$T_1(\gamma_1 K_0) = T_2(\gamma_2 K_1).$$

THEOREM 34. *If K_0 has the same m-group as K_1, and K_1 the same m-group as K_2, then K_0 has the same m-group as K_2.*

This follows from the proof of Theorem 32, with Theorem 31 replaced by Theorem 33, Theorem 4 supported by Lemma 7, and the appropriate changes in wording.

As with nuclei, it follows that complexes fall into mutually exclusive classes, two complexes having the same m-group if, and only if, they belong to the same class.

In the next section we shall discuss the relation between the formal and the topological theory of infinite complexes. It will appear that any two complexes have the same m-group, for each value of m, if they have the same homotopy type, in a sense to be defined. The converse holds for complexes of finite dimensionality. In particular, two finite complexes K_0 and K_1 have the same homotopy type if they have the same m-group for $m \geqslant \dim (K_i)+1$, with infinite additions allowed. It follows from Theorem 17 that they have the same m-group in the strictly finite sense of §§ 2–13. The analogous question for nuclei remains open, namely:

If two finite complexes have the same infinite nucleus, can they be interchanged by finite sequences of elementary deformations?

15. *The topology of infinite polyhedra.* By a *closed convex n-cell*† $C^n (n \geqslant -1)$ we shall mean a set of undefined points in a (1-1) correspondence f with the interior and boundary of a convex, polyhedral n-cell $f(C^n)$, in Cartesian n-space X^n, the set C^n being empty if $n = -1$. Two transformations, $f(C^n)$ and $f^*(C^n)$, will be said to determine the same convex n-cell if, and only if, $f^* = Tf$, where T is an affine transformation of X^n into itself. Thus C^n has the affine structure of its image $f(C^n)$ and the usual terms (interior, boundary, simplex, etc.) will mean the same when applied to C^n as to $f(C^n)$. In particular the boundary $\dot{C}^n$ of C^n consists of certain closed convex $(n-1)$-cells. The "cells on" C^n (*Seiten*), defined inductively, consist of C^n by itself if $n = -1$, and of C^n together with the cells on the $(n-1)$-cells in $\dot{C}^n$ if $n \geqslant 0$. Thus C^{-1} is on every cell. By a *polyhedral complex K* we shall mean a set of closed convex cells satisfying

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† *Cf.* O. Veblen, *Analysis Situs* (New York, 1931), 76.

the first two conditions given by Alexandroff and Hopf for a cell-complex†, namely:

(1) *each cell on any cell in K is also in K,*

(2) *the intersection of two closed cells in K is a closed cell on both of them.*

We do not require the dimensionality of the cells in K to have a finite upper bound, and the set of cells which are incident with a given one may have any cardinal number. A partition of K (*i.e.* a rectilinear sub-division), and in particular a simplicial sub-division, is defined as when K is finite.

A simplicial polyhedral complex is obviously isomorphic to some symbolic complex and conversely. Moreover, it follows from Lemma 16 and its addendum that two geometrical complexes which are isomorphic to equivalent symbolic complexes are geometrically equivalent, meaning that they have isomorphic partitions. Also it follows from an inductive argument similar to the proof of Lemma 16, and Theorem 1 in my paper on sub-divisions, that two partitions of the same geometrical complex have a common partition. Thus the elementary theorems on which combinatorial analysis situs is based apply equally well to polyhedral complexes in general as to finite, or locally finite, complexes. Except when a contrast with symbolic complexes is necessary we shall refer to K simply as a complex and, unless the contrary is implied, it is to be understood that any complex to which we refer is simplicial.

By a *topological polyhedron* $P(K)$ we shall mean the set of points in a polyhedral complex K, with the topology defined by the conditions:

(1) *each closed cell in K has the topology natural to its affine geometry,*

(2) *a set of points in K is closed if, and only if, its intersection with each closed cell is closed.*

We enumerate some of the more obvious consequences of this definition. A set of points in K is closed (open) if, and only if, its intersection with each sub-complex $L \subset K$ is closed (open) relative to the topology of $P(L)$. The sum of a finite number, and the intersection of any number, of closed sets is closed. Any sub-complex of K is closed. If $O(X, K)$ is the set of open cells in K (*i.e.* the interiors of closed cells) whose closures meet a given set of points X, then $K - O(X, K)$ is a complex and $O(X, K)$, regarded as a set of points, is open. The definition of closed sets is invariant under partition. The following is not so obvious, namely a topological polyhedron

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† *Topologie*, 126.

is a metric space if, and only if, it is locally finite. For let K be a given complex, which we may suppose to be connected and simplicial. Since it is connected, any two vertices in K are joined by a finite polygonal segment, consisting of edges of K. If K is locally finite, it follows without difficulty that it contains at most an enumerable infinity of vertices, and hence that $P(K)$ is a metric space†. Conversely, let some vertex a be incident with infinitely many simplexes in K, and let $A_1, A_2, \ldots$ be an enumerable sequence of simplexes with a on their boundaries. Assuming that $P(K)$ is metricized, let p_r be a point inside A_r, whose distance from a is less than $1/r$. According to the metric the sequence of points $p_1, p_2, \ldots$ converges to a. But the dimensionality of each simplex in K is finite and no open simplex contains more than one of the points p_r. Therefore the number of these points in any closed simplex is finite and $\{p_r\}$ is a closed set according to the topology of $P(K)$. Therefore $P(K)$ is not a metric space. A similar argument shows that any compact set of points in $P(K)$ is contained in a finite sub-complex of K. Therefore a map in $P(K)$ of any compact space is contained in a finite sub-complex of K, as when K is locally finite.

It is now to be understood that all our polyhedral complexes have this topological structure, in addition to the rectilinear geometry of each cell and the combinatorial structure of the incidence relations.

LEMMA 18. *A transformation $f(K)$, of a complex K into any topological space P, is continuous if, and only if, it is continuous throughout each closed cell in K.*

This follows at once from the definition of closed sets and a standard definition of continuous transformations, namely: $f(K) \subset P$ is continuous if, and only if, $f^{-1}(X)$ is a closed set in K, where X is any closed set in P.

Certain fundamental theorems may now be extended from finite and locally finite complexes to infinite complexes in general.

THEOREM 35. *If K is covered by a given set of open sets, there is a stellar sub-division σK such that $N(a, \sigma K) \subset U(a)$, where a is any vertex in σK and $U(p)$ denotes any one of the given open sets containing a point p.*

We first prove a sharper theorem for finite complexes. Let K be any finite complex which is covered by a given set of open sets and let

$$N(b_\lambda, L) \subset U_\lambda \quad (\lambda = 1, \ldots, m),$$

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† See, for example, Lefschetz, *Topology* (*loc. cit.*), 292.

where L is a sub-complex of K $(L \neq 1)$, $b_1, \ldots, b_m$ are the vertices of L, and U_λ is a particular one of the sets $U(b_\lambda)$. The auxiliary theorem is:

Under these conditions there is a sub-division σ, leaving L unaltered, such that $N(a, \sigma K) \subset U(a)$, where a is any vextex in σK, and

$$N(b_\lambda, \sigma K) \subset U_\lambda \quad (\lambda = 1, \ldots, m).$$

First assume that $N(b_\lambda, K) \subset U_\lambda$ for each value of $\lambda = 1, \ldots, m$. Since every open simplex A in $O(L, K)$ is incident with at least one of the vertices $b_1, \ldots, b_m$, it follows that $N(A, K)$ is contained in one or more of the neighbourhoods $U_1, \ldots, U_m$. Therefore, if a is any vertex of a sub-divison σK, which lies in the open region $O(L)$, covered by $O(L, K)$, the closed star $N(a, \sigma K)$ is contained in one or more of the sets $U_1, \ldots, U_m$. In particular $N(b_\lambda, \sigma K) \subset U_\lambda$. Let $N = N(L, s_L K)$ and let $K_0 = s_L K - O(L, s_L K)$. Then $N \subset O(L)$ and K_0 does not meet L. If $p \in K_0$, the set $U(p) . K_0$ is an open neighbourhood of p in K_0 and, by a standard theorem, there is a stellar sub-division $\sigma_1 K_0$ such that $N(a, \sigma_1 K_0) \subset U(a)$, where a is any vertex in $\sigma_1 K_0$. Also $\sigma_1 = 1$ in L, since L does not meet K_0. Then $N(a, \sigma K)$, with $\sigma = \sigma_1 s_L$, is contained in some U_λ if $a \in \sigma_1 N$, and $N(b_\lambda, \sigma K) \subset U_\lambda$. On the other hand, if $a \in \sigma K - \sigma_1 N$ we have $N(a, \sigma K) = N(a, \sigma_1 K_0) \subset U(a)$. Therefore $N(a, \sigma K) \subset U(a)$ if a is any vertex in σK, $N(b_\lambda, \sigma K) \subset U_\lambda$ and $\sigma = 1$ in L.

It remains to show that there is a sub-division σ_0, leaving L unaltered, such that $N(b_\lambda, \sigma_0 K) \subset U_\lambda$ $(\lambda = 1, \ldots, m)$. We shall assume, for convenience, that none of the sets U_λ coincides with K. This involves no loss of generality. For, if $K = L$ the theorem is trivial, and, if $K \neq L$, $U_\lambda = K$, we remove a point of $K - L$ from U_λ. After a preliminary sub-division, if necessary, let $N(L, K)$ be a normal simplicial neighbourhood of L. Then each closed simplex in K is of the form AB, where $A . L = 1$ and $B \subset L$ (possibly $A = 1$ or $B = 1$). This being so, let K be imbedded in Euclidean metric space R^n. If $B = b_\lambda B_0$ is any closed simplex in L (other than 1), the Euclidean distance $\delta(B, K - U_\lambda)$, from B to the closed set $K - U_\lambda$, is positive. Let ρ be the least of these distances, calculated for all the vertices of B and all the simplexes in L. Then

$$0 < \rho \leqslant \delta(B, K - U_\lambda)$$

if B contains b_λ. Now apply the sub-division s_L to K, placing the centre of each star $s_L(AB)$, in which $B \neq 1$, at a distance less than ρ from B. Let $A' B = A' b_\lambda B_0$ be any principal closed simplex in $N(b_\lambda, s_L K)$, where $A' . L = 1$ and $B_0 \subset L$. If $A' = 1$ we have $A' B = B \subset U_\lambda$. Otherwise

$A' = a_0 \dots a_k$, where a_i is the centre of a star $s_L(A_i B)$. Since $\delta(a_i, B) < \rho$, all the vertices of $A'B$ are contained in the sub-set of R^n given by $\delta(p, B) < \rho$. But the latter is a convex region† and therefore contains $A'B$. That is to say, $\delta(p, B) < \rho$ if $p \epsilon A'B$, and since $\rho \leqslant \delta(B, K - U_\lambda)$ we have $A'B \subset U_\lambda$. Therefore $N(b_\lambda, s_L K) \subset U_\lambda$ and the auxiliary theorem is established.

In proving the main theorem we assume that we can select a particular one from those of the given neighbourhoods containing each closed set which is in at least one of them‡. This being so, let K^n be the n-dimensional skeleton of K and assume that, if $n \leqslant m$ $(m \geqslant -1)$, there is a stellar sub-division $\sigma_n K^n$ such that:

(1) $\sigma_p = \sigma_n$ in K^p if $p < n$,

(2) $N(a, \sigma_m K^m) \subset U_a$, where a is any vertex in $\sigma_m K^m$ and U_a is a particular one of the given sets $U(a)$.

If A^{m+1} is any closed $(m+1)$-simplex in K^{m+1}, the sets $U(p) . A^{m+1}$ are open relative to A^{m+1}. It follows from the auxiliary theorem, with $\sigma_m A^{m+1}$ and $\sigma_m \dot{A}^{m+1}$ taking the place of K and L, that there is a sub-division $\sigma^* \sigma_m A^{m+1}$, leaving $\sigma_m \dot{A}^{m+1}$, and hence $\sigma_m K^m$, unaltered, such that

$$N(a, \sigma^* \sigma_m A^{m+1}) \subset U(a)$$

if $a \epsilon \sigma^* \sigma_m (A^{m+1} - \dot{A}^{m+1})$, and $N(b, \sigma^* \sigma_m A^{m+1}) \subset U_b$ if $b \epsilon \sigma_m \dot{A}^{m+1}$. Let σ_{m+1} be the resultant of σ_m, followed by all the sub-divisions σ^* corresponding to the various $(m+1)$-simplexes in K^{m+1}. Then $\sigma_{m+1} = \sigma_m$ in K^m, whence $\sigma_{m+1} = \sigma_n$ in K^n if $n < m$. If $b \epsilon \sigma_{m+1} K^m$, any closed simplex bA, in $N(b, \sigma_{m+1} K^{m+1})$, is contained in $\sigma_{m+1}(K^m + A^{m+1})$, where A^{m+1} is some closed simplex in K^{m+1} (possibly $bA \subset \sigma_{m+1} K^m$, in which case A^{m+1} may be arbitrary). Therefore $bA \subset U_b$, and it follows that $N(b, \sigma_{m+1} K^{m+1}) \subset U_b$ if $b \epsilon \sigma_{m+1} K^m$. Finally, with each vertex a, in $\sigma_{m+1} K^{m+1} - \sigma_{m+1} K^m$, we associate a particular neighbourhood U_a, selected from the neighbourhoods $U(a)$ which contain $N(a, \sigma_{m+1} K^{m+1})$.

The conditions are now as before, with m replaced by $m+1$. If we begin with $\sigma_{-1} = 1$, it follows inductively that they are satisfied by some sub-division σ_m for all values of m. Taking σ to be the sub-division given by $\sigma A = \sigma_n A$ if $A \subset K^n$, let a be any vertex in σK. Any closed simplex aA, in $N(a, \sigma K)$, is contained in $\sigma_n K^n$ for $n = \dim(aA)$. Therefore $aA \subset U_a$. It follows that $N(a, \sigma K) \subset U_a$, which is one of the given open sets, and the theorem is established.

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† Alexandroff and Hopf, *Topologie*, 598.

‡ This assumption is justified, without an appeal to the axiom of choice, if no point is in an infinite number of the given sets. For in this case the intersection of each sub-set of the given sets is open. Let all these intersections be included among the given sets. From those containing any closed set which is in at least one of them we then select that one of the given sets which is contained in all the others. This condition is satisfied in the corollary to Theorem 35 and in Theorem 36 below.

Let X_1 and X_2 be closed sets in a complex K and let $U_i = K - X_i$. If X_1 and X_2 have no common point, each point in K is either in U_1 or U_2 or both. By Theorem 35 there is a sub-division σK such that $N(a, \sigma K) \subset U_i$ ($i = 1$ or 2) and hence does not meet both X_1 and X_2, where a is any vertex in σK. That is to say $N(X_1, \sigma K)$ does not meet $N(X_2, \sigma K)$, where $N(X_i, \sigma K)$ stands for the totality of closed simplexes in σK which meet X_i. Clearly $N(X_i, \sigma K)$ is the closure of the open neighbourhood $O(X_i, \sigma K)$, and we have the corollary to Theorem 35:

COROLLARY. *A topological polyhedron is a normal topological space*[†].

THEOREM 36. *Any map of a complex K in a simplicial complex L is homotopic to a simplicial map of σK in L, where σ is a suitable sub-division of K.*

If $f(K) \subset L$ is a given map, $f^{-1}\{O(B, L)\}$ is an open set in K, where B is any open simplex in L. Each point in K is contained in at least one of these sets and, by Theorem 35, there is a sub-division σK such that $N(a, \sigma K) \subset U(a)$, where a is any vertex in σK and $U(a)$ is one of the sets $f^{-1}\{O(B, L)\}$. It follows from the argument used in the finite case that f is deformable into a simplicial map of σK in L, and from Lemma 18 that the deformation is continuous.

As in the finite case, the deformation f_t $(0 \leqslant t \leqslant 1)$ of a given map $f_0(K) \subset L$ into a simplicial map $f_1(\sigma K)$ may be chosen so that the "trajectory" $p_t = f_t(p_0)$, of any point p_0, is a rectilinear segment in the closure of the open simplex which contains $f_0(p_0)$.

THEOREM 37. *Let $f_0(K)$ be a map of a complex K in any topological space P, let L be a sub-complex of K and let g_t be a deformation of the map $f_0(L)$ into $g_1(L)$ $(0 \leqslant t \leqslant 1;\ g_0 = f_0$ in $L)$. Then there is a deformation of the complete map $f_0(K)$ which coincides with g_t in L.*

Let K^n be the complex consisting of L together with all the simplexes in $K - L$ whose dimensionalities do not exceed n. It follows from the same argument as when K is finite[†] that $g_t(L)$ $[=f_t^{-1}(K^{-1})]$ can be extended to a deformation $f_t^n(K^n)$ $(n=-1, 0, 1, \ldots)$, such that $f_t^m=f_t^n$ in K^n if $m>n$. The function $f(p, t)$, defined by

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[†] Alexandroff and Hopf, *Topologie*, 68.

$$f(p, t)=f_t^n(p) \text{ if } p \in K^n,$$

may be regarded as a transformation of the product complex

$$K_{01}=K\times\langle 0, 1\rangle$$

into P. This transformation is continuous in $K^n\times\langle 0, 1\rangle$ for each value of n, and is therefore continuous throughout each closed cell in K_{01}. Therefore it is continuous throughout K_{01}, by Lemma 18, and $f_t(K)$, given by $f_t(p)=f(p, t)$ $(p \in K)$, may be taken as the required deformation of $f_0(K)$.

As a corollary to Theorem 37 we have:

COROLLARY. *Theorem* 15 (§ 8 *above*) *applies equally well to infinite as to finite complexes.*

For, in the presence of Theorem 37, Hurewicz's argument, referred to in the proof of Theorem 15, shows that the condition A implies R_L. That is to say, if every cell in K whose boundary lies in L is homotopic, with its boundary fixed, to a cell in L (condition A), then L is a retract by deformation relative to itself (condition R_L). The remaining implications: R_L implies all the other conditions, R implies B and B implies A, are valid if K and $L\subset K$ are any topological spaces, the first two obviously and the last one by the argument given in § 8, which does not depend on the special nature of K and L.

LEMMA 19. *If a complex* K_0 *expands into* K *it is a retract by deformation of* K.

Let the transformation $K_0\rightarrow K$ be the resultant of a sequence of composite expansions $\bar{e}_1, \bar{e}_2, \ldots$, and let $K_n=\bar{e}_n K_{n-1}$ $(n=1, 2, \ldots)$. It is obvious that K_0 is a retract by deformation of K_1 and hence, by induction on n, of K_n. Therefore any cell in K_n whose boundary lies in K_0 is homotopic, with its boundary fixed, to a cell in K_0. Any cell in K, being a map of a compact space, is contained in a finite sub-complex of K, and hence in K_n for some value of n. Therefore any cell in K whose boundary lies in K_0 is homotopic, with its boundary fixed, to a cell in K_0, and the lemma follows from the corollary to Theorem 37.

[†] Alexandroff and Hopf, *Topologie*, 501.

16. *Extension of previous results.* It will now be clear, with a few indications, that many of the main results in §§3–12 can be extended to infinite complexes. Everything in §3 can be taken over provided that the theorems and proofs are restated in terms of expansion rather than contraction. In particular, a collapsible complex is a complex of the form $E(a)$, where a is a single vertex and E is an expansion. Theorem 4 is true of infinite complexes, as we have already observed. We drop Theorem 6, and the rest of §§4 and 5 is replaced by §14. Everything in §6 can be extended with trivial modifications, and we come to Theorem 12 in §7. First notice that, unless $K^{m+1}=K^m$, the transformation $K^m \rightarrow K^{m+1}$ is a composite filling of order $m+1$, where K^n is the n-dimensional skeleton of a given complex K. Therefore K^m and K^{m+1} have the same m-group, and in extending Theorem 12 we may assume that our complexes are at most 2-dimensional. Secondly, in any connected graph g there is a tree which contains all the vertices of g. For, since g is connected, any two vertices in g are contained in at least one finite, connected sub-graph. Let us define the distance between them as the minimum number of edges in such a graph, this minimum being attained by a simple segment. Let T_n be a tree in g containing those, and only those, vertices of g whose distances from a given vertex do not exceed n. Let each vertex whose distance from some vertex in T_n is unity be joined to T_n by a single edge, and let T_{n+1} be the graph consisting of T_n together with these edges. Then T_{n+1} is obviously a tree satisfying the same conditions as T_n, with n replaced by $n+1$, and a tree $T=T_1+T_2+\dots$ containing all the vertices in g is defined by induction on n. Moreover, this argument shows that any tree is collapsible.

The first part of the extended Theorem 12, namely that all complexes in the same 2-group have the same fundamental group, follows from an argument similar to the proof of Lemma 19. To prove the converse, let T be a tree containing all the vertices in K $(T \subset K)$. By the corollary to Lemma 4 and Theorem 2 we may shrink T into a point in $s_T{}^2 K$, obtaining a system of generators and relations for $\pi_1(K)$. Tietze's method of transforming two systems of generators and relations of the same abstract group into the same system can obviously be extended from finite to infinite systems, and in all remaining details the proof is the same as in the finite case.

As a complementary theorem to Theorem 12 we have

THEOREM 38. *Any group is isomorphic to the fundamental group of some complex.*

Any group G can be represented by a system of generators and relations. For we can take every element in G as a generator and as a set of relations we can take the "multiplication table". Then a complex having G as its fundamental group can be constructed by the method used when the set of relations is finite†.

We proceed to Lemma 8 and Theorem 13. Let K_0 and K_1 be two complexes of at most n dimensions which have the same n-group. Then

$$T_0(\gamma_0 K_0) = T_1(\gamma_1 K_1) = K_{01}, \text{ say,}$$

where T_0 and T_1 consist of expansions and of fillings whose orders exceed n. As in Lemma 8, we replace each elementary expansion of the form

$$K \rightarrow K + aA^{m-1},$$

where $a\dot{A}^{m-1} \subset K$, $A^{m-1} \not\subset K$ and $m > n$, by the special filling $K \rightarrow K + A^{m-1}$ of order $m-1$, followed by the filling $K + A^{m-1} \rightarrow K + aA^{m-1}$. If $m = n+1$, the n-element $a\dot{A}^{m-1}$ is contained in K^n, the n-dimensional skeleton of K. Therefore the transformation $K^n \rightarrow K^n + A^{m-1}$ is a special filling of K^n. Similarly $K^n \rightarrow K^n + aA^{k-1}$ $(k \leqslant n)$ is an expansion of K^n if $K \rightarrow K + aA^{k-1}$ is an expansion of K. After modifying the transformations T_0 and T_1 in the way just described, we find that, omitting all the fillings whose orders exceed n,

$$T_0{}^n(\gamma_0 K_0) = T_1{}^n(\gamma_1 K_1) = K_{01}^n, \tag{16.1}$$

where K_{01}^n is the n-dimensional skeleton of K_{01} and $T_i{}^n$ consists of expansions whose orders do not exceed n and of special fillings of order n. This is the generalization of Lemma 8.

If an elementary expansion of order $k \leqslant n$ is applicable after a filling f^n, of order n, it is applicable before f^n. Therefore (16.1) may be exhibited in the form

$$F_0{}^n E_0(\gamma_0 K_0) = F_1{}^n E_1(\gamma_1 K_1) = K_{01}^n,$$

where E_i is an expansion whose order does not exceed n and $F_i{}^n$ consists of special fillings of order n. Replacing each filling $K \rightarrow K + A^n$ in F_i by the expansion $K \rightarrow K + aA^n$, where $a\dot{A}^n \subset K$, we have an expansion of $E_i(\gamma_i K_i) = K_i{}^*$, say, into a complex K_i^{n+1}. By Lemma 19, $K_i{}^*$ is a retract of K_i^{n+1} and, since $K_i{}^* \subset K_{01}^n \subset K_i^{n+1}$, the complex $K_i{}^*$ is a retract of K_{01}^n, in general not by deformation. If $K \rightarrow K + A^n$ is any special filling in $F_i{}^n$ it follows that $\dot{A}^n$, being $(n-1)$-dimensional and hence in $K_i{}^*$, bounds a cell in $K_i{}^*$, since it bounds a cell in K_{01}^n. Therefore

$$K_{01}^n = D_0{}^*(K_0{}^* + \Sigma_0{}^*) = D_1{}^*(K_1{}^* + \Sigma_1{}^*),$$

---

† O. Veblen, *Analysis Situs* (*loc. cit.*), chap. V. § 24.

as in Theorem 13, where Σ_i^* is a cluster of n-spheres, in general infinite, attached to K_i^*. Clearly

$$K_i^*+\Sigma_i^*=D_i(K_i+\Sigma_i),$$

where Σ_i is a cluster of n-spheres attached to K_i, and we have

$$K_1+\Sigma_1=D(K_0+\Sigma_0),$$

as in Theorem 13. Moreover, the order of the deformation D need not exceed $n+1$.

Everything in § 8 applies, with minor alterations, to infinite complexes. In particular, if L is a retract by deformation of K it has the same m-group as K for every value of m. Half of Theorem 17, in § 9, applies to infinite complexes in general, namely the theorem that two complexes have the same m-group if they are of the same homotopy type. The converse applies to infinite complexes of finite dimensionality.

In the definition of the ring $\mathfrak{B}_n(K, p)$ and in Theorems 18 and 19 (§§ 10 and 11) K may be not only an infinite complex, but any connected, locally 0-connected† topological space. On the other hand, the proof of Theorem 20 seems to break down completely if K is infinite, even if it is locally finite. For, if the matrix $\|r_{ij}\|$ has infinitely many rows and columns, the process of reduction involves an infinite sequence of formal deformations, which, so far as I can see, might require an infinite sequence of contractions, each of which could be applied only after its predecessors.

Theorem 21 is a consequence of Theorem 20 and conversely. Therefore the two further questions left open are:

(1) *Are two complexes of infinite dimensionality of the same homotopy type if their m-groups are the same for all values of m?*

(2) *Have two infinite complexes the same nucleus if they are of the same homotopy type and if their fundamental group satisfies the condition imposed in* § 11?

Theorems 22 and 23 in § 12 can be extended, in a modified form, to locally finite sub-complexes of infinite manifolds. An n-dimensional manifold, finite or infinite, is defined as before and may be bounded or unbounded. An infinite unbounded manifold will be described as *open* (by contrast with a closed, or finite unbounded manifold). If K is a sub-complex of a manifold M (K may be infinite if M is infinite), a sub-complex

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† Lefschetz (*loc. cit.*), 91.

$U(K, M)$ of M will be called a regular neighbourhood of K if it is a manifold and if γK expands into $\gamma U(K, M)$, where γ is a suitable sub-division of $U(K, M)$. In general $U(K, M)$ is a bounded manifold and Theorem 23, as it stands, is false if the complexes are infinite. For example, if K is a point in Euclidean space R^n, a regular neighbourhood $U(K, R^n)$ might be an n-element, or some infinite bounded manifold, or the whole of R^n. However, if we take M to be a geometrical complex of the kind described in § 15, the following modification of Theorem 23 is true:

THEOREM 39. *The interiors*† *of two regular neighbourhoods of the same sub-complex of the same manifold are in a* (1-1) *semi-linear correspondence.*

In the first corollary to Theorem 23, and elsewhere, n-elements must be replaced by open (semi-linear) n-cells. Then Theorem 39 and the generalization of Theorem 22 follow from the arguments in § 12, supported by the following lemma:

LEMMA 20. *Let* $M_0, M_1, \ldots$ *be an infinite sequence of n-dimensional polyhedral manifolds such that* $M_i \to M_{i+1} = M_i + E_i^n$ *is a regular expansion. Then* $I(M_0)$ *is in a* (1-1) *semi-linear correspondence with*

$$M = \sum_{i=0}^{\infty} I(M_i),$$

where $I(M_i)$ *stands for the interior of* M_i.

It will be sufficient to outline a proof, the details of which do not involve anything new. Let f_i be a (1-1) semi-linear map of M_i on M_0. Then M_{i+1} can be mapped on M_0 in a (1-1) semi-linear transformation f_{i+1}, which coincides with f_i except in $E_i^n + U_i$, where U_i is an arbitrarily chosen regular neighbourhood in M_i of $E_i^{n-1} = M_i . E_i^n = \dot{M}_i . \dot{E}_i^n$. Therefore we may assume that $f_{i+1} = f_i$ in L_i^n, where L_i^n is an open manifold in M_i, such that, treating M and $I(M_0)$ as polyhedra, we have

(1) $L_i^n \subset L_{i+1}^n$,

(2) $M = \sum_{i=0}^{\infty} L_i^n$ and $I(M_0) = \sum_{i=0}^{\infty} f_i(L_i^n)$.

If we begin with f_0 as the identical map of M_0 on itself and

$$L_0^n = M_0'' - O(\dot{M}_0'', M_0''),$$

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† The interior $I(U)$ of a bounded polyhedral manifold U is obviously an open manifold in the sense that any rectilinear, simplical covering of $I(U)$ satisfies the combinatorial condition for an open manifold.

the map f_i, subject to the condition $f_i = f_j$ in L_i^n if $i < j$, is defined inductively for each i, and the required map, $f(M) = I(M_0)$, is given by $f(p) = f_i(p)$ if $p \in L_i^n$.

If $\Sigma(K, M)$ is taken to mean the simplicial space determined by some complex covering the interior of $U(K, M)$, the generalization of Theorem 24 presents no difficulty. Theorem 25 is true of Euclidean complexes† K and L, provided that they have the same "Euclidean nucleus", meaning that sub-divisions of K and L expand into the same Euclidean complex (it follows from the proof of Theorem 32 that this relation between K and L is transitive). The corollaries to Theorem 25 depend on Theorem 21, concerning which we are ignorant. Lemma 12 and Theorem 26 generalize automatically. If the questions Q. 2 and Q. 3 of § 10 are restated in terms of Euclidean complexes and nuclei, and of the interiors of bounded manifolds M, M_i and $M + M_i$, such that $M . M_i = \dot{M} . \dot{M}_i = E_i^{n-1}$, Theorem 27 follows from the arguments in § 12 and from

THEOREM 40: *A bounded manifold and its interior have the same Euclidean nucleus.*

If M is any bounded n-dimensional manifold the theorem will follow if we can show that some Euclidean complex expands both into a sub-division of M and into a rectilinear complex covering $I(M)$. After an initial subdivision, if necessary (*e.g.* $s_{\dot{M}}$), let $N(\dot{M}, M)$ be a normal simplicial neighbourhood of $\dot{M}$. Then, if we write M' for the first derived complex of M, $N(\dot{M}', M')$ is geometrically equivalent to the topological product $\dot{M}'_{01} = \dot{M}' \times \langle 0, 1 \rangle$. For let A_1^k, A_2^k, ... be the k-simplexes in $\dot{M}$ $(k = 0, \ldots, n-1)$ and let E_i^{n-k} and F_i^{n-k-1} be the closed cells dual to A_i^k in the complexes M^* and $\dot{M}^*$, dual to M and $\dot{M}$ (as usual the cells of M^* are composed of simplexes in M'). Let f_0 be the map of $\dot{M}'$ on $\dot{M}' \times 1$ given by $f_0(p) = p \times 1$ $(p \in \dot{M}')$, and let f_{r+1} $(0 \leqslant r \leqslant n-1)$ be a (1-1) semi-linear map of

$$N_{r+1} = \dot{M}' + E_1^{r+1} + E_2^{r+1} + \ldots \quad \text{on} \quad (\dot{M}' \times 1) + (\dot{M}'^r \times \langle 0, 1 \rangle),$$

where $\dot{M}'^r = F_1^r + F_2^r + \ldots$, which coincides with f_0 in $\dot{M}'$ and is such that $f_{r+1}(E_j^s) = F_j^{s-1} \times \langle 0, 1 \rangle$ $(1 \leqslant s \leqslant r+1;\ j = 1, 2, \ldots)$. If $r < n-1$, it follows from theorems on sub-division to which we have previously referred‡ that f_{r+1} can be extended throughout each cell E_i^{r+2} to give a map f_{r+2}, satisfying the same conditions as before with r replaced by $r+1$. It follows from induction on r that such a map exists for $r=n-1$, and f_n is the required map of $N(\dot{M}', M)$ on $\dot{M}'_{01}$.

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† *I.e.* locally finite complexes whose dimensionalities are finite.

‡ *Cf.* the proof of Theorem 6, § 4.

Let $M_0 = M' - O(\dot{M}', M')$. Then it follows from the last paragraph that we may replace M' by $M_0 + \dot{M}'_{01}$, where each point p in $\dot{M}_0$ is identified with $p \times 0$ in $\dot{M}'_{01}$. Then M_0 expands geometrically into $M_0 + \dot{M}'_{01}$, by Theorem 8. Let

$$M_i = M_0 + \{\dot{M}_0' \times \langle 0, i/(i+1) \rangle\} \quad (i = 1, 2, \ldots).$$

By Theorem 8 each of the transformations $M_i \rightarrow M_{i+1}$ is a geometrical expansion. Their resultant is an expansion of M_0 into a polyhedral complex covering the interior of $M_0 + \dot{M}'_{01}$, and the theorem is established.

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ON SUMMABLE TRIGONOMETRICAL SERIES: AN EXTENSION OF UNIQUENESS THEOREMS

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Introduction.

The aim of this paper is to investigate the nature of a summable trigonometrical series

$$\tfrac{1}{2}a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)$$

which is not a Fourier series. We assume that the series is summable (C, k) to a finite sum function, and Perron-integrable in (a, b). The condition that the sum function is finite can be relaxed in the ordinary way for an enumerable set of points. Then the main result (§ 17) states that there is only a reducible* set $\mathfrak{E}$ of singular points. In every interval complementary to $\mathfrak{E}$ the given series is a restricted Fourier series (Hobson, 2, 686–692, Wolf, 2), *i.e.* it behaves, so far as summability (C, k) is concerned, like the Fourier series of the sum function. At an isolated point x_0 of $\mathfrak{E}$, the singular part of the series is of the form

$$\sum_{m=0}^{2m+1<k} A_m \sum_{n=1}^{\infty} n^{2m+1} \sin n(x-x_0).$$

The series
$$\sum_{n=1}^{\infty} n^{2m+1} \sin nx$$

are summable (C, k) to zero, if $k > 2m+1$, for all x and, in fact, the summability is uniform in any interval excluding $x = 0$.

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* *I.e.*, a set whose derivative is enumerable.

This result easily yields (§ 21) an asymptotic formula for the allied series of the given series at $x = x_0$, namely

$$\sum_{n=1}^{[\omega]} (a_n \sin nx - b_n \cos nx) = -\sum_{n=1}^{\omega} \sum_{m=0}^{2m+1<k} A_m n^{2m+1} + o(\omega^2).$$

Hence a necessary and sufficient condition for the non-existence of isolated points of $\mathfrak{E}$ is

$$\sum_{n=1}^{[\omega]} (a_n \sin nx - b_n \cos nx) = o(\omega^2).$$

But a reducible set without isolated points does not exist, so that this is a necessary and sufficient condition for the validity of the uniqueness theorem (Verblunsky 4). Then the given series is a Fourier series.

The paper is divided into two chapters of which only the second one deals actually with trigonometrical series. The first chapter is interesting in itself as an investigation of the nature of a function whose $2m$-th symmetrical de la Vallée Poussin derivative is given.

I conceived the idea of the paper and first made its results public in the Conversation Class of Prof. Hardy and Prof. Littlewood in the spring of 1937. I am glad to acknowledge my debt to Prof. Hardy and Mr. A. C. Offord for many stimulating conversations and to thank them for the trouble that they took in revising and improving my manuscript. My stay in Cambridge was made possible by a grant received from the Denis fund of the Czecho-Slovak Government.

I.

1. If there are numbers A_k such that

$$(1.1) \qquad \overline{D}^{2m} F(x) = \overline{\lim_{h \to 0}} \frac{F(x+h) + F(x-h) - 2 \sum_{k=0}^{m-1} h^{2k} A_k/(2k)!}{2 \,.\, h^{2m}/(2m)!}$$

is finite, then we say that $\overline{D}^{2m} F(x)$ is the upper symmetrical de la Vallée Poussin derivative of $F(x)$ at the point x; and we define the lower symmetrical d.l.V.P. derivative similarly. If $\overline{D}^{2m} F(x) = \underline{D}^{2m} F(x)$ we say that $F(x)$ has the symmetrical d.l.V.P. derivative $D^{2m} F(x)$. It is evident that, if $D^{2m} \overline{F}(x)$ and $\underline{D}^{2m} F(x)$ are both finite, then $F(x)$ has derivatives $D^2 F(x)$, $D^4 F(x)$, ..., $D^{2m-2} F(x)$ and $A_0 = F(x)$, $A_2 = D^2 F(x)$, ...

Further, let

$$(1.2) \qquad \phi_{2k}(x, h) = \frac{1}{h^{2k}} \sum_{l=0}^{2k} (-1)^l \binom{2k}{l} F\big(x - (k-l)h\big),$$

be the Riemann symmetrical quotient.

2. LEMMA. *If $F(x)$ is continuous in (a, b), if*

$$(2.1)\quad \theta_m(x, h) = \frac{F(x+h)+F(x-h)-2F(x)-2\sum\limits_{k=1}^{m-1}\dfrac{h^{2k}}{(2k)!}D^{2k}F(x)}{2h^{2m}/(2m)!}$$

and

$$(2.2)\qquad \lim_{h\to 0} h\theta(x, h) = 0 \quad (a<x<b),$$

then

$$(2.3)\qquad \frac{1}{|h|}\left|\phi_{2m-2}(x, 2h)-\phi_{2m-2}(x, h)\right| \leqslant M(x),$$

where $M(x)$ is finite and depends only on x.

It is clearly sufficient to show that the limit of (2.3) as $h\to 0$ is finite. When we insert the expression

$$(2.4)\quad F(x+h)+F(x-h) = 2\sum_{k=0}^{m-1}\frac{h^{2k}}{(2k)!}D^{2k}f(x)+2\frac{h^{2m}}{(2m)!}\theta_m(x, h),$$

obtained from (2.1), into (1.2), we get

$$\begin{aligned}
&\phi_{2m-2}(x, h)\\
&= \frac{1}{2h^{2m-2}}\sum_{l=0}^{2m-2}(-1)^l\binom{2m-2}{l}\Big[F\big(x+(m-l-1)h\big)+F\big(x-(m-l-1)h\big)\Big]\\
&= \frac{1}{h^{2m-2}}\sum_{k=0}^{m-1}\frac{D^{2k}F(x)}{(2k!)}h^{2k}\sum_{l=0}^{2m-2}(-1)^l\binom{2m-2}{l}(m-l-1)^{2k}\\
&\qquad\qquad +\frac{h^2}{(2m)!}\sum_{l=0}^{2m-2}(-1)^l\binom{2m-2}{l}(m-l-1)^{2m}\theta_m\big((m-l-1)h\big)\\
&= D^{2m-2}F(x)+\frac{h^2}{(2m)!}\sum_{l=0}^{2m-2}(-1)^l\binom{2m-2}{l}(m-l-1)^{2m}\theta_m\big(x_1(m-l-1)h\big).
\end{aligned}$$

Hence, by (2.2),

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{h}\left[\phi_{2m-2}(x, 2h)-\phi_{2m-2}(x, h)\right]\\
&= \frac{h}{(2m)!}\sum_{l=0}^{2m-2}(-1)^l\binom{2m-2}{l}(m-l-1)^{2m}\Big[4\theta_m\big(x, 2(m-l-1)h\big)\\
&\qquad\qquad -\theta_m\big(x, (m-l-1)h\big)\Big]\\
&= 0.
\end{aligned}$$

3. *Definition.* We shall say that $F(x)$ satisfies conditions A if it is

(i) continuous in (a, b),

(ii) such that

$$\lim_{h\to 0} h\theta_m(x, h) = 0, \tag{3.1}$$

where $\theta_m(x, h)$ is given by (1 . 1), for all x of $(a, b)-\mathfrak{H}$, $\mathfrak{H}$ being an enumerable set.

LEMMA. *If $F(x)$ satisfies conditions A, then there is a non-diminishing sequence of closed sets $\mathfrak{A}_n^{(2m-2)}$, such that*

$$(a, b)-\mathfrak{H} \subset \lim_{n\to\infty} \mathfrak{A}_n^{(2m-2)}, \tag{3.2}$$

and $\phi_{2m-2}(x, 2^{-k})$ converges, as $k \to \infty$, uniformly in any set $\mathfrak{A}_n^{(2m-2)}$.

Let

$$\mathfrak{B}_{n,k}^{2m-2} = \underset{x}{E}[|\phi_{2m-2}(x, 2^{-k})-\phi_{2m-2}(x, 2^{-k-1})| \leqslant 2^{-k}n]. \tag{3.3}$$

This set is closed. The same is also true of

$$\mathfrak{A}_n^{(2m-2)} = \prod_{k=0}^{\infty} \mathfrak{B}_{n,k}^{(2m-2)}. \tag{3.4}$$

If $n \geqslant M(x)$, then, by (2 . 3),

$$x \in \mathfrak{A}_n^{(2m-2)}.$$

Hence
$$(a, b)-\mathfrak{H} \subset \lim_{n\to\infty} A_n^{(2m-2)}.$$

4. LEMMA. *If $F(x)$ satisfies conditions A, then there is a non-diminishing sequence of closed sets $\mathfrak{A}_n$ such that*

$$(a, b)-\mathfrak{H} \subset \lim_{n\to\infty} \mathfrak{A}_n, \tag{4.1}$$

and

$$\lim_{k\to\infty} \phi_{2l}(x, 2^{-k}) = D^{2l} F(x), \tag{4.2}$$

for $l = 1, 2, \ldots, m-1$, uniformly for x in $\mathfrak{A}_n$.

If (3 . 1) is satisfied, then $\theta_{m-1}(x, h)$, $\theta_{m-2}(x, h)$, ..., $\theta_1(x, h)$ satisfy a similar condition. Hence the preceding lemma is true also if we replace

m by $m-1$, $m-2$, ... Then

$$\mathfrak{A}_n = \prod_{l=1}^{m-1} \mathfrak{A}_n^{(2l)}$$

has all the properties required.

5. Lemma. *If* $\{\mathfrak{G}_n\}$ *is a non-diminishing sequence of closed sets in* (a, b) *such that the set complementary to* $\lim\limits_{n\to\infty} \mathfrak{G}_n$ *is enumerable, then the set* $\mathfrak{D}$ *of all isolated points of an infinity of* $\mathfrak{G}_n$ *is reducible.*

Since the isolated points of any set are enumerable, the isolated points of all $\mathfrak{G}_n$ are enumerable. We denote by $\mathfrak{C}_n$ the set of points consisting of $C(\mathfrak{G}_n)$ and of all isolated points of $\mathfrak{G}_n$. Then $\mathfrak{C}_n$ is a non-increasing sequence of open sets. Its inner limiting set

$$\lim_{n\to\infty} \mathfrak{C}_n$$

consists of points which belong to

$$(a, b) - \lim_{n\to\infty} \mathfrak{G}_n,$$

an enumerable set, or else are isolated points of an infinity of $\mathfrak{G}_n$. Hence the inner limiting set $\lim\limits_{n\to\infty} \mathfrak{C}_n$ is enumerable. A well-known result (Hobson, 1, 140) asserts that an enumerable inner limiting set is reducible.

6. Lemma. *If* $F(x)$ *satisfies conditions* A, *then in any perfect set* $\mathfrak{P} \subset (a, b)$ *there is a portion* $\mathfrak{P} \,.\, (c, d)$ *in which the* $D^{2l} F(x)$ *are continuous for* $l = 1, 2, \ldots, m-1$. *In particular, there is a set of intervals* $\{I_m\}$, *everywhere dense in* (a, b), *in which the* $D^{2l} F(x)$ *are continuous.*

In other words, the functions $D^{2l} F(x)$ are point-wise discontinuous on every perfect set (*cf.* Lebesgue, 99).

Let $$\mathfrak{H} = a_1 + a_2 + a_3 + \ldots,$$

and $$\mathfrak{H}_m = \sum_{k=1}^{m} a_k.$$

If $\mathfrak{P}$ is a perfect set, then

$$\mathfrak{P} \,.\, (\mathfrak{A}_n + \mathfrak{H}_n) \to \mathfrak{P}$$

by Lemma 4. A perfect set cannot be a limit of sets which are all non-dense in $\mathfrak{P}$ (*cf.* Hobson, 1, 134–138). Hence there is an index n_0 such that $\mathfrak{P} \,.\, (\mathfrak{A}_{n_0} + \mathfrak{H}_{n_0})$ is dense in a section of $\mathfrak{P}$. Since $\mathfrak{H}_{n_0}$ is a finite set, the same is also true of $\mathfrak{P}.\mathfrak{A}_{n_0}$. This is closed and therefore

$$\mathfrak{P}.\mathfrak{A}_{n_0}.(c, d) = \mathfrak{P}.(c, d)$$

for a certain interval (c, d). By Lemma 4, the $D^{2l}F(x)$ are continuous in $\mathfrak{A}_{n_0}$ and therefore in $\mathfrak{P}.(c, d)$.

7. LEMMA. *If $F(x)$ satisfies conditions A, then*

$$F(x) = \left(\int^x dx\right)^{2m-2} D^{2m-2} F(x)$$

in any interval in which all the $D^{2l} F(x)$ are continuous. In particular, this is true in $\{I_m\}$.

To prove the lemma we use the well-known theorem (Hobson, 2, 671):

If $F(x)$ is continuous in (a, b) and there is a finite integrable function $f(x)$ such that

$$\bar{D}^2 F(x) \geqslant f(x) \geqslant \underline{D}^2 F(x)$$

in (a, b), then

$$F(x) = \left(\int^x dx\right)^2 f(x).$$

In an interval of continuity of $D^2 F(x)$ we therefore have

$$F(x) = \left(\int^x dx\right)^2 D^2 F(x),$$

i.e.

$$\frac{d^2 F(x)}{dx^2} = D^2 F(x).$$

If certain well-known conditions are satisfied (*cf.* Hobson, 1, 359), then

$$\overline{\lim_{h\to 0}} \frac{p''(h)}{q''(h)} \geqslant \overline{\lim_{h\to 0}} \frac{p(h)}{q(h)} \geqslant \underline{\lim}_{h\to 0} \frac{p(h)}{q(h)} \geqslant \underline{\lim}_{h\to 0} \frac{p''(h)}{q''(h)}.$$

Hence, taking

$$p(h) = F(x+h) + F(x-h) - 2F(x) - 2\frac{h^2}{2!} D^2 F(x),$$

$$q(h) = 2\frac{h^4}{4!},$$

we obtain

$$\overline{\lim_{h\to 0}} \frac{D^2 F(x+h)+D^2 F(x-h)-2D^2 F(x)}{h^2}$$

$$\geqslant \underline{\lim_{h\to 0}} \frac{F(x+h)+F(x-h)-2F(x)-2(h^2/2!)D^2 F(x)}{2(h^4/4!)}$$

$$= D^4 F(x) \geqslant \underline{\lim_{h\to 0}} \frac{D^2 F(x+h)+D^2 F(x-h)-2D^2 F(x)}{h^2}$$

for x in an interval of continuity of all $D^{2l} F(x)$.

The theorem quoted now gives

$$F(x) = \left(\int^x dx\right)^2 D^2 F(x) = \left(\int^x dx\right)^4 D^4 F(x).$$

If we repeat this process a sufficient number of times, we deduce the required result.

8. LEMMA. *If* $F(x)$ *satisfies conditions* A *and the functions*

$$D^{2l} F(x), \quad l=1,\ 2,\ \dots,\ n\ (<m),$$

are continuous in the open interval (c, d), *and* $\lim_{x\to c+0} D^{2n} F(x)$ *exists and is different from* $D^{2n} F(c)$, *then* c *belongs to the reducible set* D, *formed by isolated points of an infinity of* $\mathfrak{A}_n$ (*cf. Lemmas* 4 *and* 5). *There is a similar result for the right-hand end point* d.

Remark. This lemma is the keystone of the first chapter. Its proof is elementary, but its fundamental idea is a little difficult to grasp. The following remarks are intended to show its plausibility and so to facilitate the reading of the rigorous proof.

The idea is to show that c is an isolated point of an infinity of $\mathfrak{A}_k$, or, what comes to the same thing, that there is a neighbourhood of c which cannot belong to $\mathfrak{A}_k$, when k is sufficiently large. The sequence $\phi_{2n}(x, 2^{-m})$ converges to $D^{2n} F(x)$ uniformly for $x \subset \mathfrak{A}_k$ when $m\to\infty$. In view of the discontinuity of $D^{2n} F(x)$ on the right of c, it is intuitively obvious that a neighbourhood, not belonging to all $\mathfrak{A}_k$, exists on the right. That this is also true on the other side of c is due to the symmetrical manner of the convergence of $\phi_{2n}(x, 2^{-m})$ which is expressed by (8.2).

Proof. Let us suppose that $c=0$ and that $0<nh\leqslant x$. By Lemma 7 we can apply the mean value theorem to get

$$\phi_{2n}(x,\ h) = D^{2k} F(\xi) \quad (0<x-nh<\xi<x+nh).$$

Hence

$$(8.1)\qquad \lim_{0<nh\leqslant x\to 0} \phi_{2n}(x, h) = D^{2n} F(+0) = D^{2n} F(0)+\alpha.$$

It is essential that the jump of $D^{2n} F(x)$ should exist at $x > c = 0$, and therefore that $\alpha \neq 0$. The function $\phi_{2n}(x, h)+\phi_{2n}(-x, h)$ is an even function of x and h and hence is the sum of a set of expressions of the form

$$c_n\Big(F(u)+F(-u)\Big),$$

where u is of the form

$$px+qh.$$

If we substitute for $F(u)+F(-u)$ the expression from (1.2) and (2.4), we get

$$\begin{aligned}
&\phi_{2n}(x, h)+\phi_{2n}(-x, h)\\
&\quad= \frac{1}{h^{2n}} \sum_{l=0}^{2n} (-1)^l \binom{2n}{l} \Big[F\Big(x+(n-l)h\Big)+F\Big(-x-(n-l)h\Big)\Big]\\
&\quad= \frac{2}{h^{2n}} \sum_{l=0}^{2n} (-1)^l \binom{2n}{l} \sum_{k=0}^{m-1} \frac{\Big(x+(n-l)h\Big)^{2k}}{(2k)!} D^{2k} F(0)\\
&\qquad\qquad + \frac{2}{h^{2n}} \sum_{l=0}^{2n} (-1)^l \binom{2n}{l} \frac{\Big(x+(n-l)h\Big)^{2m}}{(2m)!} \theta_m\Big(0, x+(n-l)h\Big)\\
&\quad= 2D^{2n} F(0)+2 \sum_{l=0}^{2n} (-1)^l \binom{2n}{l} \sum_{k=n+1}^{m-1} \frac{\Big(x+(n-l)h\Big)^{2k}}{h^{2n}(2k)!} D^{2k} F(0)\\
&\qquad\qquad +2 \sum_{l=0}^{2n} (-1)^l \binom{2n}{l} \frac{\Big(x+(n-l)h\Big)^{2m}}{h^{2n}(2m)!} \theta_m\Big(0, x+(n-l)h\Big).
\end{aligned}$$

If $n = m-1$, then the last double sum disappears.

With the further restriction $0 < \frac{1}{2}x \leqslant nh \leqslant x$, we obtain from (2.2)

$$\phi_{2n}(x, h)+\phi_{2n}(-x, h)-2D^{2n}F(0) = O(x^2)+O(x),$$

and finally

$$(8.2)\qquad \lim_{\frac{1}{2}x<nh\leqslant x\to 0} [\phi_{2n}(x, h)+\phi_{2n}(-x, h)] = 2D^{2n} F(0).$$

The relations (8.1) and (8.2) yield

$$(8.3)\qquad \lim_{\frac{1}{2}x<nh\leqslant x\to 0} \phi_{2n}(-x, h) = D^{2n}F(0)-\alpha.$$

We next use the obvious identity

$$(8.4)\quad \phi_{2n}(x, h)-\phi_{2n}(x, k)+\alpha \operatorname{sign} x$$
$$= [\phi_{2n}(x, h)-\phi_{2n}(0, h)]+[\phi_{2n}(0, h)-D^{2n} F(0)]$$
$$+[D^{2n} F(0)-\phi_{2n}(x, k)+\alpha \operatorname{sign} x].$$

If $\epsilon > 0$ is arbitrary, we can find $\delta(\epsilon) > 0$ such that, for $0 < h < \delta(\epsilon)$,

$$|\phi_{2n}(0, h)-D^{2n} F(0)| < \epsilon.$$

Now, h being fixed, there is a $\psi(\epsilon, h) > 0$ such that, for $0 < |x| \leqq \psi(\epsilon, h)$, we have

$$|\phi_{2n}(x, h)-\phi_{2n}(0, h)| < \epsilon$$

and, by (8.1) and (8.3),

$$|D^{2n} F(0)-\phi_{2n}(x, k)+\alpha \operatorname{sign} x| < \epsilon, \quad \tfrac{1}{2}|x| < nk \leqq x \leqq \psi(\epsilon, h).$$

Hence (8.4) gives

$$(8.5)\qquad |\phi_{2n}(x, h)-\phi_{2n}(x, k)+\alpha \operatorname{sign} x| < 3\epsilon$$

for $h < \delta(\epsilon),\ 0 < |x| < \psi(\epsilon, h),\ \frac{1}{2}|x| < nk \leqq |x|$.

Let n_0 be so large that

$$x = 0 \in \mathfrak{A}_{n_0}$$

and let $n_1 > n_0$ be an arbitrary but definite number. We shall show that $x = 0$ is an isolated point of $\mathfrak{A}_{n_1}$. Let m be so large that

$$n_1 \sum_{l=m}^{\infty} 2^{-l} < \frac{|\alpha|}{3} \quad \text{and} \quad 2^{-m} < \delta\left(\frac{|\alpha|}{9}\right).$$

If x is a point of $\mathfrak{A}_{n_1}$, then, by (4.2), (3.2), and (3.1), for $m' > m$,

$$(8.6)\qquad |\phi_{2n}(x, 2^{-m})-\phi_{2n}(x, 2^{-m'})| \leqq n_1 \sum_{l=m}^{\infty} 2^{-l} < \frac{|\alpha|}{3}.$$

We now return to (8.5) in which we put

$$\epsilon = \frac{|\alpha|}{9}, \quad h = 2^{-m}, \quad k = 2^{-m'}, \quad |x| < \frac{n}{2^m}.$$

We determine m' by the inequality

$$(8.7)\qquad \frac{|x|}{2n} \leqq 2^{-m'} < \frac{|x|}{n}.$$

Then (8.5) yields

$$|\phi_{2n}(x, 2^{-m})-\phi_{2n}(x, 2^{-m'})|>\frac{2a}{3} \tag{8.8}$$

for $0<|x|\leqslant \min\left(\psi(\tfrac{1}{9}a,\ 2^{-m}),\ 2^{-m}n\right)=\chi$, where m' is given by (8.7).

The relation (8.6), valid for all points of $\mathfrak{A}_n$, is in contradiction with (8.8) and shows that

$$(-\chi,\ \chi)\cdot\mathfrak{A}_{n_1}$$

consists only of $x=0$. Hence it is an isolated point of $\mathfrak{A}_{n_1}$. The integer n_1 being any arbitrary one greater than n_0, the point $x=0$ belongs to the set $\mathfrak{D}$ of isolated points of an infinity of $\mathfrak{A}_n$, which is reducible (*cf.* Lemma 4).

9. LEMMA. *We denote by* $\mathfrak{F}$ *the set of open intervals in* (a, b), *in which all* $D^{2l}F(x)$, $l=1, 2, \ldots, m-1$, *are continuous. If* $F(x)$ *satisfies conditions A and if*

$$D^{2m-2}F(x)=\int_{a_n}^{x}dy\int_{a_n}^{y}s(z)\,dz+c_n(x-a_n)+d_n \tag{9.1}$$

in every interval (a_n, b_n) *of* $\mathfrak{F}$, *where* $s(x)$ *is a function P-integrable in* (a, b), *then the set of points of discontinuity of all* $D^{2l}F(x)$ *is reducible.*

Let $\mathfrak{E}$ denote the set complementary to $\mathfrak{F}$. We shall show that the hypothesis of the existence of a perfect kernel $\mathfrak{P}\subset\mathfrak{E}$ leads to a contradiction. We assume therefore that

$$\mathfrak{E}=\mathfrak{P}+\mathfrak{R},$$

where $\mathfrak{P}$ is perfect and $\mathfrak{R}$ reducible. If $\mathfrak{P}$ exists, then we can find an interval (c, d) containing a perfect portion $\mathfrak{P}_1\subset\mathfrak{P}$, but devoid in its interior of points of $\mathfrak{D}+\mathfrak{R}$ (*cf.* Lemma 8). Further, by Lemma 6, $\mathfrak{P}_1$ contains a portion $\mathfrak{P}_2=\mathfrak{P}_1\cdot(p, q)$ in which all $\mathfrak{D}^{2l}F(x)$ are continuous. But we shall show that (p, q) must belong to $\mathfrak{F}$. This will be a contradiction of the fact, deduced from our hypothesis, that it contains $\mathfrak{P}_2$ a part of $\mathfrak{E}$. It will be sufficient to show that all $D^{2l}F(x)$ are continuous in (p, q). We begin with $D^{2m-2}F(x)$.

Let x_0 be any point of $\mathfrak{P}_2$. In virtue of the continuity of $D^{2m-2}F(x)$ on $\mathfrak{P}_2$, it is sufficient to show that

$$\lim_{a_k\to x_0} D^{(2m-2)}F(a_k)=D^{2m-2}F(x_0), \tag{9.3}$$

for an arbitrary sequence $a_k\subset\mathfrak{F}$ converging to x_0.

If (9.3) were not satisfied for all such sequences $\{\alpha_k\}$, we should have to distinguish two cases.

(i) All the α_k are only in a finite number of different intervals (a_n, b_n) of $\mathfrak{F}$. It is then possible to extract a partial sequence $\alpha_k' \to x_0$ which is entirely in one interval (a_0, b_0) of $\mathfrak{F}$ and for which $\lim\limits_{k\to\infty} D^{2m-2}F(\alpha_k')$ exists and is different from $D^{2m-2}F(x_0)$. The point x_0 is clearly an end point of (a_0, b_0). To fix the ideas let us suppose that $a_0 = x_0$.

By (9.1), $\lim\limits_{x\to x_0+0} D^{2m-2}F(x) = \lim\limits_{k\to\infty} D^{2m-2}F(\alpha_k')$, which is different from $D^{2m-2}F(x_0)$. Now Lemma 8 shows that x_0 must belong to $\mathfrak{D}$. This possibility has been excluded by the definition of $\mathfrak{P}_2$. Hence the assumption that (9.3) is not satisfied leads in this case to a contradiction.

(ii) The α_k lie in an infinity of different intervals (a_n, b_n) of $\mathfrak{F}$. Then we can extract a partial sequence $\alpha_k'' \to x_0$, such that all α_k'' lie in different intervals (a_k, b_k) of $\mathfrak{F}$, and $\lim\limits_{k\to\infty} D^{2m-2}F(\alpha_k'')$ exists and is different from $D^{2m-2}F(x_0)$. It is clear that

$$(9.4)\qquad (a_k, b_k) \to x_0,$$

and

$$(9.5)\qquad \lim_{k\to\infty} D^{2m-2}F(a_k) = \lim_{k\to\infty} D^{2m-2}F(b_k) = D^{2m-2}F(x_0).$$

The last relations are true because a_k, b_k are points of $\mathfrak{P}_2$ and $D^{2m-2}F(x)$ is continuous on $\mathfrak{P}_2$.

From (9.1), (9.4), and (9.5) we get

$$\lim_{k\to\infty} d_k = D^{2m-2}F(x_0), \quad \lim_{k\to\infty} c_k(b_k - a_k) = 0.$$

Hence we can readily deduce that

$$\lim_{k\to\infty} D^{2m-2}F(x_k) = D^{2m-2}F(x_0)$$

for *any* $x_k \subset (a_k, b_k)$. But this is again a contradiction to the hypothesis that $\lim\limits_{k\to\infty} D^{2m-2}F(\alpha_k'')$ exists and is different from $D^{2m-2}F(x_0)$.

We have thus proved that the set of discontinuities $\mathfrak{E}_{2m-2}$ of $D^{2m-2}F(x)$ is reducible.

If (e, f) is an interval contiguous to $\mathfrak{E}_{2m-2}$, then, by Lemma 7,

$$D^{2m-4}F(x) = \left(\int^x dx\right)^2 D^{2m-2}F(x)$$

for $x \subset \mathfrak{F} . (e, f)$, and $D^{2m-2} F(x)$ is continuous in (e, f). Hence the conditions, imposed in Lemma 9 upon $D^{2m-2} F(x)$ in (a, b), are satisfied also for $D^{2m-4} F(x)$ in (e, f). Repeating the whole argument, we prove that the set of points of discontinuity of $D^{2m-4} F(x)$ is reducible in (e, f). This being true for an arbitrary interval (e, f) contiguous to $\mathfrak{E}_{2m-2}$, we find that $\mathfrak{E}_{2m-4}$, the set of points of discontinuity of $D^{2m-4} F(x)$ in the whole interval (a, b), is reducible.

Repeating the argument a sufficient number of times, we prove the lemma for all $D^{2l} F(x)$.

10. LEMMA. *If* $\theta_m(x, h)$, *defined by* (2.1), *satisfies*

$$\varliminf_{h\to 0} \theta_m(x, h) \leqslant S(x) \leqslant \varlimsup_{h\to 0} \theta_m(x, h), \tag{10.1}$$

where $S(x)$ *denotes a Perron-integrable function in* (a, b) *which is finite everywhere except at an enumerable set of points* $\mathfrak{H}$, *at which*

$$\varliminf_{h\to 0} h\theta_m(x, h) \leqslant 0 \leqslant \varlimsup_{h\to 0} h\theta_m(x, h), \tag{10.2}$$

then

$$D^{2m-2} F(x) = \int_c^x dy \int_c^y S(z)\, dz + ex + f$$

in any interval (c, d) *in which all* $D^{2l} F(x)$, $l = 1, 2, \ldots, m-1$, *are continuous.*

By Lemma 7,

$$F(x) = \left(\int^x dx\right)^{2m-2} D^{2m-2} F(x)$$

in (c, d). Treating, therefore, $\theta_m(x, h)$ as an indeterminate form and differentiating the numerator and denominator $2m-2$ times, we get

$$\underline{D}^2\left(D^{2m-2} F(x)\right) \leqslant S(x) \leqslant \bar{D}^2\left(D^{2m-2} F(x)\right). \tag{10.3}$$

In the same way we get from (10.2)

$$\varliminf_{h\to 0} \frac{D^{2m-2} F(x+h) + D^{2m-2} F(x-h) - 2D^{2m-2} F(x)}{h} \tag{10.4}$$

$$\leqslant 0 \leqslant \varlimsup_{h\to 0} \frac{D^{2m-2} F(x+h) + D^{2m-2} F(x-h) - 2D^{2m-2} F(x)}{h}.$$

Hence for every x there is a sequence $h_k \to 0$, such that

$$\lim_{k\to\infty} \frac{D^{2m-2}F(x+h_k)-D^{2m-2}F(x)}{h_k} = \lim_{k\to 0} \frac{D^{2m-2}F(x)-D^{2m-2}F(x-h_k)}{h_k},$$

and this means that $D^{2m-2}F(x)$ has one symmetrical derivative in (c, d).

Let $\Phi(x)$ and $\phi(x)$ be a major and minor function, respectively, of $-S(x)$ in the sense of Saks (p. 191). Both are continuous and

$$\Phi(c) = \phi(c) = 0, \quad \bar{\phi}(x) \leqslant -S(x) \leqslant \underline{\Phi}(x), \quad 0 \leqslant \Phi(x)-\phi(x) < \epsilon.$$

The second relation, in which the bars denote upper and lower derivatives, is satisfied everywhere in (e, d) except at an enumerable set of points. The number ϵ, in the last relation, can be chosen beforehand.

If
$$\Phi_1(x) = \int_c^x \Phi(y)\,dy,$$
then
$$-S(x) \leqslant \lim_{h\to 0} \frac{\Phi(x+h)-\phi(x)}{h}$$
and also

$$-S(x) \leqslant \underline{\lim}_{h\to 0} \frac{\Phi(x+h)-\Phi(x-h)}{2h} \leqslant \underline{\lim}_{h\to 0} \frac{\Phi_1(x+h)+\Phi(x-h)-2\Phi_1(x)}{h^2}$$
$$= \underline{D}^2\Phi_1(x).$$

Further, $\Phi_1(x)$ has an ordinary and *a fortiori* a symmetrical derivative. We have therefore everywhere, except at points of an enumerable set,

$$\bar{D}^2\bigl(D^{2m-2}F(x)+\Phi_1(x)\bigr) \geqslant \bar{D}^2 D^{2m-2}F(x)+\underline{D}^2\phi_1(x) \geqslant 0.$$

Applying a theorem due to Hobson (2, §434, p. 665), we find that

$$D^{2m-2}F(x)+\Phi_1(x)$$

is a convex function. If $\phi_1(x)$ is defined similarly, we show in the same way that

$$D^{2m-2}F(x)+\phi_1(x)$$

is a concave function. According to the usual definition of the Perron integral

$$D^{2m-2}F(x)+\Phi_1(x) \geqslant D^{2m-2}F(x)-\int_c^x dy \int_c^y S(z)\,dz \geqslant D^{2m-2}F(x)+\phi_1(x).$$

If, for every number of a decreasing sequence $\epsilon_n \to 0$, we form the functions $\Phi(x)$ and $\phi(x)$, then the first and third members of the last inequality converge uniformly to the same limit, given by the second member, which must be convex and concave at the same time. It is, therefore, linear in (c, d). Thus the lemma is proved.

11. THEOREM A. *If* $\theta_m(x, h)$, *defined by* (2.1), *satisfies*

$$\varliminf_{h\to 0} \theta_m(x, h) \leqslant S(x) \leqslant \varlimsup_{h\to 0} \theta_m(x, h) \tag{11.1}$$

with $S(x)$ *Perron-integrable in* (a, b) *and finite except at an enumerable set of points* $\mathfrak{H}_1$, *and if further*

$$\lim_{h\to 0} h\theta_m(x, h) = 0 \tag{11.2}$$

except at an enumerable set of points $\mathfrak{H}_2$ *and*

$$\varliminf_{h\to 0} h\theta_m(x, h) \leqslant 0 \leqslant \varlimsup h\theta_m(x, h) \quad (x \subset \mathfrak{H}_2), \tag{11.3}$$

then there is a reducible set of points $\mathfrak{E} \subset (a, b)$ *such that in any interval of* $(a, b)-\mathfrak{E}$

$$F(x) = \left(\int^x dx\right)^{2m} S(x). \tag{11.4}$$

Moreover, at an isolated point x_0 *of* $\mathfrak{E}$,

$$F^{(2l+1)}(x_0-0) = F^{(2l+1)}(x_0+0) \quad (l = 0,\ 1,\ \dots,\ m-1), \tag{11.5}$$

and, for sufficiently small h,

$$F(x_0+h)-F(x_0-h) = \sum_{k=1}^{m-1} \frac{c_k}{2\,.\,(2k)!}\, h^{2k}\operatorname{sign} h + 2\sum_{k=0}^{m-1} c_{2k+1} h^{2k+1} + \left(\int_0^h dh\right)^{2m} \bigl(S(x+h)-S(x-h)\bigr), \tag{11.6}$$

where c_k *are the jumps of* $F^{(2k)}(x)$ *at* $x = x_0$.

It follows from Lemmas 9 and 10 that the set of points of discontinuity of all $D^{2l}F(x)$, $l = 1, 2, \dots, m-1$, is reducible. The relation (11.4) follows from Lemmas 10 and 7. This relation is true on both sides of x_0, but with different polynomials of integration. Without loss of generality we may suppose that $x_0 = 0$. Then

$$F(x) = c_{2m-1}x^{2m-1}+c_{2m-2}x^{2m-2}+\dots+c_0+\left(\int_0^x dx\right)^{2m} S(x) \quad (x>0),$$

$$F(x) = c'_{2m-1}x^{2m-1}+c'_{2m-2}x^{2m-2}+\dots+c_0'+\left(\int_0^x dx\right)^{2m} S(x) \quad (x<0),$$

and therefore

$$F(x)+F(-x)=\sum_{k=0}^{m-1}(c_{2k+1}-c'_{2k+1})\,x^{2k-1}\operatorname{sign}x$$
$$+\sum_{k=0}^{m-1}(c_{2k}+c'_{2k})\,x^{2k}+\left(\int_0^x dx\right)^{2m}[S(x)+S(-x)].$$

Condition (11.2) or (11.3) gives $c_{2k+1}=c'_{2k+1}$, while the continuity of $F(x)$ yields $c_0=c_0'$. Hence we get

$$F(x)-F(-x)=\sum_{k=1}^{m-1}(c_{2k}-c'_{2k})\,x^{2k}\operatorname{sign}x$$
$$+2\sum_{k=1}^{m-1}c_{2k+1}x^{2k+1}+\left(\int_0^x dx\right)^{2m}\Big(S(x)+S(-x)\Big),$$

which is the relation required.

12. *Example.* We prove that $F^{(2l+1)}(x)$ need not be, even approximately, continuous at limiting points of $\mathfrak{E}$.

We determine two sequences of numbers $\{x_k\}$, $\{q_k\}$ by the conditions

$$q_0=-3,$$
$$3x_k+q_k=-3x_{k-1}-q_{k-1}=x_k^3 \quad (k\geqslant 1),$$

and put
$$F''(x)=3x+q_k$$

in $(x_k,\ x_{k+1})$, and
$$F''(x_k)=0.$$

We have
$$|F''(x)|\leqslant x^3$$

and
$$\mathfrak{E}=\{x_k\}.$$

If now
$$F(x)=\int_0^x dy\int_0^y F''(z)\,dz \quad (1\geqslant x\geqslant 0),$$
$$F(x)=0 \qquad (-1\leqslant x\leqslant 0),$$

then
$$|F(x)|\leqslant|x|^5$$

and
$$D^4F(x)=0 \quad (-1<x<1).$$

Further
$$D^3F(x)=3 \quad (x>0),$$
$$D^3F(x)=0 \quad (x<0).$$

Remark. If we drop the symmetry in the definition of the de la Vallée Poussin derivative and put

$$\theta_m'(x, h) = \frac{F(x+h) - F(x) - \sum_{k=1}^{2m-1} \frac{h^k}{k!} A_k}{h^{2m}/(2m)!},$$

then [by (1.1)]

$$\theta_m'(x, h) + \theta_m'(x, -h) = \theta_m(x, h).$$

If, now, $\theta_m'(x, h)$ satisfies the conditions stated in Theorem A for $\theta_m(x, h)$, then this is also true of $\theta_m(x, h)$ itself. Therefore Theorem A holds good, and moreover the reducible set of singular points $\mathfrak{E}$ does not occur. Condition (11.2) for $\theta_m'(x, h)$ shows indeed that all c_k in (11.6) must be zero. Hence no point can be an isolated point of the reducible set $\mathfrak{E}$. And a reducible set without isolated points does not exist.

II.

The next two lemmas are stated in a somewhat more general form than is necessary for ordinary trigonometrical series. The main advantage of this is a formal simplification of proof.

13. LEMMA. *If S_k and s_k $(k \geqslant 0)$ are the upper and lower limits of*

$$\int_0^\omega \left(1 - \frac{\lambda}{\omega}\right)^k dA(\lambda), \tag{13.1}$$

as $\omega \to \infty$, if

$$\psi(u) = (-1)^m \frac{(2m)!}{u^{2m}} \left[\cos u - 1 + \frac{u^2}{2!} - \frac{u^4}{4!} + \ldots + (-1)^m \frac{u^{2m-2}}{(2m-2)!}\right] \tag{13.2}$$

and $2m - k \geqslant 2$, then the limits

$$\overline{\lim_{h \to 0}} \lim_{\omega \to \infty} \int_0^\omega \left(1 - \frac{\lambda}{\omega}\right)^k \psi(\lambda h)\, dA(\lambda) \tag{13.3}$$

lie in the interval

$$\left(\frac{S_k + s_k}{2} - C_{k,m} \frac{S_k - s_k}{2},\quad \frac{S_k + s_k}{2} + C_{k,m} \frac{S_k - s_k}{2}\right), \tag{13.4}$$

where $C_{k,m}$ depends only on k and m.

Remark. The Stieltjes integrals may be regarded as being defined by means of an integration by parts.

The lemma is trivial if either of the limits S_k, s_k is infinite. We therefore suppose both of them to be finite.

If we write

$$(13.5)\qquad A_n(\lambda)=\int_0^\lambda A_{n-1}(\mu)\,d\mu=\frac{1}{n\,!}\int_0^\lambda(\lambda-\mu)^n\,dA(\mu)$$

and

$$(13.6)\qquad \frac{A_k(\lambda)}{\lambda^k}=\frac{1}{k!}\,\frac{S_k+s_k}{2}+\frac{\theta(\lambda)}{k!}\,\frac{S_k-s_k}{2},$$

then it follows from the hypothesis about S_k and s_k that

$$(13.7)\qquad |\theta(\lambda)|\leqslant M \quad\text{and}\quad \overline{\lim_{\lambda\to\infty}}\,|\theta(\lambda)|=1.$$

Further we note that

$$(13.8)\qquad \lim_{\mu\to 0}\psi(u)=1,$$

and, for small values of u,

$$(13.9)\qquad \psi^{(n)}(u)=O(1).$$

For large values of u we have

$$(13.10)\qquad \begin{cases}\psi(u)=O(u^{-2}),\\ \psi^{(n)}(u)=O(u^{-n-2}) & (n\leqslant 2m-2),\\ \psi^{(n)}(u)=O(u^{-2m}) & (n\geqslant 2m-2).\end{cases}$$

By partial integration we get

$$(13.11)\qquad I(\omega,h)=\frac{1}{\omega^k}\int_0^\omega(\omega-\lambda)^k\psi(\lambda h)\,dA(\lambda)$$
$$=\frac{(-1)^{k+1}}{\omega^k}\int_0^\omega A_k(\lambda)\,\frac{d^{k+1}}{d\lambda^{k+1}}\,[(\omega-\lambda)^k\psi(\lambda h)]\,d\lambda$$
$$+k!\,A_k(\omega)\,.\,\psi(\omega h)\,\omega^{-k},$$

and further, from (13.6), (13.7), and (13.10), we have

$$(13.12)\qquad I(\omega,h)=\frac{(-1)^{k+1}}{\omega^k}\,h^{k+1}\int_0^\omega(\omega-\lambda)^k A_k(\lambda)\,\psi^{(k+1)}(\lambda h)\,d\lambda+O(\omega^{-1}).$$

If $2m-k\geqslant 2$, then, by (13.10),

$$A_k(\lambda)\,\psi^{(k+1)}(\lambda h)=O(\lambda^{-2}).$$

Hence

$$(13.13)\quad \begin{aligned}\lim_{\omega\to\infty} I(\omega, h) &= (-1)^{k+1} h^{k+1} \int_0^\infty A_k(\lambda)\, \psi^{(k+1)}(\lambda h)\, d\lambda \\ &= (-1)^{k+1} h^{k+1} \frac{S_k+s_k}{2\,.\,k!} \int_0^\infty \lambda^k\, \psi^{(k+1)}(\lambda h)\, d\lambda \\ &\qquad + (-1)^{k+1} h^{k+1} \frac{S_k-s_k}{2\,.\,k!} \int_0^\infty \theta(\lambda)\, \lambda^k\, \psi^{(k+1)}(\lambda h)\, d\lambda \\ &= I_1 + I_2.\end{aligned}$$

By repeated integrations by parts we get

$$(13.14)\qquad I_1 = \tfrac{1}{2}(S_k+s_k).$$

On the other hand, given any $a > 1$, we can find a Λ such that

$$|\theta(\lambda)| \leqslant a$$

for $\lambda > \Lambda$. Hence, by (13.7),

$$\begin{aligned}|I_2| &\leqslant \left[h^{k+1} M \int_0^\Lambda \lambda^k\, |\psi^{(k+1)}(\lambda h)|\, d\lambda + a h^{k+1} \int_\Lambda^\infty \lambda^k\, |\psi^{(k+1)}(\lambda h)|\, d\lambda\right] \frac{S_k-s_k}{2\,.\,k!} \\ &\leqslant \left[M \int_0^{\Lambda h} \lambda^k\, |\psi^{(k+1)}(\lambda)|\, d\lambda + a \int_0^\infty \lambda^k\, |\psi^{(k+1)}(\lambda)|\, d\lambda\right] \frac{S_k-s_k}{2\,.\,k!}\end{aligned}$$

and

$$(13.15)\qquad \left|\lim_{h\to 0} I_2\right| \leqslant a \int_0^\infty \lambda^k\, |\psi^{(k+1)}(\lambda)|\, d\lambda \,.\, \frac{S_k-s_k}{2\,.\,k!}.$$

Since a is any number greater than 1, we can take

$$(13.16)\qquad C_{k,m} = \frac{1}{k!} \int_0^\infty \lambda^k\, |\psi^{(k+1)}(\lambda)|\, d\lambda.$$

From (13.13), (13.14), (13.15), and (13.16) we get the desired result.

14. LEMMA. *If*

$$\int_0^\omega \left(1-\frac{\lambda}{\omega}\right)^k dA(\lambda) = o(\omega),$$

as $\omega \to \infty$, *and if* $2m-k \geqslant 2$, *then*

$$\lim_{h\to 0} \lim_{\omega\to\infty} h \int_0^\omega \left(1-\frac{\lambda}{\omega}\right)^k \psi(\lambda h)\, dA(\lambda) = 0.$$

We put
$$A_k(\lambda)=\lambda^{k+1}\theta(\lambda),$$
where $\theta(\lambda)\to 0$. Arguing in the same way as in the preceding lemma, but using a now for an arbitrary positive number, we find that

$$\left|\lim_{h\to 0}\lim_{\omega\to\infty} h\int_0^{\omega}\left(1-\frac{\lambda}{\omega}\right)^k \psi(\lambda h)\,dA(\lambda)\right|\leqslant a\int_0^{\infty}\lambda^{k+1}|\psi^{(k+1)}(\lambda h)|\,d\lambda.$$

Thus the lemma is proved.

15. *If*
$$\overline{\lim_{\omega\to\infty}}\left|\sum_{n=1}^{[\omega]}\left(1-\frac{n}{\omega}\right)^k(a_n\cos nx+b_n\sin nx)\right|$$
is finite in a set of positive measure, then
$$a_n,\ b_n=O(n^k).$$
If
$$\lim_{\omega\to\infty}\sum_{n=1}^{[\omega]}\left(1-\frac{n}{\omega}\right)^k(a_n\cos nx+b_n\sin nx)$$
exists in a set of positive measure, then
$$a_n,\ b_n=o(n^k).$$

These are well-known results. (*Cf.* Hobson, 2, §§ 52, 431.)

16. THEOREM B. *If $S_k(x)$ and $s_k(x)$, the upper and lower limits of*

$$\sum_{n=1}^{[\omega]}\left(1-\frac{n}{\omega}\right)^k(a_n\cos nx+b_n\sin nx)\quad(\omega\to\infty,\ k\geqslant 0),\tag{16.1}$$

are finite in a set of positive measure, then the series

$$F(x)=(-1)^m\sum_{n=1}^{\infty}\frac{1}{n^{2m}}(a_n\cos nx+b_n\sin nx)\quad(2m-k>1)\tag{16.2}$$

is uniformly convergent in $(0,2\pi)$, *and* $\overline{D}^{2m}F(x)$, $\underline{D}^{2m}F(x)$ (*cf.* § 1) *lie in the interval*

$$\left(\frac{S_k(x)+s_k(x)}{2}-C_{k,m}\frac{S_k(x)-s_k(x)}{2},\ \frac{S_k(x)+s_k(x)}{2}+C_{k,m}\frac{S_k(x)-s_k(x)}{2}\right),\tag{16.3}$$

where $C_{k,m}$ is given by (13.16).

The theorem is trivial except for those x for which $S_k(x)$ and $s_k(x)$ are both finite. Let x_0 be such a point.

By § 15, we have

$$a_n,\, b_n = O(n^k), \tag{16.4}$$

and the series (16.2) is therefore absolutely and uniformly convergent.

If we write

$$A(\omega) = \sum_{n=1}^{[\omega]} (a_n \cos nx_0 + b_n \sin nx_0),$$

then, by our hypothesis,

$$A_k(\omega) = \sum_{n=1}^{[\omega]} (\omega - n)^k (a_k \cos nx_0 + b_1 \sin nx_0) = O(\omega^k)$$

and therefore, for $l > 0$ (*cf.* § 13),

$$\frac{1}{\omega^k} \sum_{1}^{[\omega]} \frac{(\omega-n)^k}{n^l} (a_n \cos nx_0 + b_n \sin nx_0)$$

$$= \frac{1}{\omega^k} \int_0^{\omega} \frac{(\omega-\lambda)^k}{\lambda^l}\, dA(\lambda)$$

$$= \frac{(-1)^{k+1}}{\omega^k} \int_0^{\omega} A_k(\lambda) \frac{d^{k+1}}{d\lambda^{k+1}} \left[\frac{(\omega-\lambda)^k}{\lambda^l}\right] d\lambda + A_k(\omega)\, \omega^{-l-k}$$

$$= \frac{\Gamma(l+k+1)}{\Gamma(l)} \frac{1}{\omega^k} \int_0^{\omega} (\omega-\lambda)^k \frac{A_k(\lambda)}{\lambda^{k+l+1}}\, d\lambda + O(\omega^{-1}) + O(\omega^{-l}).$$

Hence we have

$$A_l = \lim_{\omega\to\infty} \frac{1}{\omega^k} \sum_{1}^{[\omega]} \frac{(\omega-n)^k}{n^l} (a_n \cos nx + b_n \sin nx) = \frac{\Gamma(l+k+1)}{\Gamma(l)} \int_0^{\infty} \frac{A_k(\lambda)}{\lambda^{k+l+1}}\, d\lambda.$$

We form the expression $\theta_m(x_0, h)$, given by (2.1), with the function $F(x)$, given by (16.2), and with

$$D^{(2n)} F(x_0) = A_{2m-2n}.$$

We get

$$\theta_m(x_0, h) = \lim_{\omega\to\infty} \sum_{n=1}^{[\omega]} \left(1 - \frac{n}{\omega}\right)^k (a_n \cos nx_0 + b_n \sin x_0)\, \psi(nh)$$

$$= \lim_{\omega\to\infty} \int_0^{\omega} \left(1 - \frac{\lambda}{\omega}\right)^k \psi(\lambda h)\, dA(\lambda),$$

where $\psi(n)$ is given by (13.2).

All that remains now is to apply Lemma 13 to the limit.

17. THEOREM C. *Let* $k \geqslant 0$, *let*

$$S(x) = \overline{\lim_{\omega \to \infty}} \sum_{n=1}^{[\omega]} \left(1 - \frac{n}{\omega}\right)^k (a_n \cos nx + b_n \sin nx),$$

and let $s(x)$ *be the corresponding lower limit. Further, suppose that both functions are*

(1) *P-integrable in* (a, b),

(2) *finite in* $(a, b) - \mathfrak{H}$, $\mathfrak{H}$ *being an enumerable set of points,*

(3) *such that, at the points of* $\mathfrak{H}$,

$$\sum_{1}^{[\omega]} \left(1 - \frac{n}{\omega}\right)^k (a_n \cos nx + b_n \sin nx) = o(\omega).$$

Then there is a reducible set $\mathfrak{E}$ *in* (a, b), *such that the series is a restricted Fourier series* (*cf.* Hobson, 2, 686–692), *in any interval not containing points of* $\mathfrak{E}$.

By Theorem B (§ 16), $\bar{D}^{2m} F(x)$ and $\underline{D}^{2m} F(x)$ lie between P-integrable limits and are therefore P-integrable themselves. Moreover they are finite except at points of $\mathfrak{H}$, where, by Lemma 14,

$$\lim_{h \to 0} h\theta(x, h) = 0.$$

Hence the conditions of Theorem A (§ 11) are satisfied and (11.4) shows that the series integrated $2m$ times, term by term, converges to a function which is, in any interval not containing points of $\mathfrak{E}$, the $2m$-th integral of a P-integrable function $S(x)$. It is, therefore, a restricted Fourier series.

18. LEMMA. *If* $\sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx)$ *is the Fourier series of* $f(x)$, *then, for* $k \geqslant 0$,

$$\sum_{n=1}^{[\omega]} \left(1 - \frac{n}{\omega}\right)^k (\alpha_n \sin nx - \beta_n \cos nx)$$
$$= \frac{\Gamma(k+1)}{\pi \omega^k} \int_0^\infty t^{-(1+k)} C_{2+k}(\omega t) [f(x+t) - f(x-t)]\, dt.$$

The lemma is the analogue, for conjugate series, of the formulae for the original series proved by Hobson (2, § 373, 567), and it can be proved in a similar manner.

19. THEOREM D. *If the conditions of the Theorem* C *are satisfied, with* $S(x) = s(x)$ *at a set of positive measure, then, at an isolated point* x_0 *of* $\mathfrak{E}$,

$$(19.01)\quad \bar{S}_k(x_0, \omega) = \sum_{n=0}^{[\omega]} \left(1-\frac{n}{\omega}\right)^k (a_n \sin nx_0 - b_n \cos nx_0)$$

$$= (-1)^m \sum_{n=1}^{m-1} \frac{\Gamma(k+1)\,\Gamma(2m-2n)}{\Gamma(2m-2n+k+1)} \, \frac{C_{2n}}{2(2n)!}\, \omega^{2m-2n}$$

$$+\frac{\Gamma(k+1)}{\pi\omega^k} \int_0^{\epsilon} t^{-(1+k)} C_{2+k}(\omega t)\,[S_k(x_0+t) - S_k(x_0-t)]\,dt$$

$$+A(x_0) + o(1),$$

where $k > 0$ *and* $\omega \to \infty$. *Here the* C_{2n} *are the jumps of* $D^{2n} F(x)$ *at the point* x_0.

By the preceding lemma [*cf.* (16.2)],

$$(19.02)\quad \bar{S}_k(x_0, \omega) = \frac{d^{2m}}{dx_0^{2m}} \frac{\Gamma(k+1)}{\pi\omega^k} \int_0^{\infty} t^{-(1+k)} C_{2+k}(\omega t)[F(x_0+t) - F(x_0-t)]\,dt.$$

Let the positive number ϵ be arbitrary, but so small that $(x_0 - \epsilon, x_0 + \epsilon)$ does not contain any other point of $\mathfrak{E}$ except x_0 itself. We introduce a function:

$$(19.03)\quad \begin{cases} \psi(t) = 0, & [t \subset (0,\ \tfrac{1}{2}\epsilon)] \\ \psi(t) = t^{-(k+1)} & [t \subset (\epsilon,\ \infty)]. \end{cases}$$

In $(\tfrac{1}{2}\epsilon,\ \epsilon)$ the function may be arbitrary but must be such that it and its $2m$ derivatives are continuous.

The formula (Hobson, 2, § 371)

$$(19.04)\quad C_{2+k}(t) = \frac{t^{1+k}}{\Gamma(k+1)} \int_0^1 (1-u)^k \sin tu\,du$$

shows that, if, for negative values of t, we define $\psi(t)$ so that

$$(-t)^k\, \psi(-t) = -t^k\, \psi(t),$$

then

$$(19.05)\quad [t^{-(1+k)} - \psi(t)]\, C_{2+k}(\omega t)$$

is an odd function of t with $2m$ continuous derivatives.

We now get (*cf.* Lemma 18)

$$(19.06)\qquad \bar{S}_k(x_0,\,\omega)=\frac{d^{2m}}{dx_0^{2m}}\,\frac{\Gamma(k+1)}{\pi\omega^k}\int_0^\infty \psi(t)\,C_{2+k}(\omega t)[F(x_0+t)-F(x_0-t)]\,dt$$

$$+\frac{\Gamma(k+1)}{\pi\omega^k}\int_0^\epsilon \{[t^{-(1+k)}-\psi(t)]\,C_{2+k}(wt)\}^{(2m)}$$

$$\times[F(x_0+t)-F(x_0-t)]\,dt$$

$$=I_1+I_2.$$

An asymptotic formula, deduced by partial integrations from (19.04), can be used (*cf.* Wolf, **2**, Th. 11) to show that

$$(19.07)\qquad \lim_{\omega\to\infty} I_1(x_0)=\frac{1}{\pi}\,\frac{d^{2m}}{dx_0^{2m}}\int_0^\infty t^k\psi(t)[F(x_0+t)-F(x_0-t)]\,dt$$

$$=A_1(x_0)=\frac{1}{\pi}\int_{-\infty}^\infty [t^k\psi(t)]^{(2m)}F(x_0+t)\,dt$$

under the hypothesis that a_n and b_n are $o(n^k)$. By §15, this hypothesis is satisfied since $S(x)=s(x)$ at a set of positive measure.

The integral is uniformly convergent because

$$(19.08)\qquad [t^k\psi(t)]^{(2m)}=O(t^{-2m-1})$$

for large absolute values of t and $F(t)$ is a continuous periodic function [*cf.* (16.2), (16.4)]. Hence $A_1(x_0)$ is a continuous periodic function having, moreover, continuous derivatives of all orders.

To evaluate I_2 we use (11.6); we have

$$(19.09)\qquad I_2=\frac{\Gamma(k+1)}{\pi\omega^k}\sum_{n=1}^{2m-1}\frac{c_n}{n!}\int_0^\epsilon \{[t^{-(1+k)}-\psi(t)]\,C_{2+k}(\omega t)^{(2m)}\}\,t^n\,dt$$

$$+\frac{\Gamma(k+1)}{\pi\omega^k}\int_0^\epsilon \{[t^{-(1+k)}-\psi(t)]\,C_{2+k}(\omega t)\}^{(2m)}\,dt$$

$$\times\left(\int_0^t dt\right)^{2m}[S_k(x_0+t)-S_k(x_0-t)]$$

$$=\sum_{n=1}^{2m-1} I_{2,n}+I_3.$$

By partial integration,

$$(19.10)\qquad I_{2,n}=(-1)^{n+1}\,\frac{\Gamma(k+1)}{\pi\omega^k}\cdot 2c_n\lim_{t\to 0}\left[\frac{C_{2+k}(\omega t)}{t^{1+k}}\right]^{(2m-n-1)}.$$

We have again used the fact that the difference in square brackets and its $2m$ derivatives are zero for $t=\epsilon$. To calculate the last expression use (19.04).

We thus find that

$$\lim_{t\to 0}\left[\frac{C_{2+k}(\omega t)}{t^{1+k}}\right]^{(2m-2n)}=0 \tag{19.11}$$

and

$$\begin{aligned}\lim_{t\to 0}\left[\frac{C_{2+k}(\omega t)}{t^{1+k}}\right]^{2m-2n-1} &= \omega^{k+2m-2n}\,\frac{(-1)^{m+n+1}}{\Gamma(k+1)}\int_0^1 u^{2m-2n-1}(1-u)^k\,du \\ &= (-1)^{m-n-1}\,\omega^{k+2m-2n}\,\frac{\Gamma(2m-2n)}{\Gamma(2m-2n+k+1)}.\end{aligned} \tag{19.12}$$

We obtain, therefore,

$$I_{2,\,2n+1}=(-1)^m\,\frac{\Gamma(k+1)\,\Gamma(2m-2n)}{\Gamma(2m-2n+k+1)}\,\frac{2c_n}{\pi}\,\omega^{2m-2n}. \tag{19.13}$$

The last integral I_3 can be simplified by $2m$ partial integrations, thus

$$\begin{aligned} I_3 &= \frac{\Gamma(k+1)}{\pi\omega^k}\int_0^\epsilon t^{-(1+k)}\,C_{2+k}(\omega t)\,[S_k(x_0+t)-S_k(x_0-t)]\,dt \\ &\qquad -\frac{\Gamma(k+1)}{\pi\omega^k}\int_0^\epsilon \psi(t)\,C_{2+k}(\omega t)\,[S_k(x_0+t)-S_k(x_0-t)]\,dt \\ &= I_3'+I_3''.\end{aligned} \tag{19.14}$$

In a way similar to that in which (19.07) was obtained we get

$$\begin{aligned}\lim_{\omega\to\infty} I_3''(x_0) &= -\frac{1}{\pi}\int_0^\epsilon \psi(t)\,t^k[S_k(x_0+t)-S_k(x_0-t)]\,dt \\ &= A_2(x_0).\end{aligned} \tag{19.15}$$

If we sum up the results and remember that the jump of $D^{(2n)}F(x)$ is $4c_{2n}$, we obtain the required relation.

20. THEOREM E. *If the conditions of Theorem* D *are satisfied and* (p, q) *is a closed interval not containing points of* $\mathfrak{E}$, *then*

$$\begin{aligned}\lim_{\omega\to\infty}\left\{\bar{S}_k(x,\omega)-\frac{\Gamma(k+1)}{\pi\omega^k}\int_0^\epsilon t^{-(1+k)}\,C_{2+k}(\omega t)[S(x+t)-S(x-t)]\,dt\right\} \\ = A_1(x)+A_2(x)=A(x)\end{aligned} \tag{20.1}$$

uniformly in (p, q). *The number* ϵ *is fixed but arbitrary and the functions* $A_1(x)$, $A_2(x)$ *are given by*

$$A_1(x) = \frac{1}{\pi} \frac{d^{2m}}{dx^{2m}} \int_0^\infty t^k \psi(t) \left[F(x+t) - F(x-t)\right] dt,$$

$$A_2(x) = -\frac{1}{\pi} \int_0^\epsilon \psi(t) t^k \left[S(x+t) - S(x-t)\right] dt.$$

The sum $A(x)$ *is independent of the particular choice of* $\psi(t)$.

The theorem is valid for any allied series of a restricted Fourier series, and its proof can be constructed on the lines of the proof of Theorem D, by putting $C_n = 0$.

21. THEOREM F. *If* $k > 0$, *if*

$$S(x) = \overline{\lim_{\omega \to \infty}} \sum_{n=1}^{[\omega]} \left(1 - \frac{n}{\omega}\right)^k (a_n \cos nx + b_n \sin nx) \tag{21.1}$$

and $s(x)$ *is the corresponding lower limit, and if*

(1) $S(x) = s(x)$ *at a set of points of positive measure,*

(2) $S(x)$ *and* $s(x)$ *are L-integrable and finite in* (a, b),

then (*a*)

$$\bar{S}(x) = \lim_{\omega \to \infty} \sum_{n=1}^{[\omega]} \left(1 - \frac{n}{\omega}\right)^k (a_n \sin nx - b_n \cos nx) \tag{21.2}$$

exists almost everywhere in (a, b),

(*b*) *there is a reducible set* $\mathfrak{E}$ *such that*

$$\bar{S}_k(x, \omega) = o(\omega), \tag{21.3}$$

provided that x *does not belong to* $\mathfrak{E}$ (*this is the same reducible set as that of Theorem C*),

(*c*) *at isolated points of* $\mathfrak{E}$

$$\bar{S}_k(x, \omega) = P_{\frac{1}{2}k}(\omega^2) + o(\omega). \tag{21.4}$$

It is well known that the allied series of a Fourier series is almost everywhere summable (C, k), for $k > 0$. (*Cf.* Zygmund, 7.1 (ii), 146.)

Now let $f(x)$ be an arbitrary periodic L-integrable function, such that

$$\int_0^{2\pi} f(x)\, dx = 0$$

and
$$\sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx)$$

its Fourier series.

By Lemma 18 we have

$$(21.5)\quad \sum_{n=1}^{[\omega]} \left(1-\frac{n}{\omega}\right)^k (\alpha_n \sin nx - \beta_n \cos nx) - \frac{\Gamma(k+1)}{\pi\omega^k} \int_0^{\epsilon} t^{-(1+k)} C_{2+k}(\omega t)\, [f(x+t)-f(x-t)]\, dt$$
$$= \frac{\Gamma(k+1)}{\pi\omega^k} \int_{\epsilon}^{\infty} t^{-(1+k)} C_{2+k}(\omega t) [f(x+t)-f(x-t)]\, dt.$$

Using again the asymptotic formula deduced from (19.04), we can easily show that

$$(21.6)\quad \lim_{\omega\to\infty} \frac{\Gamma(k+1)}{\pi\omega^k} \int_{\epsilon}^{\infty} t^{-(1+k)} C_{2+k}(\omega t) [f(x+t)-f(x-t)]\, dt$$
$$= \frac{1}{\pi} \int_{\epsilon}^{\infty} t^{-1} [f(x+t)-f(x-t)]\, dt$$

uniformly in x. The sum in (21.5) being almost everywhere convergent, we find, therefore, that

$$\frac{\Gamma(k+1)}{\pi\omega^k} \int_0^{\epsilon} t^{-(1+k)} C_{2+k}(\omega t) [f(x+t)-f(x-t)]\, dt$$

is almost everywhere convergent. The function $f(x)$ may be any integrable periodic function. Theorem E (20.1) now gives the result (a).

Properties (b) and (c) can be deduced from Theorems D and E. It is obviously sufficient to show that

$$\frac{1}{\omega^k} \int_0^{\epsilon} t^{-(1+k)} C_{2+k}(\omega t) [S(x+t)-S(x-t)]\, dt = o(\omega)$$

for an L-integrable $S(x)$. Using (19.05) we get

$$\omega \int_0^1 (1-u)^k du \int_0^{\epsilon} [S(x+t)-S(x-t)] \sin u\omega t\, dt,$$

and this can be easily shown to be $o(\omega)$.

22. For a P-integrable $S(x)$ we get merely

Theorem G. *If $S(x)$, $s(x)$ are P-integrable and $k > 0$, then, at all points not belonging to $\mathfrak{E}'$*

$$\bar{S}_k(x, \omega) = o(\omega^2),$$

and, at isolated points of $\mathfrak{E}$,

$$\bar{S}_k(x, \omega) = P_k(\omega^2) + o(\omega^2),$$

where $P_k(x)$ is a polynomial of order not exceeding $\frac{1}{2}k$.

The proof presents no difficulty.

23. Corollary. *If $k > 0$, and if*

(1) *$S(x)$, $s(x)$ are P-integrable in (a, b) and finite in $(a, b) - \mathfrak{H}$, where $\mathfrak{H}$ is enumerable,*

(2) *$S_k(x, \omega) = o(\omega)$ at any point of $\mathfrak{H}$,*

(3) *$\bar{S}_k(x, \omega) = o(\omega^2)$ for all $x \subset (a, b)$,*

then the series is a restricted Fourier series in (a, b). If, moreover,

$$(a, b) = (0, 2\pi),$$

then the series is a Fourier series of $S(x)$ and

$$S(x) = s(x)$$

almost everywhere.

This corollary has much in common with a result due to Verblunsky (4, Th. VII, 189).

24. If $k = 1$, then (3) is satisfied and the result holds under the first two hypotheses.

25. Theorem H. *If the conditions of Theorem C are satisfied for $(a, b) = (0, 2\pi)$, then*

$$\left\{\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} S(x)\cos nx\,dx + \lim_{p\to\infty}\sum_{l=0}^{k} n^l \sum_m \alpha_{l,m}^{(p)} \cos\left(n\xi_m - l\tfrac{1}{2}\pi\right), \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} S(x)\sin nx\,dx + \lim_{p\to\infty}\sum_{l=0}^{k} n^l \sum_m \alpha_{l,m}^{(p)} \sin\left(n\xi_m - l\tfrac{1}{2}\pi\right), \end{aligned}\right. \tag{25.1}$$

where the ξ_m are points of $\mathfrak{E}$.

Remark. The singular series

$$\sum_{n=1}^{\infty} n^{2l-1} \sin nx \quad (l=1, 2, \ldots) \tag{25.2}$$

are summable $(C, 2l)$ to zero everywhere; this is also true of

$$\sum_{n=1}^{\infty} n^{2l} \cos nx \tag{25.3}$$

for summability $(C, 2l+1)$ except at the point $x=0$. This theorem states that a trigonometrical series which is summable (C, k) to $S(x)$ consists of two parts, (a) the Fourier series of $S(x)$ and (b) singular trigonometrical series of the types (25.2) and (25.3) whose singular points form a reducible set of points.

Let the intervals I_m of the open sets $(0, 2\pi)-\mathfrak{E}$ be enumerated in decreasing order of length. For those, if any, which are of equal length we enumerate them from left to right. If we put [*cf.* (11.4)]

$$F^{(2m-k-2)}(x) = \left(\int_0^x dx\right)^{k+2} S(x)+H(x),$$

then $H(x)$ is a periodic function which is a polynomial of degree $k+1$ in every interval contiguous to $\mathfrak{E}$. $F^{(2m-k-2)}(x)$ is given as the sum of a trigonometrical series with coefficients $o(n^{-2-k})$. Hence $H(x)$ is continuous.

Now let

$$H_p(x) = H(x) \quad \text{for} \quad x \subset I_n, \quad n=1, 2, \ldots, p.$$

In every interval of $(0, 2\pi)-\sum_{n=1}^{p} I_n$ let $H_p(x)$ be linear and also let it be periodic and continuous for all x. Obviously

$$H_p(x) \to H(x) \quad (p \to \infty)$$

boundedly. In view of the order of magnitude of the coefficients of the series for $F^{(2m-k-2)}(x)$, we get

$$\frac{a_n}{n^{k+2}} = \frac{1}{\pi}\int_0^{2\pi} F^{(2m-k-2)}(x) \cos\left(nx - \frac{k+2}{2}\pi\right) dx$$

$$= \frac{1}{\pi n^{k+2}} \int_0^{2\pi} S(x) \cos nx\, dx + \frac{1}{\pi} \lim_{p\to\infty} \int_0^{2\pi} H_p(x) \left(\cos nx - \frac{k+2}{2}\pi\right) dx.$$

By actually calculating the last integral we get the relations (25.1).

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ON SKEW-GROUPS

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We consider groups $\mathfrak{G}$ with the following properties[†]:

(1) $\mathfrak{G}$ is infinite, has a finite number of generators, but has no elements of finite order other than 1.

(2) The derived group $\mathfrak{H}$ of $\mathfrak{G}$ is cyclical and contained in the central $\mathfrak{Z}$ of $\mathfrak{G}$.

We shall give a complete determination of all these groups (§ 1) and of their groups of automorphisms (§ 2).

1. Let us deal first with the case in which the central $\mathfrak{Z}$ of $\mathfrak{G}$ is itself cyclical. If it is generated by the element Z, $\mathfrak{Z} = \{Z\}$, then the derived group is generated by a power of Z, say Z^h, $\mathfrak{H} = \{Z^h\}$. We may assume that $h > 0$.

We shall call groups of this type "skew-groups", for a reason which will become apparent presently.

THEOREM I. *Let $\mathfrak{G}$ be any skew-group, let $\mathfrak{Z} = \{Z\}$ be its central, and $\mathfrak{H} = \{Z^h\}$ its derived group, $h > 0$. The factor-group of the central $\mathfrak{G}/\mathfrak{Z}$ is a free Abelian group with an even number of independent generators. These generators can be chosen so that their representatives in $\mathfrak{G}$ occur in pairs*

$$(1) \qquad P_1, Q_1;\ P_2, Q_2;\ \dots;\ P_n, Q_n,$$

[†] These groups are "S-groups" in the sense of my previous papers "On infinite soluble groups", I and II, *Proc. London Math. Soc.* (2), 44 (1938), 56–60, 336–344. They possess a *central series* and are of *class* 2. It follows from Theorem 2.321 (*loc. cit.*, p. 341) that the derived group, if cyclical, is necessarily contained in the central. The present note is independent of the results of those papers.

where the following commutator-relations hold†:

$$(2)\qquad \begin{cases} (P_i,\ Q_i) = Z^{e_i} \quad (e_i > 0,\ i = 1,\ 2,\ \dots,\ n), \\ P_i,\ P_j) = (P_i,\ Q_j) = (Q_i,\ Q_j) = 1 \quad \textit{for} \quad i \neq j. \end{cases}$$

Moreover, each e_i *divides its successors, thus*: $e_1 | e_2 | \dots | e_n$; *and* $e_1 = h$.

Proof. $\mathfrak{Z}$ contains the derived group $\mathfrak{H}$ of $\mathfrak{G}$. $\mathfrak{G}/\mathfrak{Z}$ is, therefore, Abelian and, since $\mathfrak{G}$ contains no elements of finite order other than 1, it is free Abelian‡.

Let $G_1, G_2, \dots, G_m$ be representatives in $\mathfrak{G}$ of an arbitrary, but fixed, set of independent generators of $\mathfrak{G}/\mathfrak{Z}$. Since the commutators of the G_i are all contained in $\{Z^h\}$, we have

$$(3)\qquad (G_i,\ G_j) = Z^{g_{ij}}, \quad g_{ii} = 0, \quad g_{ij} \equiv 0 (h).$$

The exponents of the m^2 commutators form a skew- symmetric matrix

$$(4)\qquad \mathbf{G} = (g_{ij}).$$

To any set of independent generators of $\mathfrak{G}/\mathfrak{Z}$, represented in $\mathfrak{G}$ by

$$(5)\qquad G_1^*,\ G_2^*,\ \dots,\ G_m^*,$$

there now correspond two matrices:

(i) a unimodular matrix

$$(6)\qquad \mathbf{A} = (a_{ij}),$$

describing the transition from the generators§ $G_1, G_2, \dots, G_m$ to the $G_1^*, G_2^*, \dots, G_m^*$, so that

$$(7)\qquad G_i^* = \prod_{j=1}^{m} G_j^{a_{ij}} Z^{u_i};$$

(ii) a skew-symmetric matrix

$$(8)\qquad \mathbf{G}^* = (g_{ij}^*),$$

describing the commutators of the G_i^*, so that

$$(9)\qquad (G_i^*,\ G_j^*) = Z^{g_{ij}^*}.$$

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† We denote the commutator $P^{-1}Q^{-1}PQ$ of the elements P and Q by $(P,\ Q)$.

‡ If $P^k \in \mathfrak{Z}$, $Q \in \mathfrak{G}$, then $1 = (P^k, Q) = (P, Q)^k$. This implies that $k = 1$.

§ Where the expression would be too cumbersome, and where there is no risk of confusion, we use the term " generator " instead of " representative in $\mathfrak{G}$ of a generator."

$\mathbf{G}^*$ is simply the congruent transform of $\mathbf{G}$ by $\mathbf{A}$, *i.e.* we have the formula

$$(10) \qquad \mathbf{G}^* = \mathbf{A} \,.\, \mathbf{G} \,.\, \mathbf{A}^t,$$

where $\mathbf{A}^t$ denotes the transposed matrix (a_{ji}). This is easily proved. We have

$$(11) \qquad Z^{g_{ij}^*} = (G_i{}^*,\, G_j{}^*) = \left(\prod_{k=1}^{m} G_k^{a_{ik}},\ \prod_{l=1}^{m} G_l^{a_{jl}}\right).$$

Now, since the derived group is contained in the central, the commutator of products is equal to the product of the commutators of the factors. So we get further

$$(12) \qquad Z^{g_{ij}^*} = \prod_{k=1}^{m} \prod_{l=1}^{m} (G_k^{a_{ik}},\ G_l^{a_{il}}) = Z^{\sum_{k=1}^{m} \sum_{l=1}^{m} g_{kl} \cdot a_{ik} \cdot a_{jl}},$$

and finally, with $a_{lj}^t = a_{jl}$,

$$(13) \qquad g_{ij}^* = \sum_{k=1}^{m} \sum_{l=1}^{m} a_{ik} \cdot g_{kl} \cdot a_{lj}^t,$$

which is (10).

Starting from an arbitrary set of independent generators for $\mathfrak{G}/\mathfrak{Z}$, represented in $\mathfrak{G}$ by G_1, G_2, ..., G_m, we now simplify the commutator-matrix $\mathbf{G} = (g_{ij})$ by a series of elementary steps, until a new set of generators with a commutator-matrix of the form required in Theorem I is obtained.

(i) The first step consists in forming from the G_i a pair of new generators, say P_1 and Q_1, the commutator of which is $(P_1, Q_1) = Z^h$, *i.e.* generates the derived group.

Let P, Q, and R be any three of the G_i, and let

$$(14) \qquad (P, Q) = Z^a, \quad (P, R) = Z^b.$$

If $d = (a, b)$ is the greatest common divisor of a and b, and if we represent d as a linear combination $d = ax + by$, then

$$(15) \qquad (P,\, Q^x \,.\, R^y) = Z^{ax+by} = Z^d.$$

Moreover, x and y are co-prime. $Q^x \,.\, R^y$ can therefore be chosen as a new generator instead of Q.

By repeated application of this process we can form from the

$$G_1,\ G_2,\ \ldots,\ G_m$$

two new generators, P_1 and Q_1, such that $(P_1, Q_1) = Z^{e_1}$, where e_1 is the greatest common divisor of the g_{ij}, *i.e.* $e_1 = \pm h$. (By interchanging P_1 and Q_1, if necessary, we obtain $e_1 = h$.) For this purpose we need only to run through the steps of the Euclidean algorithm which gives e_1 as a linear combination of the g_{ij}.

(ii) The next step is to introduce for the remaining $m-2$ generators (which we call again $G_3, \ldots, G_m$) new ones which are permutable with both P_1 and Q_1.

If

$$(P_1, G_i) = Z^{a_i}, \quad (Q_1, G_i) = Z^{b_i}, \tag{16}$$

then

$$\begin{cases} (P_1, P_1^{b_i/e_1} Q^{-a_i/e_1} G_i) = Z^{-(a_i/e_1)\cdot e_1 + a_i} = 1, \\ (Q_1, P_1^{b_i/e_1} Q_1^{-a_i/e_1} G_i) = Z^{b_i/e_1(-e_1)+b_i} = 1. \end{cases} \tag{17}$$

The elements $G_i{}^* = P_1^{b_i/e_1} Q_1^{-a_i/e_1} . G_i$, $i > 2$, together with P_1 and Q_1, again represent obviously a complete set of independent generators for $\mathfrak{G}/\mathfrak{Z}$. The relations

$$(G_i, G_j) = Z^{g_{ij}} \quad (i, j = 3, \ldots, m),$$

where

$$g_{ij} \equiv 0 (e_1),$$

go over into

$$(G_i{}^*, G_j{}^*) = Z^{g_{ij}^*} \quad (i, j = 3, \ldots, m),$$

where a simple calculation gives

$$g_{ij}^* = g_{ij} + \frac{a_j b_i - a_i b_j}{e_1}, \tag{18}$$

so that†

$$g_{ij}^* \equiv 0 \quad (e_1) \quad (i, j = 3, \ldots, m).$$

(iii) The two processes described under (i) and (ii) may now be repeated without altering P_1 or Q_1, or the fact that the remaining generators are permutable with both P_1 and Q_1. If, after step (ii) is carried out, the generators are $P_1, Q, G_3{}^*, \ldots, G_m{}^*$, we can form from the $G_i{}^*$, $i = 3, \ldots, m$, a pair of new generators, say P_2 and Q_2, where

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† This is trivial at the first stage, because all commutators in $\mathfrak{G}$ are powers of $Z^h = Z^{e_1}$. At the later stages, however, considered under (iii), the modulus e_1 is replaced by a higher one; but the corresponding a_i, b_i, and g^{ij} are then divisible by the higher modulus, so that the congruence still holds.

$$(19)\qquad \begin{cases} (P_1,\, P_2) = (P_1,\, Q_2) = (Q_1,\, Q_2) = 1, \\ (P_2,\, Q_2) = Z^{e_2}, \quad e_2 > 0, \quad e_1 | e_2, \end{cases}$$

and e_2 is the greatest common divisor of all g_{ij}^{*} $(i,\, j = 3,\, \ldots,\, m)$.

Then we can introduce new generators $G_5^{**},\, \ldots,\, G_m^{**}$ such that

$$(20)\qquad \begin{cases} (P_i,\, G_j^{**}) = (Q_i,\, G_j^{**}) = 1 & (i = 1,\, 2;\; j = 5,\, \ldots,\, m), \\ (G_i^{**},\, G_j^{**}) = Z^{g_{ij}^{**}} & (i,\, j = 5,\, \ldots,\, m), \end{cases}$$

where $g_{ij}^{**} \equiv 0 \ (e_2)$.

By repeating these processes we finally arrive at a set of generators, $2m = n$ in number,

$$(1)\qquad P_1,\ Q_1;\ \ P_2,\ Q_2;\ \ \ldots;\ \ P_n,\ Q_n,$$

where

$$(2)\qquad \begin{cases} (P_i,\, Q_i) = Z^{e_i}, \quad e_i > 0, \quad e_1 = h, \quad e_1 | e_2 | \ldots | e_n; \\ (P_i,\, P_j) = (P_i,\, Q_j) = (Q_i Q_j) = 1 \quad \text{for} \quad i \neq j. \end{cases}$$

Thus Theorem I is proved.

As the proof has shown, this theorem is substantially equivalent to

THEOREM II. *Every non-singular skew-symmetric matrix with integer coefficients* $\mathbf{G} = (g_{ij})$ *can be carried into the "diagonal" form by a congruent transformation with a unimodular matrix* $\mathbf{A} = (a_{ij})$; *thus*

$$(21)\qquad \mathbf{AGA}^t = \begin{pmatrix} 0 & e_1 & & & & & \\ -e_2 & 0 & & & & & \\ & & 0 & e_2 & & & \\ & & -e_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & e_n \\ & & & & & -e_n & 0 \end{pmatrix},$$

where $e_i > 0$, *and* $e_1 | e_2 | \ldots | e_n$†.

The integers e_i, each repeated twice, are the elementary divisors of the "diagonal" matrix $\mathbf{AGA}^t$. We shall call them the "elementary divisors" of $\mathfrak{G}$. They characterise $\mathfrak{G}$ completely. We have, in fact,

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* *Cf.* E. Cahen, *Théorie des nombres*, 1 (Paris, 1914). 282.

THEOREM III. *Two skew-groups* $\mathfrak{G}$ *and* $\mathfrak{G}^*$ *with the elementary divisors* e_i *and* e_i^*, *respectively, are isomorphic if and only if* $e_i = e_i^*$ *for* $i = 1, 2, \ldots, n$. *Corresponding to every set of positive integers* e_i, *where* $e_1 | e_2 | \ldots | e_n$, *there exists a skew-group* $\mathfrak{G}$ *which has the* e_i *as its elementary divisors.*

Proof. All the statements in Theorem III are more or less obvious. If two skew-groups $\mathfrak{G}$ and $\mathfrak{G}^*$ have the same elementary divisors, and if in $\mathfrak{G}$ and $\mathfrak{G}^*$ generators are chosen in accordance with Theorem I, then an isomorphism between $\mathfrak{G}$ and $\mathfrak{G}^*$ is established by the correspondences

$$P_i \longleftrightarrow P_i^*, \quad Q_i \longleftrightarrow Q_i^*, \quad Z \longleftrightarrow Z^*. \tag{22}$$

On the other hand, the elementary divisors have been introduced as invariant features of the skew-groups, viz. as greatest common divisors of certain sets of numbers. They depend only on the group itself and not on the particular choice of generators. Therefore, if the elementary divisors of two skew-groups $\mathfrak{G}$ and $\mathfrak{G}^*$ are not equal, the groups cannot be isomorphic.

Finally, if any set of positive integers e_i is given, with the relations $e_1 | e_2 | \ldots | e_n$, we consider the group $\mathfrak{G}$ defined by $2n+1$ generators[†]

$$P_1,\ Q_1;\quad P_2,\ Q_2;\quad \ldots;\quad P_n,\ Q_n;\quad Z; \tag{1'}$$

with the relations

$$\left\{\begin{array}{l} (P_i,\ Q_i) = Z^{e_i} \quad (i = 1,\ 2,\ \ldots,\ n); \\[1em] (P_i,\ P_j) = (P_i,\ Q_j) = (Q_i,\ Q_j) = 1, \quad \text{for} \quad i \neq j; \\[1em] (Z_1 P_i) = (Z_1 Q_i) = 1 \quad (i = 1,\ 2,\ \ldots,\ n). \end{array}\right. \tag{2'}$$

The factor-group of the central of $\mathfrak{G}$ is the free Abelian group generated by the P_i and Q_i. Since the orders of the P_i and Q_i are infinite here, they are also infinite in $\mathfrak{G}$. To see that Z has infinite order, we consider the subgroup of $\mathfrak{G}$ generated by P_1 and Q_1 alone. This subgroup is isomorphic with the group of matrices

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \tag{23}$$

with integer coefficients a, b, and c. In it Z^{e_1} corresponds to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

[†] If $e_1 = 1$, the last generator Z is superfluous.

which has, of course, infinite order. Therefore Z^{e_1}, and likewise Z, has infinite order in $\mathfrak{G}$. So we see that $\mathfrak{G}$ is a skew-group which has the e_i as its elementary divisors. Thus Theorem III is proved.

If the central of a group $\mathfrak{G}$ with the properties (1) and (2)† is not cyclical itself, but has, besides Z, other generators, say $Z_1, Z_2, \ldots, Z_e$, then $\{Z_1, Z_2, \ldots, Z_e\}$ forms a direct Abelian factor of $\mathfrak{G}$. The most general group with the properties (1) and (2) is, therefore, the direct product of a skew-group and an arbitrary free Abelian group.

If $\mathfrak{G}$ is a skew-group and if generators for $\mathfrak{G}$ are chosen in accordance with Theorem I, they form a uniqueness-basis for $\mathfrak{G}$; *i.e.* every element G of $\mathfrak{G}$ can be represented once and only once in the form

$$G = P_1^{x_1} Q_1^{y_1} \ldots P_n^{x_n} Q_n^{y_n} Z^g. \tag{24}$$

For the product of G with

$$G' = P_1^{x_1'} Q_1^{y_1'} \ldots P_n^{x_n'} Q_n^{y_n'} Z^g \tag{25}$$

we have the rule

$$GG' = P_1^{x_1'+x_1'} Q_1^{y_1+y_1'} \ldots P_n^{x_n+x_n'} Q_n^{y_n'} Z^{g''}, \tag{26}$$

where
$$g'' = g+g' - \sum_{i=1}^{n} e_i x_i' y_i.$$

The commutator of G and G' is given by $(G, G') = Z^c$, where

$$c = \Sigma\, e_i(x_i y_i' - x_i' y_i). \tag{27}$$

2. The last formula (27) enables us to determine the groups of automorphisms of the skew-groups. We denote by $\mathfrak{S}_{e_1 e_2 \ldots e_n} = \mathfrak{S}_{e_i}$ the skew-group with the elementary divisors $e_1, e_2, \ldots, e_n$, by $\mathfrak{A}_{e_1 e_2 \ldots e_n} = \mathfrak{A}_{e_i}$ its group of automorphisms.

An automorphism of $\mathfrak{S}_{e_i}$ is completely given by its effect on the basis elements, *i.e.* by a set of correspondences

$$\tag{28} \left\{ \begin{aligned} &P_i \longleftrightarrow P_i^* = P_1^{a_{i1}} Q_1^{b_{i1}} \ldots P_n^{a_{in}} Q_n^{b_{in}} Z^{u_i}, \\ &Q_i \longleftrightarrow Q_i^* = P_1^{c_{i1}} Q^{d_{i1}} \ldots P_n^{c_{in}} Q_n^{d_{in}} Z^{v_i}, \\ &\qquad\quad Z \longleftrightarrow Z^* = Z^{\pm 1}. \qquad\qquad (i = 1, 2, \ldots, n). \end{aligned} \right.$$

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† *Cf.* p. 358.

The automorphisms for which $Z^* = Z$ form a (self-conjugate) subgroup of $\mathfrak{A}_{e_i}$ of index 2. We denote it by $\mathfrak{A}'_{e_i}$. In $\mathfrak{A}'_{e_i}$ the automorphisms

$$P_i^* = P_i Z^{u_i}, \quad Q_i^* = Q_i Z^{v_i}, \tag{29}$$

which consist in multiplying every element by an element of the central $\mathfrak{Z}$ of $\mathfrak{S}_{e_i}$ form a subgroup $\mathfrak{C}_{e_i}$. The elements of $\mathfrak{C}_{e_i}$ can be described as those elements of $\mathfrak{A}_{e_i}$ which give the identical automorphism for $\mathfrak{S}_{e_i}/\mathfrak{Z}$. $\mathfrak{C}_{e_i}$ is a free Abelian group of $2n$ generators. It contains as subgroup of index $e_1^2 . e_2^2 \dots e_n^2$ the group of inner automorphisms of $\mathfrak{S}_{e_i}$† (which is, of course, isomorphic to the free Abelian group $\mathfrak{S}_{e_i}/\mathfrak{Z}$). Evidently $\mathfrak{C}_{e_i}$ is a self-conjugate subgroup of $\mathfrak{A}'_{e_i}$. We denote the factor group $\mathfrak{A}'_{e_i}/\mathfrak{C}_{e_i}$ by $\mathfrak{B}_{e_i}$.

The elements of $\mathfrak{B}_{e_i}$ give automorphisms for the free Abelian group $\mathfrak{S}_{e_i}/\mathfrak{Z}$. They have a true representation by unimodular matrices

$$\mathbf{B} = \begin{pmatrix} a_{11} & \dots & b_{1n} \\ \vdots & & \\ c_{n1} & \dots & d_{nn} \end{pmatrix}, \tag{30}$$

where the a, b, c, d have the same meaning as in (28).

THEOREM IV. *$\mathfrak{B}_{e_i}$ consists of all unimodular matrices* $\mathbf{B}$ *which, when applied simultaneously to two sets of cogredient variables* $(x_1, y_1, \dots, x_n, y_n)$ *and* $(x_1', y_1', \dots, x_n', y_n')$, *leave the skew-symmetric bilinear form*

$$\sum_{i=1}^{n} e_i(x_i y_i' - x_i' y_i)$$

absolutely invariant.

Proof. Let G and G' be any two elements of $\mathfrak{S}_{e_i}$. An automorphism for $\mathfrak{S}_{e_i}/\mathfrak{Z}$ represented by the matrix (30) gives rise to the correspondences

$$\left\{ \begin{aligned} G = P_1^{x_1} Q_1^{y_1} \dots P_n^{x_n} Q_n^{y_n} Z^g \longleftrightarrow G^* &= P_1^{*x_1} Q_1^{*y_1} \dots P_n^{*x_n} Q_n^{*y_n} Z^{*g} \\ &= P_1^{\xi_1} Q_1^{\eta_1} \dots P_n^{\xi_n} Q_n^{\eta_n} Z^{\delta}, \\ G' = P_1^{x_1'} Q_1^{y_1'} \dots P_n^{x_n'} Q_n^{y_n'} Z^{g'} \longleftrightarrow G'^* &= P_1^{\xi_1'} Q_1^{\eta_1'} \dots P_n^{\xi_n'} Q_n^{\eta_n'} Z^{\delta'}. \end{aligned} \right. \tag{31}$$

The ξ_j, η_j, ξ_j, and η_j' are found to be

$$\left\{ \begin{aligned} \xi_j &= \sum_{i=1}^{n} (a_{ij} x_i + c_{ij} y_i), & \eta_j &= \sum_{i=1}^{n} (b_{ij} x_i + d_{ij} y_i), \\ \xi_j' &= \sum_{i=1}^{n} (a_{ij} x_i' + c_{ij} y_i'), & \eta_j' &= \sum_{i=1}^{n} (b_{ij} x_i' + d_{ij} y_i') \quad (i = 1, 2, \dots, n), \end{aligned} \right. \tag{32}$$

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† Because, *e.g.*, $Q_1^{-1} P_1 Q_1 = P_1 Z^{e_1}$, etc.

i.e.

$$(33)\qquad \begin{cases} (\xi_1, \eta_1, \ldots, \xi_n, \eta_n) = (x_1, y_1, \ldots, x_n, y_n)\,.\,\mathbf{B} \\ (\xi_1', \eta_1', \ldots, \xi_n', \eta_n') = (x_1', y_1', \ldots, x_n', y_n')\,.\,\mathbf{B}. \end{cases}$$

Now the commutator of G and G' must correspond to the commutator of G^* and G'^*. This gives

$$(34)\qquad (G, G') = Z^c \longleftrightarrow (G^*, G'^*) = Z^{c^*}$$

or $c = \pm c^*$, according as $Z^* = Z^{\pm 1}$. But for the matrices of $\mathfrak{B}_{e_i}$ we have $Z^* = Z$. The formula (27) now gives

$$(35)\qquad \sum_{i=1}^{n} e_i(x_i y_i' - x_i' y_i) \equiv \sum_{i=1}^{n} e_i(\xi_i \eta_i' - \xi_i' \eta_i).$$

Conversely, a unimodular matrix **B** which satisfies the identity (35) [when the relations (33) are taken into account] obviously gives an automorphism for $\mathfrak{S}_{e_i}/\mathfrak{Z}$.

If we substitute the expressions (32) in the identity (35), we obtain the relations which hold between the coefficients of a unimodular matrix, if (and only if) it belongs to $\mathfrak{B}_{e_i}$.

We get

$$(36)\qquad \sum_{i=1}^{n} e_i(x_i y_i' - x_i' y_i) \equiv \sum_{j=1}^{n} e_j \Bigg[\sum_{i,k} \begin{vmatrix} a_{ij} & b_{ij} \\ a_{kj} & b_{kj} \end{vmatrix} x_i x_k' + \sum_{i,k} \begin{vmatrix} c_{ij} & d_{ij} \\ c_{kj} & d_{kj} \end{vmatrix} y_i y_j' + \sum_{i,k} \begin{vmatrix} a_{ij} & b_{ij} \\ c_{kj} & d_{kj} \end{vmatrix} (x_i y_k' - x_k' y_i) \Bigg].$$

This gives the equations

$$(37)\qquad \sum_{j=1}^{n} e_j \begin{vmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{vmatrix} = e_i \quad (i = 1,\ 2,\ \ldots,\ n),$$

$$(38)\qquad \sum_{j=1}^{n} e_j \begin{vmatrix} a_{ij} & b_{ij} \\ a_{kj} & b_{kj} \end{vmatrix} = \sum_{j=1}^{n} e_j \begin{vmatrix} c_{ij} & d_{ij} \\ c_{kj} & d_{kj} \end{vmatrix} = \sum_{j=1}^{n} e_j \begin{vmatrix} a_{ij} & b_{ij} \\ c_{kj} & d_{kj} \end{vmatrix} = 0$$

for $(i \neq j)$.

THEOREM V. *$\mathfrak{B}_{e_i}$ can be generated*† *by the matrices which represent the*

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† The group $\mathfrak{B}_{1,1,\ldots,1}$ occurs in various parts of mathematics. Generators for $\mathfrak{B}_{1,1,\ldots,}$ were given by Kronecker in his paper "Ueber bilineare Formen", *Werke*, I, 143–162; *Cf.* also Clebsch und Gordan, *Theorie der Abelschen Funktionen* (1866), 304–308.

following substitutions:

$$\mathbf{S}_i:\quad \begin{array}{l} x_i' = y_i, \\ y_i' = -x_i \qquad (i = 1, 2, \ldots, n). \end{array}$$

$$\mathbf{T}_i:\quad x_i' = x_i + y_i \qquad (i = 1, 2, \ldots, n).$$

$$\mathbf{V}_{ij}:\quad \begin{array}{l} x_i' = x_i + y_j, \\ x_j' = x_j + \dfrac{e_j}{e_i}\, y_i \quad (1 \leqslant i < j \leqslant n). \end{array}$$

(*All other variables are left unchanged.*)

Proof. Let **B** be any element of $\mathfrak{B}_e$. The matrices $\mathbf{S}_i$ and $\mathbf{T}_i$ permit all unimodular processes within any "pair" of rows or columns of **B**. The matrices $\mathbf{V}_{ij}$, in connection with the $\mathbf{S}_i$, permit the addition of a multiple of any of the columns in the i-th pair to any of the columns in the j-th pair, $(i < j)$, provided that a suitable multiple of the other column in the j-th pair is added to the other column in the i-th pair. We show that these processes are sufficient to reduce **B** step by step to the unit-matrix.

(i) By unimodular processes within the pairs of columns alone we can arrange that, in the first row of the matrix, all elements in the even columns vanish. Suppose that the first pair of rows is, at this stage, of the form

$$\left\{\begin{array}{llllllll} a_{11} & 0 & a_{12} & 0 & \ldots & a_{1n} & 0 \\ c_{11} & d_{11} & c_{12} & d_{12} & \ldots & c_{1n} & d_{1n}. \end{array}\right. \tag{39}$$

The relations (37) for $i = 1$ now give

$$e_1 a_{11} d_{11} + e_2 a_{12} d_{12} + \ldots + e_n a_{1n} d_{1n} = e_1, \tag{40}$$

which will be used later.

(ii) We now add the first column to the fourth, and the third, multiplied by e_2/e_1, to the second, and obtain as the first row

$$a_{11},\quad \frac{e_2}{e_1} a_{12},\quad a_{12},\quad a_{11},\quad a_{13},\quad 0,\quad \ldots,\quad a_{1n},\quad 0. \tag{41}$$

Then by unimodular processes within the first pair of columns we can obtain in the first place the greatest common divisor of a_n and $e_2/e_1\, a_{12}$, and in the second place a zero, thus

$$\left(a_{11} \quad \frac{e_2}{e_1} a_{12}\right),\quad 0,\quad a_{12},\quad a_{11},\quad a_{13},\quad 0,\quad \ldots,\quad a_m,\quad 0. \tag{42}$$

We now add the first column to the sixth, and the fifth, multiplied by e_3/e_1, to the second, and obtain as the first row

$$(43)\quad \left(a_{11},\ \frac{e_2}{e_1}a_{12}\right),\ \frac{e_3}{e_1}a_{13},\ a_{12},\ a_{11},\ a_{13},\ \left(a_{11},\ \frac{e_2}{e_1}a_{12}\right),\ \ldots,\ a_{1n},\ 0.$$

Then again we apply the unimodular processes which give in the second place a zero, and in the first place the greatest common divisor of $(a_{11},\ e_2/e_1\,a_{12})$ and $e_3/e_1\,a_{13}$ which is $(a_{11},\ e_2/e_1\,a_{12},\ e_3/e_1 a_{13})$.

Repeating these processes n times we obtain finally in the first place of the first row the greatest common divisor $(a_{11},\ e_2/e_1\,a_{12},\ \ldots,\ e_n/e_1\,a_{1n})$. But, from (39), we infer that

$$(44)\quad \begin{cases} a_{11}\,.\,d_{11}+\dfrac{e_2}{e_1}\,a_{12}\,.\,d_{12}+\ldots+\dfrac{e_n}{e_1}\,a_{1n}\,.\,d_{1n}=1, \\[2ex] \left(a_{11},\ \dfrac{e_2}{e_1}\,a_{12},\ \ldots,\ \dfrac{e_n}{e_1}\,a_{1n}\right)=1. \end{cases}$$

(iii) By subtracting suitable multiples of the first column from the third, fourth, ..., $2n$-th column we can now arrange that all coefficients in the first row vanish except the first two, and we thus get

$$(45)\qquad 1\ b\quad 0\ 0\ \ldots\ 0\ 0.$$

By subtracting the first column b times from the second we obtain

$$(46)\qquad 1\ 0\quad 0\ 0\ \ldots\ 0\ 0.$$

The relations (37) for $i=1$ indicate now that the four coefficients in the first pair of rows and columns have the determinant 1. The first pair of rows has, therefore, the form

$$(47)\quad \begin{cases} 1\ \ 0\quad 0\quad 0\quad \ldots\ 0\quad 0, \\ c\ \ 1\quad c'_{12}\ d'_{12}\ \ldots\ c'_{1n}\ d'_{1n}. \end{cases}$$

By subtracting suitable multiples of the second column from the following ones, we obtain

$$(48)\quad \begin{cases} 1\ \ 0\quad 0\ 0\ \ldots\ 0\ 0, \\ c'\ 1\quad 0\ 0\ \ldots\ 0\ 0, \end{cases}$$

and, by subtracting the first row c' times from the second, we obtain

$$(49)\quad \begin{cases} 1\ 0\quad 0\ 0\ \ldots\ 0\ 0, \\ 0\ 1\quad 0\ 0\ \ldots\ 0\ 0. \end{cases}$$

(iv) The relations (38) now show at once that all other elements in the first pair of columns vanish. We proceed with the remaining $2n-2$ rows and columns in the same way and obtain finally the unit-matrix. This concludes the proof.

If some of the e_i are equal, say $e_i = e_j$, then the exchange of the i-th and j-th pair of generators is an automorphism for $\mathfrak{S}_{e_i}$. This is expressed by the fact that $\mathfrak{B}_{e_i}$ contains the matrix†

$$(50) \qquad (\mathbf{V}_{ij}\,\mathbf{S}_i\,\mathbf{S}_j)^3 = \begin{pmatrix} 0 & \frac{e_j}{e_i}-1 & 1 & 0 \\ 1-\frac{e_j}{e_i} & 0 & 0 & 2-\frac{e_j}{e_i} \\ \frac{e_j}{e_i} & 0 & 0 & \frac{e_j}{e_i}-1 \\ 0 & \frac{e_j}{e_i}\left(2-\frac{e_j}{e_i}\right) & 1-\frac{e_j}{e_i} & 0 \end{pmatrix},$$

which, for $e_i = e_j$, reduces to

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

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† We have written only the i-th and j-th pair of rows and columns. In all other places, there are, of course, 1's in the main diagonal and zero's elsewhere.

ON CONJUGATE FUNCTIONS

By J. COSSAR.

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1. The following theorem is due to M. Riesz:

If $p > 1$ *and if*[†] $f(x)$ *is* L^p, *then there is a function* $g(x)$, *which is also* L^p, *and which almost everywhere satisfies the equations*

$$\text{(A)} \qquad g(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(t)}{t-x}\, dt, \quad f(x) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{g(t)}{t-x}\, dt.$$

Moreover, there is a number A_p, *depending on* p *but not on* f, *such that*

$$\text{(B)} \qquad \int_{-\infty}^{\infty} |g|^p\, dx \leqslant A_p \int_{-\infty}^{\infty} |f|^p\, dx.$$

The proof given by Riesz of this property of the class L^p of real functions depends on the theory of functions of a complex variable. A proof independent of this theory, due to Titchmarsh, is based on reciprocal relations between infinite series. Proofs of the case $p = 2$ [in which A_p may be taken to be 1 and there is then equality in (B)] have been given by Hardy[‡] and Titchmarsh, the second of these depending on the theory of Fourier transforms.

The object of this paper is to give a further proof of the theorem for even integral values of p. Riesz has shown[§] that the complete theorem can be deduced from this special case.

[†] Throughout this paper "f is L^p" means "f is $L^p(-\infty, \infty)$".

[‡] G. H. Hardy, "On Hilbert Transforms", *Quart. J. of Math.*, 3 (1932), 102–112. References will be found there to the other investigations mentioned. Another "complex" proof is given by P. Stein, "On a theorem of M. Riesz", *Journal London Math. Soc.*, 8 (1933), 242–247. See also E. C. Titchmarsh, *Theory of Fourier integrals* (Oxford, 1937), Ch. V.

[§] M. Riesz, "Sur les maxima des formes bilinéaires et sur les fonctionelles linéaires", *Acta Math.*, 49 (1926), 465–497 (§ 29).

2. The methods used are those of the theory of functions of a real variable. In particular, as in Hardy's paper, considerable use is made of the theory of mean convergence.

A new proof is given in §§ 3–6 for the case $p=2$; and the general case of even integral p is deduced from this and the relations of Lemma 17. This proof originated in a suggestion of Prof. Hardy that it should be possible to establish these relations by some such method as is employed here. I am further indebted to Prof. Hardy for criticism that has led me to make the argument clearer by greater use of direct calculation, particularly in § 3.

3. This paragraph contains some elementary calculations. Here and elsewhere the symbol "$P_{a,b,\dots}$", standing before an integral sign, indicates that the principal value is to be taken with respect to the points a, b, ... of the range of integration.

The numbers δ, δ' are positive.

We write

$$\begin{aligned}\lambda(x,\delta)&=x^{-1}\quad(|x|>\delta),\\&=0\qquad(|x|<\delta).\end{aligned}$$

LEMMA 1. $$I(\theta)=\int_0^\infty \log\left|\frac{u+\theta}{u-\theta}\right|\frac{du}{u}=\frac{\pi^2}{2}\operatorname{sgn}\theta.$$

It will be sufficient to prove this when $\theta>0$. Then

$$\begin{aligned}I(\theta)=\int_0^\infty \log\left|\frac{u+1}{u-1}\right|\frac{du}{u}&=2\int_0^1\log\frac{1+u}{1-u}\frac{du}{u}\\&=4\int_0^1\left(1+\frac{u^2}{3}+\frac{u^4}{5}+\dots\right)du=\frac{\pi^2}{2}.\end{aligned}$$

LEMMA 2. $$\begin{aligned}J(x,\delta)=P_0\int_{-\infty}^\infty \log\left|\frac{u-x+\delta}{u-x-\delta}\right|\frac{du}{u}&=\pi^2\quad(|x|<\delta),\\&=0\quad(|x|>\delta).\end{aligned}$$

We have

$$\begin{aligned}J(x,\delta)&=\int_0^\infty \log\left|\frac{u-x+\delta}{u-x-\delta}\,\frac{u+x+\delta}{u+x-\delta}\right|\frac{du}{u}\\&=I(x+\delta)-I(x-\delta)=\tfrac{1}{2}\pi^2\{\operatorname{sgn}(x+\delta)-\operatorname{sgn}(x-\delta)\},\end{aligned}$$

by Lemma 1, and this gives the result stated.

LEMMA 3. $\frac{1}{\pi^2} P_t \int_{-\infty}^{\infty} \log \left| \frac{u-x+\delta}{u-x-\delta} \right| \frac{du}{(u-t)(u-x)} = -\lambda(t-x, \delta)$

$(|t-x| \neq \delta)$.

When $t=x$, change the variable to v, where $v=u-x$; the integrand then becomes odd, and so the integral vanishes. When $t \neq x$, since

$$\frac{1}{(u-t)(u-x)} = \frac{1}{t-x}\left(\frac{1}{u-t} - \frac{1}{u-x}\right),$$

the integral is equal to

$$(t-x)^{-1}\{J(x-t, \delta) - J(0, \delta)\},$$

and, by Lemma 2, this gives the result stated.

LEMMA 4. *Let*

$$K(t) = K(t, \delta, \delta') = \int_{-\infty}^{\infty} \lambda(x-t, \delta)\lambda(x, \delta')\, dx \quad (\delta' \leqslant \delta).$$

Then K is an even function of t, $K>0$, and

$$tK = \log\frac{t+\delta}{t-\delta} + \log\frac{t+\delta'}{t-\delta'} \qquad (t \geqslant \delta+\delta'),$$

$$= \log\left(\frac{t}{\delta}+1\right) + \log\left(\frac{t}{\delta'}+1\right) \quad (\delta-\delta' \leqslant t \leqslant \delta+\delta'),$$

$$= \log\frac{\delta+t}{\delta-t} \qquad (0 < t \leqslant \delta-\delta').$$

If, in the integral expression for $K(-t)$, we put $x=-x'$ and observe that λ is an odd function of its first argument, we see that $K(-t)=K(t)$.

From the definition of K, it is plain that $K(0)>0$. That $K(t)>0$ for $t>0$ (and so also for $t<0$, since K is even) is clear from the logarithmic formulae for tK, which we now prove.

The integrand in the expression for K is equal to zero inside the intervals $(-\delta', \delta')$, $(t-\delta, t+\delta)$, and to

$$\frac{1}{t}\left(\frac{1}{x-t} - \frac{1}{x}\right)$$

outside both intervals. These intervals may be distinct, they may interlace, or the second may contain the first. The evaluation of tK in each case is a matter of simple calculation.

LEMMA 5. $$\int_{-\infty}^{\infty} K\,dt = \pi^2 \quad (\delta' \leqslant \delta).$$

The integrand being even, we consider the integral over $(0, \infty)$. By Lemma 4, it is equal to $L_1+L_2+L_3$, where

$$L_1 = \int_{\delta+\delta'}^{\infty} \log\frac{t+\delta'}{t-\delta'}\frac{dt}{t} + \int_{\delta-\delta'}^{\delta+\delta'} \log\frac{t+\delta'}{\delta'}\frac{dt}{t}, \qquad L_2 = \int_0^{\infty} \log\left|\frac{t+\delta}{t-\delta}\right|\frac{dt}{t},$$

$$L_3 = \int_{\delta-\delta'}^{\delta+\delta'} \log\left|\frac{t-\delta}{\delta}\right|\frac{dt}{t} = \int_{1-\delta'/\delta}^{1+\delta'/\delta} \log|t-1|\frac{dt}{t}.$$

By Lemma 1, $L_2 = \frac{1}{2}\pi^2$. A simple calculation shows that when $\delta \neq \delta'$ the derivative of L_1+L_3, with respect to δ, is zero, so that L_1+L_3 is independent of δ $(\delta > \delta')$. Its value is seen to be zero by making $\delta \to \infty$. By continuity, the value is zero also when $\delta = \delta'$. This completes the proof.

LEMMA 6. *When* $\delta' \to 0$, $tK \to \log\left|\frac{t+\delta}{t-\delta}\right| \quad (|t| \neq \delta)$.

This follows from Lemma 4.

4. The next two lemmas deal with singular integrals.

LEMMA 7. *If* $\phi(\tau)$ *is a bounded continuous function, then*

$$\int_{-\infty}^{\infty} \phi(\tau)\,K(\tau, \delta, \delta')\,d\tau \to \pi^2\phi(0) \quad (0 < \delta' \leqslant \delta \to 0).$$

By Lemma 5, we have to prove that

$$\int_0^{\infty} \psi(\tau)\,K(\tau, \delta, \delta')\,d\tau \to 0,$$

where $$\psi(\tau) = \phi(\tau)+\phi(-\tau)-2\phi(0).$$

Let $\epsilon > 0$, and choose η so that

$$|\psi(\tau)| < \epsilon \quad (0 \leqslant \tau \leqslant \eta).$$

Then, by Lemma 5, since K is positive and even,

$$\left|\int_0^{\eta} \psi K\,d\tau\right| < \epsilon\int_0^{\eta} K\,d\tau < \tfrac{1}{2}\epsilon\pi^2.$$

Also, if B is the upper bound of $|\phi|$, it follows from Lemma 4 that, when $\delta+\delta' \leqslant \eta$, we have

$$\left|\int_{\eta}^{\infty} \psi K\, d\tau\right| \leqslant 4B \int_{\eta}^{\infty} \left(\log \frac{\tau+\delta}{\tau-\delta} + \log \frac{\tau+\delta'}{\tau-\delta'}\right) \frac{d\tau}{\tau} = 4B\left(\int_{\eta/\delta}^{\infty} + \int_{\eta/\delta'}^{\infty}\right) \log \frac{\tau+1}{\tau-1} \frac{d\tau}{\tau},$$

and this tends to zero with δ, δ'. This completes the proof.

LEMMA[†] 8. *If $p > 1$ and χ is L^p, then*

$$\frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{\chi(t)}{t-x} \log \left|\frac{t-x+\delta}{t-x-\delta}\right| dt \to \chi(x),$$

when $\delta \to 0$, for almost all x and in particular at every point of continuity of χ.

5. Lemmas 9, 10 are well-known results in the theory of mean-square convergence. The number a may be ∞.

LEMMA 9. *If*

$$\lim_{\sigma,\, \sigma' \to a} \int_{-\infty}^{\infty} h_{\sigma}(x)\, h_{\sigma'}(x)\, dx$$

exists, then h_σ converges in mean-square as $\sigma \to a$.

LEMMA 10. *If*

$$\underset{\sigma \to a}{\text{l.i.m.}}\, h_{\sigma}(x) = h(x), \quad \underset{\sigma \to a}{\text{l.i.m.}}\, k_{\sigma}(x) = k(x),$$

then

$$\lim_{\sigma \to a} \int_{-\infty}^{\infty} h_{\sigma} k_{\sigma}\, dx = \int_{-\infty}^{\infty} hk\, dx.$$

6. In this paragraph, $f(x)$ is a function of L^2 and $g(x)$ is the "conjugate" or "transform" of $f(x)$, defined in Lemma 11, and denoted also by $\bar{f}(x)$.

LEMMA 11. *As $\delta \to 0$,*

$$g(x, \delta) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t)\, \lambda(t-x, \delta)\, dt$$

converges in mean-square to a function $g(x)$, and

$$\int_{-\infty}^{\infty} f^2\, dx = \int_{-\infty}^{\infty} g^2\, dx.$$

[†] Hardy, *loc. cit.*, Theorem 3, p. 109, and footnote, p. 110.

Let
$$f_n(x) = f(x) \qquad (|x| \leqslant n),$$
$$ = 0 \qquad (|x| > n),$$

and write
$$g_n(x, \delta) = \frac{1}{\pi} \int_{-\infty}^{\infty} f_n(t)\, \lambda(t-x, \delta)\, dt.$$

Then

$$\int_{-\infty}^{\infty} g_n(x, \delta)\, g_{n'}(x, \delta')\, dx$$
$$= \frac{1}{\pi^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} f_{n'}(u)\, \lambda(u-x, \delta')\, du \int_{-\infty}^{\infty} f_n(t) \lambda(t-x, \delta)\, dt.$$

When the integrands on the right-hand side are replaced by their moduli, each of the inner integrals clearly remains $O(|x|^{-1})$ for large $|x|$, so that the repeated integral is absolutely convergent. Writing $t = u + \tau$ in the third integral and integrating in a different order, we have

$$\int_{-\infty}^{\infty} g_n(x, \delta)\, g_{n'}(x, \delta')\, dx$$
$$= \frac{1}{\pi^2} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} f_{n'}(u) f_n(u+\tau)\, du \int_{-\infty}^{\infty} \lambda(u-x, \delta')\, \lambda(u-x+\tau, \delta)\, dx$$
$$= \frac{1}{\pi^2} \int_{-\infty}^{\infty} \phi_{n, n'}(\tau)\, K(\tau, \delta, \delta')\, d\tau,$$

where
$$\phi_{n, n'}(\tau) = \int_{-\infty}^{\infty} f_{n'}(u) f_n(u+\tau)\, du.$$

It is clear that
$$\phi_{n, n'}(\tau) \to \phi(\tau) \quad (n, n' \to \infty),$$

where
$$\phi(\tau) = \int_{-\infty}^{\infty} f(u) f(u+\tau)\, du;$$

and, by the inequality of Schwarz, this convergence is bounded in τ. Moreover, by Lemmas 4 and 5, $K > 0$ and

$$\int_{-\infty}^{\infty} K(\tau, \delta, \delta')\, d\tau < \infty.$$

Hence

$$\lim_{n, n' \to \infty} \int_{-\infty}^{\infty} g_n(x, \delta)\, g_{n'}(x, \delta')\, dx = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \phi(\tau)\, K(\tau, \delta, \delta')\, d\tau.$$

Taking $\delta' = \delta$, we see by Lemma 9 that, as $n \to \infty$, $g_n(x, \delta)$ converges in mean-square; and the mean-square limit is the ordinary limit, $g(x, \delta)$, since this exists. Therefore, by Lemma 10,

$$\int_{-\infty}^{\infty} g(x, \delta)\, g(x, \delta')\, dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x, \delta)\, g_n(x, \delta')\, dx$$

$$= \frac{1}{\pi^2} \int_{-\infty}^{\infty} \phi(\tau)\, K(\tau, \delta, \delta')\, d\tau.$$

It follows, by Lemma 7, that

$$\lim_{\delta' \leqslant \delta \to 0} \int_{-\infty}^{\infty} g(x, \delta)\, g(x, \delta')\, dx = \phi(0).$$

Thus, by Lemma 9, $g(x, \delta)$ converges in mean-square to a function $g(x)$; and, by Lemma 7,

$$\int_{-\infty}^{\infty} g^2\, dx = \lim_{\delta\to 0} \int_{-\infty}^{\infty} g^2(x, \delta)\, dx = \phi(0) = \int_{-\infty}^{\infty} f^2(u)\, du.$$

LEMMA 12. *If $F(t)$ is L^2 and $G(t)$ is its conjugate, then*

$$\int_{-\infty}^{\infty} fF\, dt = \int_{-\infty}^{\infty} gG\, dt.$$

With an obvious notation we have

$$g(t) = \text{l.i.m.}\, g(t, \delta), \quad G(t) = \text{l.i.m.}\, G(t, \delta).$$

Hence
$$g(t) \pm G(t) = \text{l.i.m.}\, \{g(t, \delta) \pm G(t, \delta)\},$$

and this is the transform of $f \pm F$. We thus have

$$\int_{-\infty}^{\infty} (f \pm F)^2\, dt = \int_{-\infty}^{\infty} (g \pm G)^2\, dt,$$

and the result is now plain.

LEMMA 13. *If*

$$F(t) = \lambda(t-x, \delta), \quad G(t) = \frac{1}{\pi(t-x)} \log \left| \frac{t-x+\delta}{t-x-\delta} \right|,$$

then $\bar{F} = G$, $\bar{G} = -F$.

We have

$$\bar{F}(t) = \frac{1}{\pi} \underset{\delta'\to 0}{\text{l.i.m.}} \int_{-\infty}^{\infty} \lambda(u-x, \delta)\, \lambda(u-t, \delta')\, du = \frac{1}{\pi} \underset{\delta'\to 0}{\text{l.i.m.}}\, K(x-t, \delta, \delta').$$

But†, by Lemma 6,

$$\lim_{\delta'\to 0} K(x-t, \delta, \delta') = \pi G(t).$$

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† Strictly, if $t \neq x$, $x \pm \delta$. Statements such as "$g = \bar{f}$" assert equivalence only. Further instances occur later, *e.g.*, in the proof of Lemma 16.

This proves the first part. Again,

$$\bar{G}(t) = \frac{1}{\pi} \underset{\delta' \to 0}{\text{l.i.m.}} \int_{-\infty}^{\infty} \frac{1}{\pi(u-x)} \log \left| \frac{u-x+\delta}{u-x-\delta} \right| \lambda(u-t, \delta')\, du$$

$$= \frac{1}{\pi^2} P_t \int_{-\infty}^{\infty} \log \left| \frac{u-x+\delta}{u-x-\delta} \right| \frac{du}{(u-x)(u-t)},$$

and, by Lemma 3, this is equal to $-F(t)$.

LEMMA 14. *For almost all x, f and g satisfy the equations* (A) *of* § 1.

Apply Lemma 12, taking F, G to be the functions of Lemma 13. Then

$$\int_{-\infty}^{\infty} f(t) \lambda(t-x, \delta)\, dt = \int_{-\infty}^{\infty} g(t) \frac{1}{\pi(t-x)} \log \left| \frac{t-x+\delta}{t-x-\delta} \right| dt,$$

and, by applying Lemma 8, with $\chi = g$, we obtain, for almost all x, the first of the equations (A). The second is obtained similarly, by replacing F, G by G, $-F$ in the equation of Lemma 12; this is permissible since, by Lemma 13, the transform of G is $-F$.

7. The case $p = 2$ of the theorem of § 1 is a consequence of Lemmas 11 and 14. We now proceed to establish the theorem in the case $p = 2k$, where k is any positive integer.

8. The object of this paragraph is the proof of Lemma 17.

LEMMA 15. *Let k be a positive integer, and let*

$$(\xi + i\eta)^k = R_k(\xi, \eta) + i I_k(\xi, \eta).$$

Then

$$(a)\ \ \xi R_k - \eta I_k = R_{k+1}, \quad \xi I_k + \eta R_k = I_{k+1}, \qquad (b)\ \ R_k^2 - I_k^2 = R_{2k}.$$

LEMMA 16. *Suppose that f has a bounded derivative and vanishes outside a finite interval, that F is L^2, and that $g = \bar{f}$, $G = \bar{F}$. Then $fF - gG$ is L^2 and*

$$\overline{fF - gG} = fG + gF.$$

If $(-X, X)$ is an interval outside which f vanishes,

$$\pi g(x) = \int_{-2X}^{2X} \frac{f(t) - f(x)}{t-x}\, dt + f(x)\, P \int_{-2X}^{2X} \frac{dt}{t-x}.$$

Now, on the right-hand side, the first integrand lies between the bounds of f', and the second term is equal to

$$f(x)\log\left|\frac{2X-x}{2X+x}\right|;$$

this is bounded for all x since f is bounded and vanishes outside $(-X, X)$.

Thus g is bounded, as well as f. Hence the functions fF, gG, $fF-gG$ are L^2. Then (the suppressed arguments all being x) we have

$$\overline{fF}=\frac{1}{\pi}\,P\int_{-\infty}^{\infty}\frac{f(t)\,F(t)}{t-x}\,dt=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)-f(x)}{t-x}\,F(t)\,dt+\frac{1}{\pi}\,f(x)\,P\int_{-\infty}^{\infty}\frac{F(t)}{t-x}\,dt$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}G_1(t)\,G(t)\,dt+fG,$$

by Lemma 12, where $G_1(t)$ is the transform of the function (of t)

$$\frac{f(t)-f(x)}{t-x}.$$

Now

$$\pi G_1(t)=P_t\int_{-\infty}^{\infty}\frac{f(u)-f(x)}{(u-x)(u-t)}\,du$$

$$=\frac{1}{t-x}\,P_t\int_{-\infty}^{\infty}\{f(u)-f(x)\}\left(\frac{1}{u-t}-\frac{1}{u-x}\right)du$$

$$=\frac{1}{t-x}\,P_{t,x}\int_{-\infty}^{\infty}f(u)\left(\frac{1}{u-t}-\frac{1}{u-x}\right)du$$

$$-\frac{f(x)}{t-x}\,P_{t,x}\int_{-\infty}^{\infty}\left(\frac{1}{u-t}-\frac{1}{u-x}\right)du$$

$$=\pi\,\frac{g(t)-g(x)}{t-x}+0.$$

Hence

$$\overline{fF}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(t)-g(x)}{t-x}\,G(t)\,dt+fG$$

$$=\frac{1}{\pi}\,P\int_{-\infty}^{\infty}\frac{g(t)\,G(t)}{t-x}\,dt-\frac{1}{\pi}\,g(x)\,P\int_{-\infty}^{\infty}\frac{G(t)}{t-x}\,dt+fG$$

$$=\overline{gG}+gF+fG,$$

since $\bar{G}=-F$; and the result follows.

LEMMA 17. *Suppose that f, g are defined as in Lemma 16. Then, for each positive integer k, $R_k(f, g)$ is L^2, and*

$$\overline{R_k(f, g)} = I_k(f, g).$$

The function $R_k(f, g)$ is a polynomial in the bounded functions f, g, and so, since $f(x)$ vanishes and $g(x)$ is plainly $O(|x|^{-1})$ for large $|x|$, R_k is L^2.

The second part will be proved by induction. It is clearly true when $k = 1$. Suppose it to be true when $k = m$. Then in Lemma 16 we may take

$$F = R_m(f, g), \quad G = I_m(f, g);$$

so that $fR_m - gI_m$ is L^2 and $\overline{fR_m - gI_m} = fI_m + gR_m$, or, by Lemma 15, (*a*),

$$\overline{R_{m+1}(f, g)} = I_{m+1}(f, g).$$

9. LEMMA 18. *If f is L^{2k}, where k is a positive integer, then the function*

$$g(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(t)}{t-x}\, dt$$

exists almost everywhere; and there is a number A_{2k}, depending only on k, such that

$$\int_{-\infty}^{\infty} g^{2k}\, dx \leqslant A_{2k} \int_{-\infty}^{\infty} f^{2k}\, dx.$$

(*a*) Suppose first that f has a bounded derivative and vanishes outside a finite interval. Then g exists and, by Lemma 17, $R_k(f, g)$ is L^2 and $\overline{R_k(f, g)} = I_k(f, g)$. Hence

$$\int_{-\infty}^{\infty} R_k{}^2\, dx = \int_{-\infty}^{\infty} I_k{}^2\, dx,$$

or, by Lemma 15, (*b*),

$$\int_{-\infty}^{\infty} R_{2k}\, dx = 0,$$

whence[†]

$$\int_{-\infty}^{\infty} g^{2k}\, dx = \int_{-\infty}^{\infty} \left\{ \binom{2k}{2} f^2 g^{2k-2} - \binom{2k}{4} f^4 g^{2k-4} + \dots \right\} dx$$
$$< \int_{-\infty}^{\infty} \left\{ \binom{2k}{2} f^2 g^{2k-2} + \binom{2k}{6} f^6 g^{2k-6} + \dots \right\} dx.$$

[†] *Cf.* M. Riesz, "Sur les fonctions conjuguées", *Math. Zeitschrift*, 27 (1927), 221.

Write
$$\int_{-\infty}^{\infty} g^{2k}\,dx = X^{2k}\int_{-\infty}^{\infty} f^{2k}\,dx;$$

then, by Hölder's inequality, for $1 \leqslant r \leqslant k-1$,
$$\int_{-\infty}^{\infty} f^{2r} g^{2k-2r}\,dx \leqslant \left(\int_{-\infty}^{\infty} f^{2k}\,dx\right)^{r/k} \left(\int_{-\infty}^{\infty} g^{2k}\,dx\right)^{1-r/k} = X^{2k-2r}\int_{-\infty}^{\infty} f^{2k}\,dx.$$

Thus
$$X^{2k} < \binom{2k}{2} X^{2k-2} + \binom{2k}{6} X^{2k-6} + \ldots,$$

so that X does not exceed the positive root of the equation
$$x^{2k} = \binom{2k}{2} x^{2k-2} + \binom{2k}{6} x^{2k-6} + \ldots.$$

This completes the proof for the type of function f considered.

(b) We next remove the restriction that f should have a bounded derivative, and suppose only that f vanishes outside a finite interval. The existence of g follows from the fact that f is L^{2k}, and is therefore L^2.

There is a sequence $\{f_n\}$, each function of which has a bounded derivative and vanishes outside the interval outside which f vanishes, such that
$$\int_{-\infty}^{\infty} (f-f_n)^{2k}\,dx \to 0 \quad (n \to \infty).$$

This entails

$$(*) \qquad \int_{-\infty}^{\infty} f_n^{2k}\,dx \to \int_{-\infty}^{\infty} f^{2k}\,dx$$

and also, by Hölder's inequality (the effective range of integration being finite),
$$\int_{-\infty}^{\infty} (f-f_n)^2\,dx \to 0,$$

so that
$$\int_{-\infty}^{\infty} (g-g_n)^2\,dx \to 0,$$

where $g_n = \bar{f}_n$. There is, therefore, a sequence of integers $\{n_i\}$ such that
$$g_{n_i}(x) \to g(x) \quad (i \to \infty)$$

for almost all x. It then follows that
$$\int_{-\infty}^{\infty} g^{2k}\,dx \leqslant \varliminf_{i\to\infty} \int_{-\infty}^{\infty} g_{n_i}^{2k}\,dx \leqslant \varliminf_{i\to\infty} A_{2k} \int_{-\infty}^{\infty} f_{n_i}^{2k}\,dx = A_{2k}\int_{-\infty}^{\infty} f^{2k}\,dx.$$

The first inequality is that of Fatou, the second follows from part (*a*) of the proof, and the last step is a consequence of (*).

(*c*) Finally, we suppose only that f is L^{2k}. It is clear that g exists, since f is L^2 in every finite interval and the integral defining g converges (absolutely) at $\pm\infty$ by Hölder's inequality.

Let
$$\begin{aligned} f_n(x) &= f(x) \quad (|x| \leqslant n), \\ &= 0 \qquad (|x| > n), \end{aligned}$$

and let $g_n = \bar{f}_n$. Then

$$g_n(x) = \frac{1}{\pi} P \int_{-n}^{n} \frac{f(t)}{t-x}\, dt \to g(x) \quad (n \to \infty)$$

for almost all x. It follows that

$$\int_{-\infty}^{\infty} g^{2k}\, dx \leqslant \varliminf_{n\to\infty} \int_{-\infty}^{\infty} g_n^{2k}\, dx \leqslant \varliminf_{n\to\infty} A_{2k} \int_{-\infty}^{\infty} f_n^{2k}\, dx \leqslant A_{2k} \int_{-\infty}^{\infty} f^{2k}\, dx.$$

The first inequality is that of Fatou, the second follows from part (*b*) of the proof, and the last follows from the inequality $f_n^{2k} \leqslant f^{2k}$.

10. To complete the proof of M. Riesz's theorem in the case $p = 2k$, we have now, in view of Lemma 18, only to prove the second of the equations (A) of § 1.

With the notation of part (*c*) of the proof of Lemma 18 we have, by Hölder's inequality,

$$\left| \int_{-\infty}^{\infty} \{g(t) - g_n(t)\} \lambda(t-x, \delta)\, dt \right|^{2k} \leqslant \int_{-\infty}^{\infty} (g-g_n)^{2k}\, dt \left(\int_{-\infty}^{\infty} |\lambda|^{2k/(2k-1)}\, dt \right)^{2k-1}.$$

The first factor on the right-hand side does not exceed, by Lemma 18,

$$A_{2k} \int_{-\infty}^{\infty} (f-f_n)^{2k}\, dx,$$

which tends to zero as $n \to \infty$; and the second factor is finite and independent of n. Hence

$$\int_{-\infty}^{\infty} g(t) \lambda(t-x, \delta)\, dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(t) \lambda(t-x, \delta)\, dt.$$

The factors of the last integrand are both L^2, so that, by Lemma 12, they may be replaced by their transforms, which are $-f_n(t)$ and, by Lemma 13,

$$\frac{1}{\pi(t-x)} \log \left| \frac{t-x+\delta}{t-x-\delta} \right|.$$

Therefore

$$\int_{-\infty}^{\infty} g(t)\,\lambda(t-x,\,\delta)\,dt = -\frac{1}{\pi} \lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{f_n(t)}{t-x} \log \left| \frac{t-x+\delta}{t-x-\delta} \right| dt$$

$$= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t-x} \log \left| \frac{t-x+\delta}{t-x-\delta} \right| dt,$$

and, by Lemma 8, with $\chi = f$, this tends to $-\pi f(x)$ for almost all x as $\delta \to 0$.

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THE METHOD OF IMAGES APPLIED TO THE PROBLEM OF TORSION*

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1. *Introduction.*

This paper deals with the problem of torsion of a uniform beam of isotropic material under no body forces. The method of images gives an immediate formal solution for the following cross-sections: (i) the equilateral triangle, (ii) the rectangle, (iii) the isosceles right-angled triangle, (iv) the 30°-60°-90° triangle. The chief interest of the present paper is the unification provided by the use of a single method. The solutions obtained in each of the four cases agree with those derived by other methods†. The torsional rigidities have been computed for all but (iv): to demonstrate the method of images we shall investigate (iv) and derive the stress components, warping function and finally the torsional rigidity. The application of the method to each of (i), (ii), (iii), and (iv) follows precisely the same lines.

We take rectangular axes of x and y in a plane of the cross-section and denote the warping function and its conjugate by ϕ and ψ respectively‡. If we denote the stress components by E_{ij} $(i, j = 1, 2, 3)$ the only non-vanishing ones are E_{13} and E_{23}, and they are given in terms of ϕ by

$$(1.1) \qquad E_{13} = \mu\tau\left(\frac{\partial\phi}{\partial x} - y\right), \qquad E_{23} = -\mu\tau\left(\frac{\partial\phi}{\partial y} + x\right),$$

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† Saint-Venant, *Mém. des Savants étrangers*, 14 (1856), 233–560; solutions for (i), (ii). Galerkin, B. G., *Bull. de l'Acad. des sci. de Russie*, 13 (1919), 111–118; solution for (iii). Kolossof, G., *Comptes rendus*, 178 (1924), 2057–2060; solution for (iii). Seth, B. R., *Quart. J. of Math.* (Oxford series), 5 (1934), 161–171; solutions for (i), (iii), (iv).

‡ For a detailed account of this general theory, see A. E. H. Love, *Theory of elasticity* (4th ed., 1934), 310–322.

where μ is the rigidity and τ is the angle of twist per unit length. We write

$$\chi = \psi - \tfrac{1}{2}(x^2+y^2), \tag{1.2}$$

where χ is a function* satisfying

$$\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} = -2, \tag{1.3}$$

with the condition

$$\chi = 0 \tag{1.4}$$

on the boundary of the cross-section. In terms of χ the stress components are

$$E_{13} = \mu\tau \frac{\partial \chi}{\partial y}, \qquad E_{23} = -\mu\tau \frac{\partial \chi}{\partial x}. \tag{1.5}$$

From Love, 322, we have for the torsional rigidity

$$C = 2\mu \iint \chi \, dx \, dy. \tag{1.6}$$

2. *The method of images and the complex stress.*

To agree with later notation we denote the cross-section of the beam by B_{00}. We have to determine χ to satisfy (1.3) throughout B_{00} and (1.4) on the bounding curve of B_{00}. It is a well-known result in the theory of potential† that (1.3) is satisfied at all points inside B_{00} by

$$\chi(x, y) = -\frac{1}{\pi} \iint_{B_{00}} \log r \, dx' \, dy' + \chi_0(x, y), \tag{2.1}$$

where x' and y' are current coordinates over B_{00}, r is the distance from (x, y) to (x', y'), and χ_0 is any plane harmonic function. In (2.1) the part involving the double integral is precisely the logarithmic potential at (x, y) of a surface distribution over B_{00} of density $1/\pi$. For certain sections it is possible to satisfy the boundary condition (1.4) by taking for χ_0 the logarithmic potential at (x, y) of a doubly infinite set of distributions external to B_{00} of surface density σ/π, where $\sigma = \pm 1$. The only sections to which this procedure can be applied‡ are the four mentioned in § 1; and for any one of these sections χ_0 will have the value

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* This function is denoted by Φ by Love, 315.

† See O. D. Kellogg, *Foundations of potential theory* (1929), 172–175.

‡ See W. Burnside, *Theory of groups* (2nd ed., Cambridge, 1911), 410–418.

$$\chi_0(x, y) = -\frac{1}{\pi} \sum_{m,n}{}' \iint_{B_{mn}} \log r \, dx' \, dy', \tag{2.2}$$

where B_{mn} $(m \neq 0, n \neq 0)$ denotes the general external distribution, the dash on the Σ signifies that the term $m = 0$, $n = 0$ is omitted from the summation, and σ is constant $(= \pm 1)$ over each of the distributions. Thus χ is known if the double summation in (2.2) can be performed. This process is found difficult to carry out and a less direct method is employed.

Since χ is built up from integrations over a number of surface distributions, we can consider the contribution to χ from an element of area $dx' \, dy'$ situated either inside or outside B_{00}, and of surface density σ/π. Thus the contribution to χ, which we shall denote by χ_1, is

$$\chi_1 = -\frac{\sigma}{\pi} \log r \, dx' \, dy'. \tag{2.3}$$

Since χ_1 is plane harmonic in x and y, it possesses a conjugate χ_1', where

$$\frac{\partial \chi_1}{\partial x} = \frac{\partial \chi_1'}{\partial y}, \qquad \frac{\partial \chi_1}{\partial y} = -\frac{\partial \chi_1'}{\partial x}. \tag{2.4}$$

Also χ_1 and χ_1' are respectively the real and imaginary parts of an analytic function of the complex variable z $[= x + iy, i = \sqrt{(-1)}]$ and so

$$\chi_1 + i\chi_1' = -\frac{\sigma}{\pi} \log (z' - z) \, dx' \, dy', \tag{2.5}$$

where $z' = x' + iy'$. Upon differentiating (2.5) partially with respect to x and using (2.4), we obtain

$$\frac{\partial \chi_1}{\partial x} - i \frac{\partial \chi_1}{\partial y} = \frac{\sigma}{\pi} \frac{dx' \, dy'}{z' - z}, \tag{2.6}$$

and integration of (2.6) over all distributions in the plane produces

$$\frac{\partial \chi}{\partial x} - i \frac{\partial \chi}{\partial y} = \frac{1}{\pi} \sum_{m,n} \iint_{B_{mn}} \frac{\sigma}{z' - z} \, dx' \, dy'. \tag{2.7}$$

We introduce the components of stress into (2.7) by means of (1.5) and define the *complex stress* S by the relation

$$S = E_{13} - i E_{23}. \tag{2.8}$$

Hence we have

$$S(z) = \frac{i\mu\tau}{\pi} \sum_{m,n} \iint_{B_{mn}} \frac{\sigma}{z'-z}\, dx'\, dy'. \tag{2.9}$$

We now proceed to reduce the double integrals in (2.9) to contour integrals.

We shall denote by $\bar{g}$ the conjugate complex of any complex function g. If we suppose that w is a general (not necessarily analytic) function without poles and with continuous partial derivatives in a region F, then

$$\iint_F \left(\frac{\partial w}{\partial x'} - i\frac{\partial w}{\partial y'}\right) dx'\, dy' = i \oint_F w\, d\bar{z}'. \tag{2.10}$$

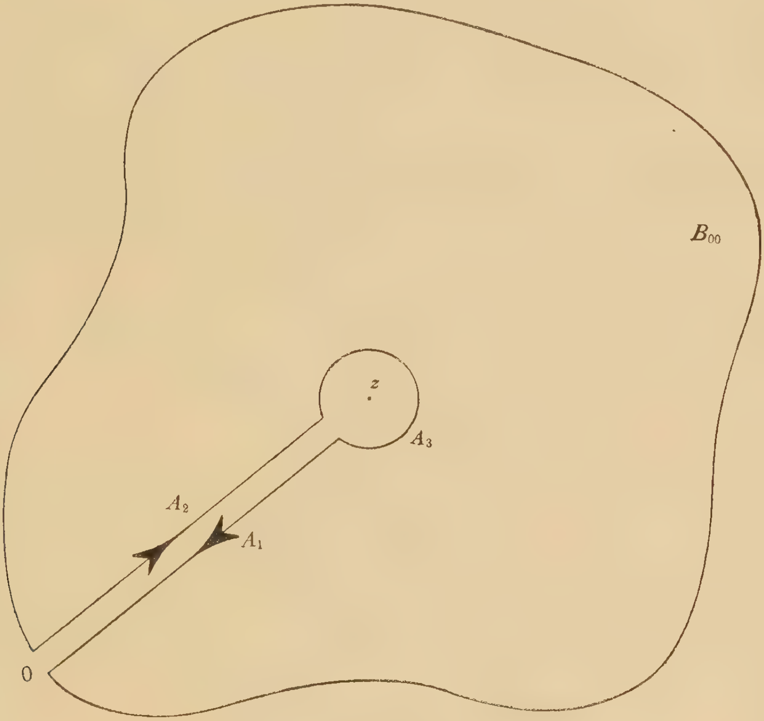

Fig. 1.

Now

$$\left(\frac{\partial}{\partial x'} - i\frac{\partial}{\partial y'}\right) \log(z'-z) = \frac{2}{\bar{z}'-\bar{z}}, \tag{2.11}$$

and hence, for any region B_{mn} ($m \neq 0$, $n \neq 0$) which does not contain z,

$$\iint_{B_{mn}} \frac{dx'\, dy'}{z'-z} = \frac{i}{2} \oint_{B_{mn}} \log(\bar{z}'-\bar{z})\, d\bar{z}', \tag{2.12}$$

any continuous determination of $\log(\bar{z}'-\bar{z})$ being used. For the region B_{00} containing z we make a cut from z to the origin (Fig. 1), which is chosen on the boundary of B_{00}, and obtain

$$\iint_{B_{00}} \frac{dx'\,dy'}{z'-z} = \frac{i}{2} \lim_{A_3 \to 0} \oint_{B_{00}+A_1+A_2+A_3} \log(z'-z)\,d\bar{z}', \tag{2.13}$$

and, since

$$\lim_{A_3 \to 0} \oint_{A_3} \log(z'-z)\,d\bar{z}' = 0, \tag{2.14}$$

$$[\log(z'-z)]_{A_2} = [\log(z'-z)]_{A_1} + 2\pi i, \tag{2.15}$$

we deduce

$$\iint_{B_{00}} \frac{dx'\,dy'}{z'-z} = -\pi\bar{z} + \frac{i}{2} \oint_{B_{00}} \log(z'-z)\,d\bar{z}'. \tag{2.16}$$

Using (2.12) and (2.16), we can write (2.9) in the form

$$\frac{S\pi}{i\mu\tau} = -\pi\bar{z} + \frac{i}{2} \sum_{m,n} \oint \sigma \log(z'-z)\,d\bar{z}'. \tag{2.17}$$

This may also be written in the form

$$T = \sum_{m,n} \oint_{B_{mn}} \sigma \log(z'-z)\,d\bar{z}', \tag{2.18}$$

where

$$T = -\frac{2\pi}{\mu\tau}(S + i\mu\tau\bar{z}). \tag{2.19}$$

Now T is an analytic function and, assuming that we can differentiate the series term by term three times, we have

$$\frac{d^3 T}{dz^3} = 2 \sum_{m,n} \oint_{B_{mn}} \sigma \frac{d\bar{z}'}{(z-z')^3}. \tag{2.20}$$

In all cases capable of treatment by the method of images, the image or external distributions B_{mn} group themselves into a double infinity of identical groups. Each group is hexagonal or rectangular in shape and the centres of the groups lie at the corners of a network of parallelograms covering the plane. If we define the sides of one of these parallelograms by the complex vectors $2\omega_1$, $2\omega_2$, then

$$\Omega_{mn} = 2m\omega_1 + 2n\omega_2 \tag{2.21}$$

is the vector defining the centre of the group m, n. It is to be noted that m, n now specify a group of distributions rather than a single one, and, for the distributions belonging to this group, we put

$$z'' = z' - \Omega_{mn}, \tag{2.22}$$

so that, for any one of these distributions,

$$\oint \frac{d\bar{z}'}{(z-z')^3} = \oint \frac{d\bar{z}''}{(z-\Omega_{mn}-z'')^3}, \tag{2.23}$$

the latter integral being taken around the corresponding distribution belonging to the group containing the distribution B_{00}. This group will be called the *fundamental group* and the distribution B_{00} will be called the *fundamental distribution.*

We now compute the contribution to d^3T/dz^3 from all the edges, one in each group, corresponding to an edge PQ in the fundamental group. This contribution is

$$\begin{aligned} 2 \sum_{m,n=-\infty}^{\infty} \sigma \int_{PQ} \frac{d\bar{z}''}{(z-z''-\Omega_{mn})^3} &= 2\sigma \int_{PQ} \sum_{m,n=-\infty} \frac{d\bar{z}''}{(z-z''-\Omega_{mn})^3} \\ &= -\sigma \int_{PQ} \wp'(z-z'')\, d\bar{z}'', \end{aligned} \tag{2.24}$$

where $\wp$ is the Weierstrass elliptic function, $\wp'$ is its first derivative and $2\omega_1$ and $2\omega_2$ are its periods, so that

$$\wp(z+2\omega_1) = \wp(z), \quad \wp(z+2\omega_2) = \wp(z). \tag{2.25}$$

Then (2.20) becomes

$$\frac{d^3T}{dz^3} = -\Sigma\sigma \int_{PQ} \wp'(z-z'')\, d\bar{z}'', \tag{2.26}$$

where the summation now extends over all the edges of each distribution in the fundamental group. If the argument of PQ is an angle θ, then

$$d\bar{z}'' = dz''\, e^{-2\theta i}, \tag{2.27}$$

and so

$$\frac{d^3T}{dz^3} = \Sigma\sigma[\wp(z-z_Q) - \wp(z-z_P)]\, e^{-2\theta i}, \tag{2.28}$$

where the summation extends over all the edges of each distribution in the fundamental group and in each distribution the sense from P to Q is counter-clockwise.

For reference we quote the Fourier expansion

$$(2.29)\qquad \frac{\pi^2}{12}\left(1-\frac{3x^2}{a^2}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{a}\qquad(-a\leqslant x\leqslant a),$$

and the summations arising from (2.29) when $x=0$ and $x=a$,

$$(2.30)\qquad \frac{\pi^2}{12}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2},$$

$$(2.31)\qquad \frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

We also quote the expansions*

$$(2.32)\qquad \begin{cases} \wp(z)=-\dfrac{\eta_1}{\omega_1}+\left(\dfrac{\pi}{2\omega_1}\right)^2\operatorname{cosec}^2\dfrac{\pi z}{2\omega_1}-\dfrac{2\pi^2}{\omega_1^2}\displaystyle\sum_{n=1}^{\infty}\dfrac{nq^{2n}}{1-q^{2n}}\cos\dfrac{n\pi z}{\omega_1}, \\ \wp(z+\omega_1)=-\dfrac{\eta_1}{\omega_1}+\left(\dfrac{\pi}{2\omega_1}\right)^2\sec^2\dfrac{\pi z}{2\omega_1}-\dfrac{2\pi^2}{\omega_1^2}\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^n nq^{2n}}{1-q^{2n}}\cos\dfrac{n\pi z}{\omega_1}, \\ \wp(z+\omega_2)=-\dfrac{\eta_1}{\omega_1}-\dfrac{2\pi^2}{\omega_1^2}\displaystyle\sum_{n=1}^{\infty}\dfrac{nq^{n}}{1-q^{2n}}\cos\dfrac{n\pi z}{\omega_1}, \\ \wp(z+\omega_1+\omega_2)=-\dfrac{\eta_1}{\omega_1}-\dfrac{2\pi^2}{\omega_1^2}\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^n nq^{n}}{1-q^{2n}}\cos\dfrac{n\pi z}{\omega_1}, \end{cases}$$

where

$$(2.33)\qquad q=e^{\pi\omega_2 i/\omega_1},$$

$$(2.34)\qquad \eta_1=\frac{1}{\omega_1}+\sideset{}{'}\sum_{m,\,n=-\infty}^{\infty}\left[\frac{1}{\omega_1-\Omega_{mn}}+\frac{1}{\Omega_{mn}}+\frac{\omega_1}{\Omega_{mn}^2}\right],$$

where the dash on the Σ-sign means that the term $m=0$, $n=0$ is to be omitted. These formulae hold for $z=\delta\omega_1+\zeta\omega_2$, where δ and ζ are real and $|\zeta|<\frac{1}{2}$.

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* See J. Tannery and J. Molk, *Fonctions elliptiques*, 4 (Paris, 1902), 101.

3. *The complex stress for the* 30°-60°-90° *triangle.*

Fig. 2 shows the part of the display of distributions adjacent to the origin. The fundamental distribution is hatched and the distributions group themselves into identical hexagons in each of which the distributions have alternate positive and negative signs.

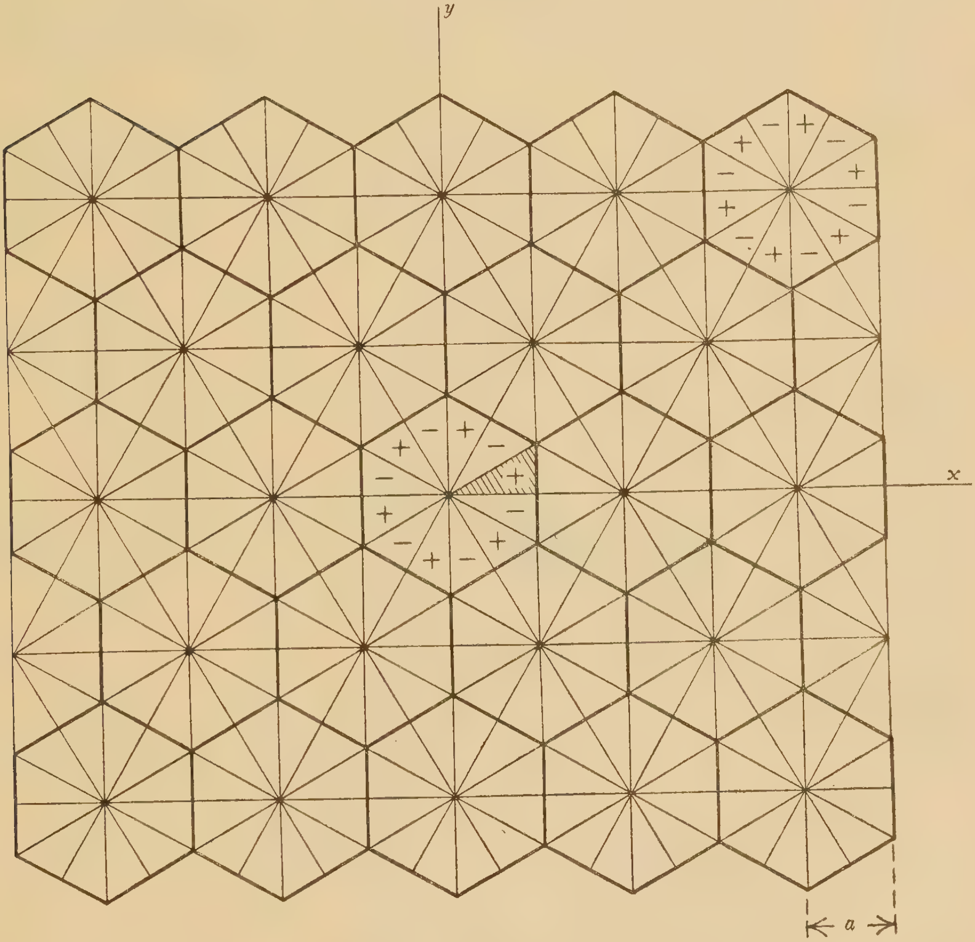

Fig. 2.

Fig. 3 shows the fundamental hexagon, the fundamental distribution being hatched. To compute d^3T/dz^3 by means of (2.28) we must add the contributions from each of the three sides of each of the twelve triangles in the fundamental hexagon. The twelve straight lines radiating from O

will then have to be considered twice and all others in Fig. 3 once. If we put $\alpha = e^{\frac{1}{3}\pi i}$, then, since $1-\alpha+\alpha^2=0$, we obtain

$$(3.1)\quad \frac{d^3T}{dz^3} = 4\{\wp(z-z_1)-\alpha\wp(z-z_3)+\alpha^2\wp(z-z_5) +\wp(z-z_7)-\alpha\wp(z-z_9)+\alpha^2\wp(z-z_{11})\} +3\{\wp(z-z_4)-\alpha\wp(z-z_6)+\alpha^2\wp(z-z_8) +\wp(z-z_{10})-\alpha\wp(z-z_{12})+\alpha^2\wp(z-z_2)\}.$$

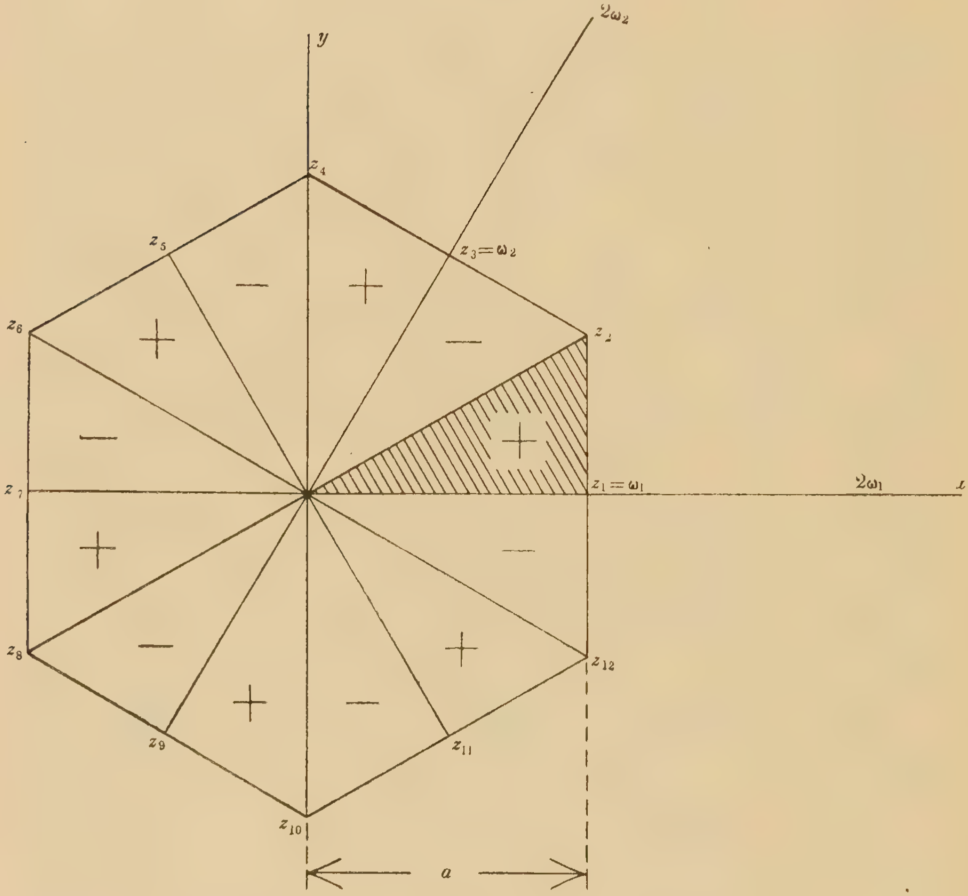
$2\omega_2$
y
z_4
z_5
$z_3=\omega_2$
z_6
z_2
z_7
$z_1=\omega_1$
$2\omega_1$
x
z_8
z_{12}
z_9
z_{11}
z_{10}
a
+
−
+
−
+
−
+
−
+
−
+
−

Fig. 3.

As stated before, the centres of the hexagons lie at the corners of a network of parallelograms, and

$$(3.2)\qquad \omega_1 = a, \quad \omega_2 = a\alpha,$$

whence

$$(3.3)\qquad \begin{cases} z_1=\omega_1, & z_3=\omega_2, & z_5=-\omega_1+\omega_2,\\ z_7=-\omega_1, & z_9=-\omega_2, & z_{11}=\omega_1-\omega_2,\\ z_2=z_2, & z_4=-z_2+2\omega_2, & z_6=z_2-2\omega_1,\\ z_8=-z_2, & z_{10}=z_2-2\omega_2, & z_{12}=-z_2+2\omega_1. \end{cases}$$

Recalling the periodicity relations (2.25), we reduce (3.1) to

$$(3.4)\qquad \frac{1}{8}\frac{d^3T}{dz^3}=\wp(z+\omega_1)-a\wp(z+\omega_2)+a^2\wp(z+\omega_1+\omega_2).$$

With the help of (2.32) we obtain

$$(3.5)\qquad \frac{1}{8}\frac{d^3T}{dz^3}=\left(\frac{\pi}{2\omega_1}\right)^2\sec^2\frac{\pi z}{2\omega_1}+\frac{2\pi^2}{\omega_1{}^2}\sum_{n=1}^{\infty}\frac{(-1)^n nq^n}{1+q^n}\cos\{n\pi z/\omega_1\}$$

$$+\frac{4a\pi^2}{\omega_1{}^2}\sum_{n=1}^{\infty}\frac{(2n-1)\,q^{2n-1}}{1-q^{2(2n-1)}}\cos\{(2n-1)\,\pi z/\omega_1\},$$

where

$$(3.6)\qquad q=e^{\pi\omega_2 i/\omega_1}.$$

We note that

$$(3.7)\qquad \int_0^z\log\cos\frac{\pi z}{2\omega_1}\,dz=\int_0^z\log\{\tfrac{1}{2}(e^{\pi iz/2\omega_1}+e^{-\pi iz/2\omega_1})\}\,dz$$

$$=\int_0^z\log\{\tfrac{1}{2}e^{-\pi iz/2\omega_1}(1+e^{\pi iz/\omega_1})\}\,dz$$

$$=A''z^2+B''z+D''-\frac{\omega_1 i}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}e^{n\pi iz/\omega_1},$$

where A'', B'', and D'' are certain complex constants; and hence, upon integrating (3.5) three times, we obtain

$$(3.8)\qquad \tfrac{1}{8}T-Az^2-Bz-D$$

$$=\frac{\omega_1 i}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}e^{n\pi iz/\omega_1}-\frac{2\omega_1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{n^2(1+q^n)}\sin\{n\pi z/\omega_1\}$$

$$-\frac{4a\omega_1}{\pi}\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(2n-1)^2(1-q^{2(2n-1)})}\sin\{(2n-1)\,\pi z/\omega_1\}$$

$$=\frac{\omega_1 i}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\,\frac{\cosh\{n\pi i(2z-\omega_2)/(2\omega_1)\}}{\cosh\{\pi ni\,\omega_2/(2\omega_1)\}}$$

$$-\frac{2\omega_1 ia}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\,\frac{\sinh\{(2n-1)\,\pi iz/\omega_1\}}{\sinh\{(2n-1)\,\pi i\omega_2/\omega_1\}},$$

where A, B, and D are unknown complex constants; to determine them, we know that the stress must vanish at the corners, *i.e.*

$$(3.9)\qquad S(0)=S(a)=S(z_2)=0,\quad z_2=2a\beta/\sqrt{3},\quad \beta=e^{\frac{1}{6}\pi i}.$$

With the help of (2.19), (2.30), and (2.31), these three conditions give rise to three linear equations in A, B, and D; and upon solving them we obtain

$$(3.10)\qquad Aa=-B=\frac{\pi}{16}(-\sqrt{3}+5i)+\frac{3P}{4\pi}(-\sqrt{3}+i),\quad D=-\frac{\pi a i}{12},$$

where

$$(3.11)\qquad P=\sum_{n=1}^{\infty}\frac{P_n}{n^2},\quad P_{2n-1}=\frac{-\cosh(2n-1)\xi+(-1)^n\sqrt{3}\sinh 2(2n-1)\xi}{\cosh 3(2n-1)\xi},$$

$$P_{2n}=\frac{-\cosh 2n\xi+(-1)^{n+1}\cosh 4n\xi}{(-1)^n+\cosh 6n\xi},$$

$$\xi=\pi/(2\sqrt{3}).$$

Now P is a real constant and we are able to perform the summation indicated in (3.11) in the following manner. Since the normal stress along Oz_1 (Fig. 3) vanishes, we must have

$$(3.12)\qquad I[S(x)]=0\quad (0\leqslant x\leqslant a),$$

where $R[g]$ and $I[g]$ denote the real and imaginary parts respectively of g. By (2.19) we then have

$$(3.13)\qquad I[T(x)]=-2\pi x,$$

and, by (2.29) and (3.8), we find that (3.13) is satisfied if $P=-\frac{1}{12}\pi^2$; hence (3.10) reduces to

$$(3.14)\qquad Aa=-B=\tfrac{1}{4}\pi i,\quad D=-\tfrac{1}{12}\pi a i.$$

Thus the complex stress $S=E_{13}-iE_{23}$ for the beam of which the cross-section is a 30°-60°-90° triangle is given by

$$(3.15)\qquad \frac{S}{\mu\tau}=-i\bar{z}+\tfrac{1}{3}ia+iz-\frac{iz^2}{a}$$

$$-\frac{4\omega_1 i}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\frac{\cosh\{n\pi i(2z-\omega_2)/(2\omega_1)\}}{\cosh\{n\pi i\,\omega_2/(2\omega_1)\}}$$

$$+\frac{8a\omega_1 i}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\frac{\sinh\{(2n-1)\pi i z/\omega_1\}}{\sinh\{(2n-1)\pi i\omega_2/\omega_1\}},$$

where

μ is the rigidity,

τ is the angle of twist per unit length,

a is the length of the side opposite the 60° angle and the axes of x and y are shown in Fig. 3.

4. *The warping function for the* 30°-60°-90° *triangle.*

Since the warping function ϕ and its conjugate ψ are plane harmonic we can define an analytic complex warping function Φ by the relation

$$\Phi = \phi + i\psi. \tag{4.1}$$

Hence, using (1.1), we obtain

$$\begin{aligned}\frac{d\Phi}{dz} &= \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y} \\ &= i\bar{z} + (E_{13} - iE_{23})/(\mu\tau) = i\bar{z} + S/(\mu\tau) \\ &= \tfrac{1}{3}ia + iz - \frac{iz^2}{a} - \frac{4\omega_1 i}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\frac{\cosh\{n\pi i(2z-\omega_2)/(2\omega_1)\}}{\cosh\{n\pi i\,\omega_2/(2\omega_1)\}} \\ &\qquad + \frac{8a\omega_1 i}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\frac{\sinh\{(2n-1)\pi iz/\omega_1\}}{\sinh\{(2n-1)\pi i\omega_2/\omega_1\}}.\end{aligned} \tag{4.2}$$

Upon integrating once, we obtain

$$\begin{aligned}\Phi = F + \tfrac{1}{3}iaz + \tfrac{1}{2}iz^2 - \frac{iz^3}{3a} - \frac{4\omega_1^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\frac{\sinh\{n\pi i(2z-\omega_2)/(2\omega_1)\}}{\cosh\{n\pi i\,\omega_2/(2\omega_1)\}} \\ + \frac{8a\omega_1^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\frac{\cosh\{(2n-1)\pi iz/\omega_1\}}{\sinh\{(2n-1)\pi i\omega_2/\omega_1\}},\end{aligned} \tag{4.3}$$

where F is an arbitrary complex constant; and upon taking the real part of (4.3) we find that

$$\phi = F' + (3x^2 - y^2 - 3ax - a^2)\frac{y}{3a} - \frac{4a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\phi_n(x, y), \tag{4.4}$$

where

$$\phi_{2n-1}(x, y) = \frac{\begin{array}{l}[\sqrt{3}(-1)^n \cosh\{(2n-1)\pi y/a\} \\ \qquad +\sinh\{(2n-1)\pi(-2y+a\sqrt{3})/(2a)\}] \cos\{(2n-1)\pi x/a\}\end{array}}{\cosh 3(2n-1)\xi},$$

$$\phi_{2n}(x, y) = \frac{\begin{array}{l}[(-1)^n \sinh\{2n\pi y/a\} \\ \qquad -\sinh\{2n\pi(-2y+a\sqrt{3})/(2a)\}] \cos\{2n\pi x/a\}\end{array}}{(-1)^n+\cosh 6n\xi},$$

$$\xi = \pi/(2\sqrt{3}),$$

and F' is an arbitrary constant: interpreted physically, the presence of F' means that the displacement parallel to the generators of the beam is determined apart from a rigid body translation.

Upon taking the imaginary part of (4.3) and using the condition that ψ vanishes at the origin, we find that $I[F] = 0$, whence we determine ψ completely. Using (1.2), we then find that

$$(4.5) \qquad \chi = (x-a)(-x^2-ax+3y^2)/(3a) - \frac{4a^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \chi_n(x, y),$$

where

$$\chi_{2n-1}(x, y) = \frac{\begin{array}{l}[\sqrt{3}(-1)^{n+1} \sinh\{(2n-1)\pi y/a\} \\ \qquad +\cosh\{(2n-1)\pi(-2y+a\sqrt{3})/(2a)\}] \sin\{(2n-1)\pi x/a\}\end{array}}{\cosh 3(2n-1)\xi},$$

$$\chi_{2n}(x, y) = -\frac{\begin{array}{l}[(-1)^n \cosh\{2n\pi y/a\} \\ \qquad +\cosh\{2n\pi(-2y+a\sqrt{3})/(2a)\}] \sin\{2n\pi x/a\}\end{array}}{(-1)^n+\cosh 6n\xi},$$

$$\xi = \pi/(2\sqrt{3}).$$

5. *The torsional rigidity for the* 30°-60°-90° *triangle**.

The torsional rigidity of the beam is given by (1.6) and hence we must multiply (4.5) through by $2\mu \, dx \, dy$ and integrate over the cross-section.

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* I am indebted to the referee for suggestions which have simplified the deduction of (5.1).

We note the following easy integrations:

$$\iint x\,dx\,dy=\frac{a^3}{3\sqrt{3}},\quad \iint y^2\,dx\,dy=\frac{a^4}{36\sqrt{3}},$$

$$\iint xy^2\,dx\,dy=\frac{a^5}{45\sqrt{3}},\quad \iint x^3\,dx\,dy=\frac{a^5}{5\sqrt{3}},$$

$$\iint \sin\{n\pi x/a\}\cosh\{n\pi(-2y+a\sqrt{3})/(2a)\}\,dx\,dy$$
$$=\frac{a^2}{4n^2\pi^2}\left[3(-1)^n\sinh n\xi+\{1+4(-1)^{n+1}\}\sinh 3n\xi\right],$$

$$\iint \sin\{(2n-1)\pi x/a\}\sinh\{(2n-1)\pi y/a\}\,dx\,dy$$
$$=\frac{a^2}{4(2n-1)^2\pi^2}\left[-5+3\cosh 2(2n-1)\xi\right],$$

$$\iint \sin\{2n\pi x/a\}\cosh\{2n\pi y/a\}\,dx\,dy=-\frac{3a^2}{16n^2\pi^2}\sinh 4n\xi,$$
$$\xi=\pi/(2\sqrt{3})\quad (n=1,\ 2,\ \ldots),$$

whence we obtain for the torsional rigidity of the beam with a 30°-60°-90° cross-section

$$C/(\mu a^4)=V+W, \tag{5.1}$$

where

$$V=7/(90\sqrt{3}),$$

$$W=\frac{2}{\pi^5}\left[3(H_1+H_2+H_3+H_4+\sqrt{3}H_5)+5(H_6+\sqrt{3}H_7)\right],$$

$$H_1=\sum_{n=1}^{\infty}\frac{1}{(2n)^5}\,\frac{\sinh 2n\xi}{(-1)^n+\cosh 6n\xi},$$

$$H_2=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)^5}\,\frac{\sinh 4n\xi}{(-1)^n+\cosh 6n\xi},$$

$$H_3=-\sum_{n=1}^{\infty}\frac{1}{(2n)^5}\,\frac{\sinh 6n\xi}{(-1)^n+\cosh 6n\xi},$$

$$H_4=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\,\frac{\sinh(2n-1)\xi}{\cosh 3(2n-1)\xi},$$

$$H_5=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^5}\,\frac{\cosh 2(2n-1)\xi}{\cosh 3(2n-1)\xi},$$

$$H_6=-\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\,\frac{\sinh 3(2n-1)\xi}{\cosh 3(2n-1)\xi},$$

$$H_7=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^5}\,\frac{1}{\cosh 3(2n-1)\xi}\qquad [\xi=\pi/(2\sqrt{3})].$$

By arithmetical calculation we find that

$$H_1 = 0{\cdot}000\,816\,290\,4, \quad H_5 = -0{\cdot}412\,450\,4,$$

$$H_2 = 0{\cdot}005\,110\,188, \quad H_6 = -0{\cdot}995\,894\,0,$$

$$H_3 = -0{\cdot}032\,675\,96, \quad H_7 = 0{\cdot}131\,087\,0,$$

$$H_4 = 0{\cdot}135\,882\,5,$$

and we have, finally,

$$V = 0{\cdot}044\,905\,02, \quad W = -0{\cdot}036\,990\,95,$$

whence, for the torsional rigidity C, we have

$$(5.2) \qquad c/(\mu a^4) = 0{\cdot}007\,914\,1,$$

correct to five significant figures, where μ is the rigidity and a is the length of the side opposite the 60° angle.

As a check on the accuracy of the above value of C, it may be mentioned that this value was also obtained by calculations in which the method of images was applied in quite a different way. This other method was, however, less easy to apply than the one developed in the present paper.

6. *The complex stress for the equilateral triangle.*

The application of the method of images to the equilateral triangle is so simple as to merit some consideration.

The part of the display of image distributions adjacent to the origin can be seen in Fig. 2 if we delete all straight lines inclined to the axis of x at angles of 0°, 60°, and 120°: the fundamental triangle then occupies the hatched region and its reflection in the axis of x.

Fig. 4 shows the fundamental hexagon, the fundamental triangle being hatched. Proceeding as in § 3, we find that

$$(6.1) \qquad d^3\,T/dz^3 = 0,$$

whence

$$(6.2) \qquad T = Az^2 + Bz + D,$$

where A, B, and D are arbitrary complex constants. To determine them we know that the stress must vanish at the three corners of the cross-section of the beam, whence

$$(6.3) \qquad A = 3\pi i/a, \quad B = -4\pi i, \quad D = 0,$$

and for the complex stress $S = E_{13} - iE_{23}$ we have

$$S/(\mu\tau) = i\bar{z} + 2iz - 3iz^2/(2a), \tag{6.4}$$

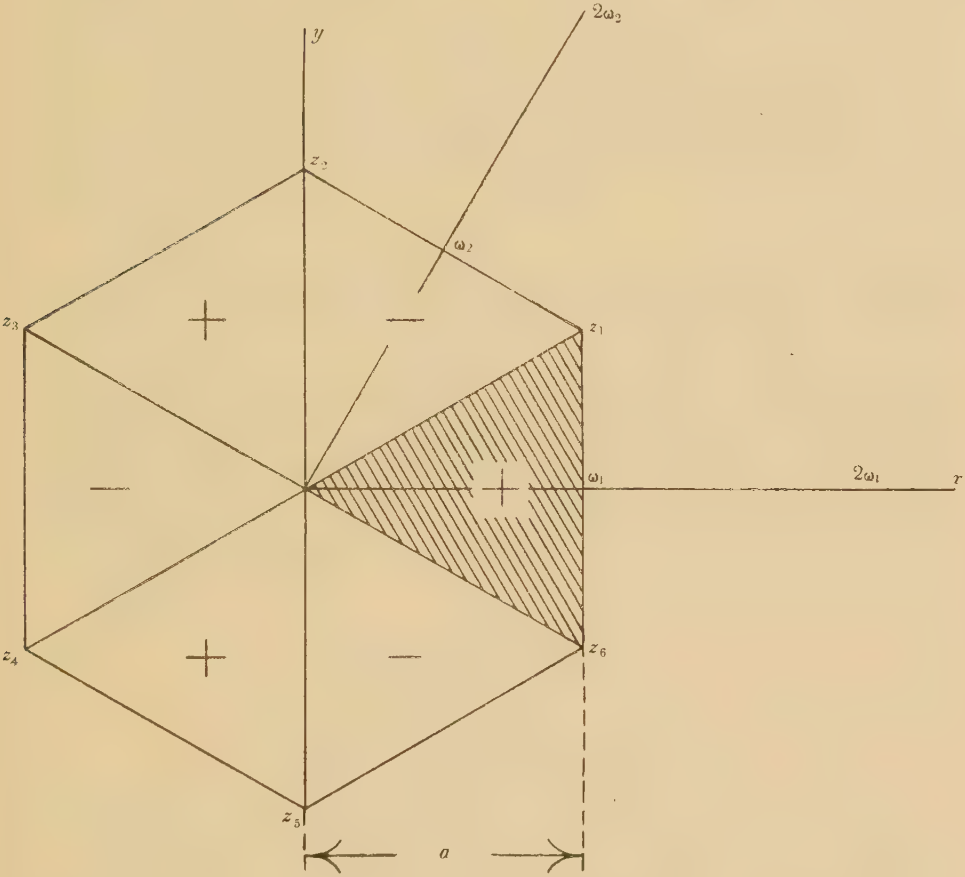

Fig. 4.

where

μ is the rigidity,

τ is the angle of twist per unit length,

a is the altitude of the triangle, and the axes are those shown in Fig. 4.

I have to thank Dr. J. L. Synge, both for suggesting the method of images as a means of solving certain torsion problems and also for much assistance.

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SOME THEOREMS ON RIESZ AND CESARO SUMS

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1. It is well known* that, for $\kappa \geqslant 0$, the Riesz means (R, n, κ) and the Cesàro means (C, κ) of a series are equivalent in the sense that, if the series is summable by one method, then it is summable to the same sum by the other method. This theorem is, of course, equivalent to the result that, if $A^{(\kappa)}(\omega) = o(\omega^\kappa)$ as $\omega \to \infty$, then $A_n^{(\kappa)} = o(n^\kappa)$ as $n \to \infty$, and conversely, where $A^{(\kappa)}(\omega)$ and $A_n^{(\kappa)}$ denote respectively the Riesz and Cesàro sums of the series $\sum_{n=0}^{\infty} a_n$; that is to say, where†

$$A^{(\kappa)}(\omega) = \sum_{r=0}^{[\omega]} a_r(\omega - r)^\kappa; \tag{1}$$

$$A_n^{(\kappa)} = \sum_{r=0}^{n} a_r \frac{(\kappa+1)(\kappa+2)\,\dots\,(\kappa+n-r)}{(n-r)!}. \tag{2}$$

As is well known, the following more general result, though not explicitly stated by Riesz in the paper cited, is established by his proof:

Theorem A. *Suppose that* $r > 0$. *If* $A^{(\kappa)}(\omega) = o(\omega^r)$ *as* $\omega \to \infty$, *then* $A_n^{(\kappa)} = o(n^r)$ *as* $n \to \infty$, *and conversely. The result continues to hold when* o *is replaced by* O.

In view of the well-known Tauberian theorems in which one sided conditions on the Riesz or Cesàro means are imposed, it is perhaps of some interest to investigate whether Theorem A continues to hold when O is replaced by O_L. We shall see that it does not, except in the trivial cases

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* M. Riesz, "Une méthode de sommation équivalente à la méthode des moyennes arithmétiques", *Comptes rendus*, 152 (1911), 1651–1654.

† An empty product is, throughout the paper, taken to mean 1. Thus, in (2), the term corresponding to $r = n$ is a_n.

$\kappa=0$, $\kappa=1$, but that, for certain values of κ, one part of the theorem holds. In order to determine whether, for a given value of κ, the result "If $A^{(\kappa)}(\omega)=O_L(\omega^r)$, then $A_n^{(\kappa)}=O_L(n^r)$" is true, we shall see that it is sufficient to investigate the truth of the proposition "If $A^{(\kappa)}(\omega)\geqslant 0$ for all* ω, then $A_n^{(\kappa)}\geqslant 0$ for all n"; for the equivalence of these two propositions will be established by Theorem 1. We shall therefore consider the latter proposition. Similar remarks apply to the converse results.

The results which we shall obtain are contained in the following theorems:

THEOREM 1. *For a given value of* κ, *let* $P(\kappa)$ *denote the proposition "If* $A^{(\kappa)}(\omega)\geqslant 0$ *for all* ω, *then* $A_n^{(\kappa)}\geqslant 0$ *for all* n", *and let* $Q(\kappa)$ *denote the converse proposition.* Let $P'(\kappa)$ *denote the proposition "If, for any* $r>0$, $A^{(\kappa)}(\omega)=O_L(\omega^r)$, *then* $A_n^{(\kappa)}=O_L(n^r)$" *and let* $Q'(\kappa)$ *denote the converse proposition. Then*

(i) $P(\kappa)$ *and* $P'(\kappa)$ *are equivalent; i.e. they are both true for the same values of* κ.

(ii) $Q(\kappa)$ *and* $Q'(\kappa)$ *are equivalent.*

THEOREM 2.

(i) $P(\kappa)$ *is true if* $0\leqslant\kappa\leqslant 1$;

(ii) $P(\kappa)$ *is false if* $\kappa>1$;

(iii) $Q(\kappa)$ *is true if* κ *is an integer*;

(iv) $Q(\kappa)$ *is false if* κ *is not an integer.*

Of the results of Theorem 2, (ii) and (iii) are almost trivial, but I state them for the sake of completeness. The greater part of this paper is occupied with the proof of (i). In order to prove this, it is convenient to obtain a rather more general result. Let $s(u)$ be any measurable function which is bounded in any finite interval. Suppose that $\kappa>0$. Let

$$R^{(\kappa)}(u)=\kappa\int_0^u (u-t)^{\kappa-1}s(t)\,dt, \tag{3}$$

and let

$$S^{(\kappa)}(u)=\frac{1}{\Gamma(\kappa+1)}\int_0^u g(u-t)\,s(t)\,dt, \tag{4}$$

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* Throughout this paper, ω, n, and x are taken to be positive. Thus "for all ω" means "for all positive ω".

where $$g(x)=\frac{dG(x)}{dx},\qquad G(x)=\frac{\Gamma(\kappa+x)}{\Gamma(x)}. \tag{5}$$

In the special case in which

$$s(u)=\sum_{n=0}^{[u]} a_n,$$

it may easily be verified that

$$R^{(\kappa)}(u)=A^{(\kappa)}(u),\qquad S^{(\kappa)}(n+1)=A_n^{(\kappa)}.$$

Hence $R^{(\kappa)}(u)$ and $S^{(\kappa)}(u)$ may conveniently be described as the Riesz and Cesàro sums of the function $s(u)$. We shall prove the following result, which evidently includes Theorem 2 (i), apart from the trivial case in which $\kappa=0$.

THEOREM 3. *Let $R^{(\kappa)}(u)$ and $S^{(\kappa)}(u)$ be defined by equations* (3) *and* (4), *and let $0<\kappa\leqslant 1$. If $R^{(\kappa)}(u)\geqslant 0$ for all u, then $S^{(\kappa)}(u)\geqslant 0$ for all u.*

2. In order to prove Theorem 1(i), we first note that, if $P(\kappa)$ is true, then $P'(\kappa)$ follows at once with the aid of Theorem A. Next suppose that $P(\kappa)$ is false. Then there is a series $\sum_{n=0}^{\infty} a_n$, such that $A^{(\kappa)}(\omega)\geqslant 0$ for all ω, but such that, for some integer m, $A_m^{(\kappa)}<0$. We now define a series $\sum_{n=0}^{\infty} b_n$ by taking

$$b_n=\sum_{\nu=0}^{[n/p]} \lambda_\nu a_{n-\nu p},$$

where p is any integer greater than m, and where the λ's are positive. Let the corresponding Riesz and Cesàro sums be denoted by $B^{(\kappa)}(\omega)$ and $B_n^{(\kappa)}$. Then

$$B^{(\kappa)}(\omega)=\sum_{\nu=0}^{[\omega/p]} \lambda_\nu A^{(\kappa)}(\omega-\nu p)\geqslant 0 \quad \text{for all } \omega.$$

On the other hand, if $f(n)$ is any given positive function of n, we can make $B_n^{(\kappa)}<-f(n)$ for all n of the form $m+sp$ (s any integer) by suitable choice of the λ's. For

$$B_{m+sp}^{(\kappa)}=\sum_{\nu=0}^{s} \lambda_{s-\nu} A_{m+\nu p}^{(\kappa)},$$

and, $\lambda_0, \lambda_1, \ldots, \lambda_{s-1}$ having previously been fixed, we can make this expression less than $-f(m+sp)$ by making λ_s sufficiently large. In particular, we can choose $f(n)$ so that $B_n^{(\kappa)}$ is not $O_L(n^r)$ for any given r. Hence $P'(\kappa)$ is false. Theorem 1(i) has now been established, and Theorem 1(ii) can be proved in a similar manner.

3. We now come to the proof of Theorem 3. The case $\kappa=1$ is trivial, and we may therefore suppose that $0<\kappa<1$. We require a number of lemmas; in all of these, the condition $0<\kappa<1$ is supposed satisfied. We define $G(x)$ and $g(x)$ throughout by equations (5), and write

$$g'(x)=\frac{dg(x)}{dx},\qquad G_1(x)=\int_0^x G(t)\,dt,$$

$$h(x)=\frac{\sin\pi\kappa}{\pi}\left\{x^{-\kappa}\,\Gamma(\kappa)+\int_0^x (x-t)^{-\kappa}\,g'(t)\,dt\right\}.$$

LEMMA 1.

$$g(v)=\int_0^v h(u)(v-u)^{\kappa-1}\,du.$$

We have

$$\begin{aligned}\int_0^v h(u)(v-u)^{\kappa-1}\,du&\\ &=\frac{\sin\pi\kappa}{\pi}\int_0^v (v-u)^{\kappa-1}\,du\left\{u^{-\kappa}\,\Gamma(\kappa)+\int_0^u (u-t)^{-\kappa}\,g'(t)\,dt\right\}\\ &=\frac{\sin\pi\kappa}{\pi}\left\{[\Gamma(\kappa)]^2\,\Gamma(1-\kappa)+\int_0^v g'(t)\,dt\int_t^v (v-u)^{\kappa-1}(u-t)^{-\kappa}\,du\right\}\\ &=\Gamma(\kappa)+\int_0^v g'(t)\,dt\\ &=g(v),\quad\text{since}\quad g(0)=\Gamma(\kappa).\end{aligned}$$

LEMMA 2.

(i) $$\frac{g(x)}{G(x)}\leqslant\frac{1}{x}-\frac{2-\kappa}{2(x+\kappa)}+\frac{\kappa}{2(x+1)},$$

(ii) $$\frac{xg(x)}{\kappa G(x)}\geqslant 1+\frac{1-\kappa}{2(x+1)}+\frac{(1-\kappa)(2-\kappa)}{3(x+\kappa)(x+1)}+\frac{(1-\kappa^2)(2-\kappa)}{6(x+\kappa)(x+1)^2(x+1+\kappa)},$$

(iii) $$\frac{x^2}{\kappa}\,\frac{d}{dx}\left[\frac{g(x)}{G(x)}\right]\geqslant-1-\frac{1-\kappa}{x+1}-\frac{(1-\kappa)(3-2\kappa)}{2(x+1)^2}-\frac{(1-\kappa)(2-\kappa)[2(2-\kappa)x+3(1-\kappa-\kappa^2)]}{3(x+\kappa)(x+1)^2(x+1+\kappa)}.$$

We have

$$\begin{aligned}\log G(x)&=\log\Gamma(x+\kappa)-\log\Gamma(x)\\ &=\lim_{n\to\infty}\left\{\kappa\log n+\sum_{r=0}^{n}\,[\log(x+r)-\log(x+r+\kappa)]\right\}.\end{aligned}$$

Differentiating with respect to x, we have

$$\frac{g(x)}{G(x)}=\sum_{r=0}^{\infty}\left\{\frac{1}{x+r}-\frac{1}{x+r+\kappa}\right\}. \tag{6}$$

It follows that

$$\frac{g(x)}{G(x)}-\left\{\frac{1}{x}-\frac{2-\kappa}{2(x+\kappa)}+\frac{\kappa}{2(x+1)}\right\}$$

$$=-\tfrac{1}{2}\sum_{r=1}^{\infty}\left\{\frac{\kappa}{x+r-1+\kappa}-\frac{2-\kappa}{x+r}+\frac{2-\kappa}{x+r+\kappa}-\frac{\kappa}{x+r+1}\right\}$$

$$=-\tfrac{1}{2}\kappa(1-\kappa)(2-\kappa)\sum_{r=1}^{\infty}\frac{1}{(x+r-1+\kappa)(x+r)(x+r+\kappa)(x+r+1)}\leqslant 0,$$

and the first part of the lemma follows at once. But we can also deduce from (6) that

$$\frac{g(x)}{G(x)}-\left\{\frac{1}{x}-\frac{1}{x+\kappa}+\frac{4+3\kappa-\kappa^2}{6(x+1)}-\frac{2-\kappa}{3(x+1+\kappa)}+\frac{\kappa(1+\kappa)}{6(x+2)}\right\}$$

$$=\tfrac{1}{6}\sum_{r=1}^{\infty}\left\{\frac{2-3\kappa+\kappa^2}{x+r}-\frac{2(1+\kappa)}{x+r+\kappa}+\frac{2(2+\kappa-\kappa^2)}{x+r+1}-\frac{2(2-\kappa)}{x+r+1+\kappa}+\frac{\kappa(1+\kappa)}{x+r+2}\right\}$$

$$=\tfrac{1}{3}(1+\kappa)\kappa(1-\kappa)(2-\kappa)$$

$$\times\sum_{r=1}^{\infty}\frac{1}{(x+r)(x+r+\kappa)(x+r+1)(x+r+1+\kappa)(x+r+2)}\geqslant 0.$$

Hence

$$\frac{xg(x)}{G(x)}\geqslant x\left\{\frac{1}{x}-\frac{1}{x+\kappa}+\frac{4+3\kappa-\kappa^2}{6(x+1)}-\frac{2-\kappa}{3(x+1+\kappa)}+\frac{\kappa(1+\kappa)}{6(x+2)}\right\}$$

$$=\kappa+\frac{\kappa}{x+\kappa}-\frac{4+3\kappa-\kappa^2}{6(x+1)}+\frac{(1+\kappa)(2-\kappa)}{3(x+1+\kappa)}-\frac{\kappa(1+\kappa)}{3(x+2)}$$

$$=\kappa\left[1+\frac{1-\kappa}{2(x+1)}+\frac{(1-\kappa)(2-\kappa)}{3(x+\kappa)(x+1)}+\frac{(1-\kappa^2)(2-\kappa)}{3(x+\kappa)(x+1)(x+1+\kappa)(x+2)}\right].$$

Since

$$\frac{1}{x+2}\geqslant\frac{1}{2(x+1)},$$

the second part of the lemma follows. Finally, we deduce from (6) that

$$\frac{g(x)}{G(x)}-\left\{\frac{1}{x}-\frac{6-\kappa-\kappa^2}{6(x+\kappa)}+\frac{1+\kappa}{3(x+1)}-\frac{2-3\kappa+\kappa^2}{6(x+1+\kappa)}\right\}$$

$$=-\tfrac{1}{6}\sum_{r=1}^{\infty}\left\{\frac{\kappa(1+\kappa)}{x+r-1+\kappa}-\frac{2(2-\kappa)}{x+r}+\frac{2(2+\kappa-\kappa^2)}{x+r+\kappa}-\frac{2(1+\kappa)}{x+r+1}+\frac{2-3\kappa+\kappa^2}{x+r+1+\kappa}\right\}$$

$$=-\tfrac{1}{3}(1+\kappa)\kappa(1-\kappa)(2-\kappa)\times\sum_{r=1}^{\infty}\frac{1}{(x+r-1+\kappa)(x+r)(x+r+\kappa)(x+r+1)(x+r+1+\kappa)},$$

and is thus an increasing function of x. Hence

$$\frac{d}{dx}\left[\frac{g(x)}{G(x)}\right]\geqslant\frac{d}{dx}\left\{\frac{1}{x}-\frac{6-\kappa-\kappa^2}{6(x+\kappa)}+\frac{1+\kappa}{3(x+1)}-\frac{2-3\kappa+\kappa^2}{6(x+1+\kappa)}\right\}.$$

We deduce that

$$x^2\frac{d}{dx}\left[\frac{g(x)}{G(x)}\right]\geqslant-\kappa-\frac{(6-\kappa-\kappa^2)(2\kappa x+\kappa^2)}{6(x+\kappa)^2}+\frac{(1+\kappa)(2x+1)}{3(x+1)^2}-\frac{(2-3\kappa+\kappa^2)[2(1+\kappa)x+(1+\kappa)^2]}{6(x+1+\kappa)^2}$$

$$=-\kappa-\frac{\kappa(1-\kappa)}{x+1}-\frac{\kappa(1-\kappa)(3-2\kappa)}{2(x+1)^2}+\tfrac{1}{6}(2-\kappa)\left\{-\frac{(3+\kappa)(2\kappa x+\kappa^2)}{(x+\kappa)^2}+\frac{2(1+3\kappa)x+(1+9\kappa-6\kappa^2)}{(x+1)^2}-\frac{(1-\kappa)[2(1+\kappa)x+(1+\kappa)^2]}{(x+1+\kappa)^2}\right\}.$$

It may be verified that the expression in curly brackets is equal to

$$-\frac{2\kappa(1-\kappa)[2(2-\kappa)x+3(1-\kappa-\kappa^2)]}{(x+\kappa)(x+1)^2(x+1+\kappa)}+\frac{\kappa^2(1-\kappa)[2\kappa x^2+2(1-\kappa+\kappa^2)x+3(1-\kappa^2)]}{(x+\kappa)^2(x+1)^2(x+1+\kappa)^2}.$$

Since the second term is evidently positive, the result follows.

LEMMA 3. $$\frac{(1+\kappa)\,G_1(x)}{xG(x)}\geqslant 1-\frac{1-\kappa}{2(x+1)}.$$

We deduce by integration from Lemma 2(i) that, if $t < x$, then

$$\log G(t) - \log G(x) \geqslant \log t - \log x - \tfrac{1}{2}(2-\kappa)\{\log(t+\kappa) - \log(x+\kappa)\}$$
$$+ \tfrac{1}{2}\kappa\{\log(t+1) - \log(x+1)\},$$

and hence that

$$\frac{G(t)}{G(x)} \geqslant \frac{t(t+\kappa)^{\frac{1}{2}\kappa-1}(t+1)^{\frac{1}{2}\kappa}}{x(x+\kappa)^{\frac{1}{2}\kappa-1}(x+1)^{\frac{1}{2}\kappa}}.$$

It follows that

$$\frac{G_1(x)}{G(x)} \geqslant \frac{\int_0^x t(t+\kappa)^{\frac{1}{2}\kappa-1}(t+1)^{\frac{1}{2}\kappa}\,dt}{x(x+\kappa)^{\frac{1}{2}\kappa-1}(x+1)^{\frac{1}{2}\kappa}},$$

and hence that

$$\frac{(1+\kappa)\,G_1(x)}{xG(x)} - \left\{1 - \frac{1-\kappa}{2(x+1)}\right\} \geqslant \frac{A(x)}{x^2(x+\kappa)^{\frac{1}{2}\kappa-1}(x+1)^{\frac{1}{2}\kappa}}, \tag{7}$$

where

$$A(x) = (1+\kappa)\int_0^x t(t+\kappa)^{\frac{1}{2}\kappa-1}(t+1)^{\frac{1}{2}\kappa}\,dt - x^2(x+\kappa)^{\frac{1}{2}\kappa-1}(x+1)^{\frac{1}{2}\kappa}\left\{1 - \frac{1-\kappa}{2(x+1)}\right\}.$$

Now it may easily be verified that

$$\frac{dA(x)}{dx} = \tfrac{1}{4}(1-\kappa)^2(2-\kappa)\,x^2(x+\kappa)^{\frac{1}{2}\kappa-2}(x+1)^{\frac{1}{2}\kappa-2} \geqslant 0.$$

Since $A(0) = 0$, it follows that $A(x)$ is positive, and the lemma follows at once with the aid of equation (7).

LEMMA 4. *Let*

$$F(x) = (1+\kappa)\kappa(1-\kappa)\,G_1(x) - 3\kappa(1-\kappa)\,xG(x) + 3(1-\kappa)\,x^2 g(x) + x^3 g'(x).$$

Then $F(x) \geqslant 0$.

We first note that, as a corollary of Lemma 2 (ii),

$$\frac{1}{\kappa}\left\{\frac{xg(x)}{G(x)}\right\}^2 \geqslant \kappa + \frac{\kappa(1-\kappa)}{x+1} + \frac{2\kappa(1-\kappa)(2-\kappa)}{3(x+\kappa)(x+1)} + \frac{\kappa(1-\kappa)^2}{4(x+1)^2}. \tag{8}$$

We now deduce from Lemma 3, Lemma 2 (ii), equation (8) and Lemma 2 (iii) that

$$\frac{F(x)}{\kappa x\, G(x)} = (1+\kappa)(1-\kappa)\,\frac{G_1(x)}{xG(x)} - 3(1-\kappa) + 3(1-\kappa)\,\frac{xg(x)}{\kappa G(x)} + \frac{1}{\kappa}\left\{\frac{xg(x)}{G(x)}\right\}^2 + \frac{x^2}{\kappa}\,\frac{d}{dx}\left\{\frac{g(x)}{G(x)}\right\}$$

$$\geqslant (1-\kappa)\left\{\frac{(1-\kappa)(2-\kappa)}{(x+\kappa)(x+1)} + \frac{(1-\kappa^2)(2-\kappa)}{2(x+\kappa)(x+1)^2(x+1+\kappa)} + \frac{2\kappa(2-\kappa)}{3(x+\kappa)(x+1)} + \frac{\kappa(1-\kappa)}{4(x+1)^2} - \frac{3-2\kappa}{2(x+1)^2} - \frac{(2-\kappa)[2(2-\kappa)\,x+3(1-\kappa-\kappa^2)]}{3(x+\kappa)(x+1)^2(x+1+\kappa)}\right\}$$

$$= \frac{(1-\kappa)(2-\kappa)[(3-\kappa)\,x^2+(-1-3\kappa+2\kappa^2)\,x+(6+11\kappa-4\kappa^2+3\kappa^3)]}{12(x+\kappa)(x+1)^2(x+1+\kappa)}.$$

The expression in square brackets is greater than or equal to

$$2x^2-4x+6>0,$$

and the lemma follows.

LEMMA 5. $h(x) \geqslant 0.$

We have

$$h(x) = \frac{\sin \pi\kappa}{\pi}\left\{x^{-\kappa}\,\Gamma(\kappa) + \int_0^x (x-t)^{-\kappa}\, g'(t)\, dt\right\}.$$

Hence

$$xh(x) = \frac{\sin \pi\kappa}{\pi}\left\{x^{1-\kappa}\,\Gamma(\kappa) + \int_0^x (x-t)^{1-\kappa}\, g'(t)\, dt + \int_0^x (x-t)^{-\kappa}\, tg'(t)\, dt\right\}.$$

On integrating the first integral by parts, we obtain

$$xh(x) = \frac{\sin \pi\kappa}{\pi}\int_0^x (x-t)^{-\kappa}\{(1-\kappa)\, g(t) + tg'(t)\}\, dt.$$

By a similar argument we can now deduce that

$$x^2 h(x) = \frac{\sin \pi\kappa}{\pi}\int_0^x (x-t)^{-\kappa}\{-\kappa(1-\kappa)\, G(t) + 2(1-\kappa)\, tg(t) + t^2 g'(t)\}\, dt,$$

and, after again repeating the process, we obtain the result

$$x^3 h(x) = \frac{\sin \pi\kappa}{\pi}\int_0^x (x-t)^{-\kappa}\, F(t)\, dt.$$

The lemma now follows at once from Lemma 4.

We are now in a position to prove Theorem 3. By Lemma 1,

$$S^{(\kappa)}(u)=\frac{1}{\Gamma(\kappa+1)}\int_0^u s(u-v)\,g(v)\,dv$$
$$=\frac{1}{\Gamma(\kappa+1)}\int_0^u s(u-v)\,dv\int_0^v h(t)(v-t)^{\kappa-1}\,dt$$
$$=\frac{1}{\Gamma(\kappa+1)}\int_0^u h(t)\,dt\int_t^u s(u-v)(v-t)^{\kappa-1}\,dv$$
$$=\frac{1}{\kappa\Gamma(\kappa+1)}\int_0^u h(t)\,R^{(\kappa)}(u-t)\,dt,$$

and the result follows at once with the aid of Lemma 5.

4. In order to prove Theorem 2 (ii), it is sufficient to consider the case in which

$$a_N=1,\quad a_{N+1}=-\alpha,\quad a_{N+2}=\beta$$

and $a_n=0$ for all other values of n. It is evident that $A^{(\kappa)}(\omega)\geqslant 0$ for $\omega<N+1$. Further, if $0\leqslant\theta\leqslant 1$,

$$A^{(\kappa)}(N+1+\theta)=(1+\theta)^\kappa-\alpha\theta^\kappa,$$

which is greater than 0 provided that

$$\alpha<2^\kappa. \tag{9}$$

Further, α having been chosen so as to satisfy (9), we can evidently make $A^{(\kappa)}(\omega)\geqslant 0$ for $\omega>N+2$ by taking β sufficiently large. On the other hand,

$$A_{N+1}^{(\kappa)}=\kappa+1-\alpha,$$

which is less than 0 if

$$\alpha>\kappa+1. \tag{10}$$

If $\kappa>1$,

$$2^\kappa>\kappa+1,$$

and hence α can be chosen so as to satisfy both (9) and (10). The result follows.

It now remains to investigate the proposition $Q(\kappa)$. We have

$$A^{(\kappa)}(\omega)=\sum_{r=0}^{[\omega]}a_r(\omega-r)^\kappa$$
$$=\sum_{r=0}^{[\omega]}(\omega-r)^\kappa\sum_{m=0}^{r}(-1)^{r-m}A_m^{(\kappa)}\frac{(\kappa+1)\kappa\ldots(\kappa+2-r+m)}{(r-m)!}$$
$$=\sum_{m=0}^{[\omega]}A_m^{(\kappa)}c^{(\kappa)}(\omega-m),$$

where
$$c^{(\kappa)}(x) = \sum_{r=0}^{[x]} (-1)^r (x-r)^\kappa \frac{(\kappa+1)\kappa \dots (\kappa+2-r)}{r!}. \tag{11}$$

It is evident that, if $c^{(\kappa)}(x) \geqslant 0$ for all x, then $Q(\kappa)$ is true. On the other hand, if $c^{(\kappa)}(x)$ can assume negative values, $Q(\kappa)$ is false, as may be seen at once by taking
$$A_N^{(\kappa)} = 1, \quad A_n^{(\kappa)} = 0 \quad \text{for} \quad n \neq N.$$

We therefore have to investigate $c^{(\kappa)}(x)$. It may easily be verified that, if $x \leqslant 1$,
$$c^{(\kappa)}(x) = \kappa \int_0^x c^{(\kappa-1)}(t)\,dt$$

and, if $x \geqslant 1$,
$$c^{(\kappa)}(x) = \kappa \int_{x-1}^x c^{(\kappa-1)}(t)\,dt. \tag{12}$$

It follows at once by induction that $c^{(\kappa)}(x) \geqslant 0$ when κ is an integer. This proves Theorem 2(iii). In order to prove Theorem 2(iv), we require two more lemmas.

LEMMA 6.
$$\int_1^\infty \frac{c^{(\kappa)}(x)}{x^s}\,dx,$$

when extended by analytic continuation, is an integral function of s.

If $\Re s > \kappa+1$, we have (the various inversions of integration and summation being justified by absolute convergence)

$$\int_1^\infty \frac{c^{(\kappa)}(x)}{x^s}\,dx = \int_1^\infty \frac{dx}{x^s} \sum_{r=0}^{[x]} (-1)^r (x-r)^\kappa \frac{(\kappa+1)\kappa \dots (\kappa+2-r)}{r!}$$
$$= \int_1^\infty x^{\kappa-s}\,dx + \sum_{r=1}^\infty (-1)^r \frac{(\kappa+1)\kappa \dots (\kappa+2-r)}{r!} \int_r^\infty \frac{(x-r)^\kappa}{x^s}\,dx$$
$$= \frac{1}{s-\kappa-1} + \frac{\Gamma(\kappa+1)\,\Gamma(s-\kappa-1)}{\Gamma(s)} \sum_{r=1}^\infty (-1)^r \frac{(\kappa+1)\kappa \dots (\kappa+2-r)}{r!}\, r^{\kappa+1-s}$$
$$= \frac{1}{s-\kappa-1} + \frac{\Gamma(\kappa+1)}{\Gamma(s)} \sum_{r=1}^\infty (-1)^r \frac{(\kappa+1)\kappa \dots (\kappa+2-r)}{r!} \int_0^\infty e^{-xr} x^{s-\kappa-2}\,dx$$
$$= \frac{1}{s-\kappa-1} + \frac{\Gamma(\kappa+1)}{\Gamma(s)} \int_0^\infty x^{s-\kappa-2}\,dx \sum_{r=1}^\infty (-1)^r e^{-xr} \frac{(\kappa+1)\kappa \dots (\kappa+2-r)}{r!}$$
$$= \frac{1}{s-\kappa-1} + \frac{\Gamma(\kappa+1)}{\Gamma(s)} \int_0^\infty x^{s-\kappa-2}\{(1-e^{-x})^{\kappa+1}-1\}\,dx.$$

Now
$$\int_1^\infty x^{s-\kappa-2}\{(1-e^{-x})^{\kappa+1}-1\}\,dx$$

converges uniformly for s in any finite domain, and is therefore an integral function. Again, for x small, we can expand

$$(1-e^{-x})^{\kappa+1}$$

in ascending powers of x,

$$(1-e^{-x})^{\kappa+1}=\sum_{r=1}^{\infty} a_r x^{\kappa+r},$$

say. We see that

$$\int_0^1 x^{s-\kappa-2}\left\{(1-e^{-x})^{\kappa+1}-\sum_{r=1}^{N} a_r x^{\kappa+r}\right\}dx$$

is regular for $\Re s>-N$, since the integrand is

$$O\{x^{\Re s+N-1}\}$$

for x small. We now deduce that

$$\int_0^\infty x^{s-\kappa-2}\{(1-e^{-x})^{\kappa+1}-1\}\,dx,$$

when extended by analytic continuation, is a meromorphic function with simple poles at $s=\kappa+1$, 0, -1, -2, -3, ..., the residue at $s=\kappa+1$ being -1. The lemma is now evident.

LEMMA 7.

$$(x+1)\,c^{(\kappa)}(x+1)=(x-\kappa-1)\,c^{(\kappa)}(x)+c^{(\kappa+1)}(x+1).$$

We have
$$c^{(\kappa)}(x+1)=\sum_{r=0}^{[x+1]} A(x,r)(\kappa+2-r);$$

$$c^{(\kappa)}(x)=-\sum_{r=0}^{[x+1]} A(x,r)\,r;$$

and
$$c^{(\kappa+1)}(x+1)=\sum_{r=0}^{[x+1]} A(x,r)(\kappa+2)(x+1-r),$$

where
$$A(x,r)=(-1)^r\frac{(\kappa+1)\kappa\ldots(\kappa+3-r)}{r!}(x+1-r)^{\kappa}.$$

But
$$(x+1)(\kappa+2-r)+r(x-\kappa-1)-(\kappa+2)(x+1-r)=0$$

and the lemma follows.

We are now in a position to prove Theorem 2(iv). We suppose that for some κ which is not an integer $c^{(\kappa)}(x)\geqslant 0$ for all x, and show that this leads to a contradiction. In the first place, it follows from Lemma 6, with the aid of a known theorem* on Dirichlet integrals, that

$$\int_1^\infty \frac{c^{(\kappa)}(x)}{x^s}\,dx \tag{13}$$

converges for all s. On the other hand, it follows from equation (12) that $c^{(\kappa+1)}(x) \geqslant 0$ for all x, and hence from Lemma 7 that

$$(x+1)\,c^{(\kappa)}(x+1) \geqslant (x-\kappa-1)\,c^{(\kappa)}(x).$$

We deduce at once that, if $x > \kappa+1$,

$$c^{(\kappa)}(x+n) \geqslant \frac{(x+n-\kappa-2)(x+n-\kappa-3)\dots(x-\kappa-1)}{(x+n)(x+n-1)\dots(x+1)}\,c^{(\kappa)}(x).$$

Hence, for x in any given finite interval contained in $x > \kappa+1$, and for any positive integer n,

$$c^{(\kappa)}(x+n) \geqslant An^{-\kappa-2}\,c^{(\kappa)}(x), \tag{14}$$

where A is a positive constant. But, since κ is not an integer, it is evidently false that $c^{(\kappa)}(x)$ is identically zero for x sufficiently large. Hence we can find numbers X, δ, h such that $X > \kappa+1$, $\delta > 0$, and such that when $X < x < X+\delta$, $c^{(\kappa)}(x) \geqslant h > 0$. We deduce with the aid of (14) that for $X < x < X+\delta$, and n any integer,

$$c^{(\kappa)}(x+n) \geqslant Ahn^{-\kappa-2}.$$

It follows that the integral (13) diverges for $\Re s \leqslant -\kappa-1$, and our contradiction has been obtained. Theorem 2(iv) now follows.

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* E. Landau, "Über einen Satz von Tschebyschef", *Math. Annalen*, 61 (1905), 527–550 (548).

THE GEOMETRICAL INVARIANTS OF ALGEBRAIC LOCI

(*Second Paper*)

By J. A. Todd.

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Introduction.

In a previous paper[†] with the above title I defined certain systems of equivalence $\{X_h\}$ for each dimension h from 0 to $d-1$ on an algebraic V_d, which I there termed *canonical systems* of dimension h. The proof of the invariance of these systems which I gave in that paper depended, for $h > 0$, on the assumption of an unproved criterion of equivalence for varieties of dimension h on V_d. After that paper was written, but before it appeared in print, Eger published two notes[‡] in which he defined canonical systems from a different point of view. These notes merely give an account of Eger's results, and his proofs have not yet been published, but it would appear from his account that these proofs are of a function-theoretic character, and moreover require the existence of certain simple integrals attached to the V_d in question. It therefore seems desirable to give a purely geometric discussion of the matter which will establish the existence of the canonical systems without making use of any unproved assumption. This is the object of the present paper, which essentially replaces the earlier one with the same title. I should like, however, to make my obligations to Eger quite clear, since it seems likely that the geometrical reasoning given below is closely related to the function-theoretic considerations employed by Eger.

[†] J. A. Todd, *Proc. London Math. Soc.* (2), 43 (1937), 127.

[‡] M. Eger, *Comptes rendus*, 204 (1937), 92, 217.

1. As a starting point we recall that on a non-singular variety V_d of d dimensions the canonical system of dimension $d-1$ may be defined by taking an irreducible linear system $|S|$ of V_{d-1}'s of freedom d and constructing its Jacobian variety S_J. If $|S|$ is free from base points it is then easily proved that the linear system $|X_{d-1}|$ defined by

$$|X_{d-1}| \equiv |S_J - (d+1) S| \tag{1}$$

is independent of $|S|$, the proof being a simple generalization of the corresponding result for $d=2$. The system $|X_{d-1}|$ is easily shown to have the property expressed by the relation of (linear) equivalence on S

$$X_{d-2}[S] \equiv (S \, . \, X_{d-1} + S) \tag{2}$$

which we call the *relation of adjunction.*

Suppose now that we have, on V_d, d irreducible linear pencils $|S_i|$ $(i=1, \ldots, d)$, and seek the variety $J_{d-1}[S_1, S_2, \ldots, S_d]$ which is the locus of points P such that the d tangent $[d-1]$'s to the varieties of the pencils $|S_i|$ which pass through P have a common line. We shall prove that

$$J_{d-1}[S_1, S_2, \ldots, S_d] \equiv X_{d-1} + 2(S_1 + S_2 + \ldots + S_d). \tag{3}$$

The proof of (3) is by induction on d. When $d=1$, (3) is equivalent to (1). We therefore assume that (3) holds on varieties of dimension $d-1$.

Let S be a particular variety of the pencil $|S_1|$ and let $|\Sigma_j|$ denote the linear system cut by $|S_j|$ on S for $j = 2, 3, \ldots, d$. The variety

$$J_{d-1}[S_1, S_2, \ldots, S_d]$$

contains the base V_{d-2}, (S_1^2), of the pencil $|S_1|$. For the tangent $[d-1]$'s at a point P of (S_1^2) to the varieties of the pencils $|S_2|, \ldots, |S_d|$ which pass through P meet in a line, and just one member of the pencil $|S_1|$ can be made to touch this line at P. At a point Q common to

$$J_{d-1}[S_1, S_2, \ldots, S_d]$$

and S which does not lie on (S_1^2) the tangent $[d-1]$'s at Q to the varieties of $|S_2|, \ldots, |S_d|$ which pass through Q meet the tangent $[d-1]$ to S at Q in the tangent $[d-2]$'s to $|\Sigma_2|, \ldots, |\Sigma_d|$ at Q, and these $[d-2]$'s meet in the common tangent line to the varieties of $|S_1|, |S_2|, \ldots, |S_d|$ which pass through Q. The locus of Q is therefore the Jacobian variety

$$J_{d-2}[\Sigma_2, \Sigma_3, \ldots, \Sigma_d]$$

of the pencils $|\Sigma_j|$. But, by the inductive hypothesis,

$$J_{d-2}[\Sigma_2, \Sigma_3, \ldots, \Sigma_d] \equiv X_{d-2}[S] + 2(\Sigma_2 + \Sigma_3 + \ldots + \Sigma_d),$$

and hence

$$\begin{aligned}(S\,.\,J_{d-1}[S_1, S_2, \ldots, S_d]) &\equiv (S_1{}^2) + X_{d-2}[S] + 2(S\,.\,S_2 + S_3 + \ldots + S_d) \\ &\equiv (S\,.\,X_{d-1} + 2[S_1 + S_2 + \ldots + S_d])\end{aligned}$$

by (2), the equivalences being on S.

The right and left sides of (3) thus cut out equivalent V_{d-2}'s on each variety S of the pencil $|S_1|$, and so by a well-known criterion of linear equivalence† the two sides of (3) are linearly equivalent on V_d, which proves the theorem.

2. In seeking to extend these results, and to define invariant systems of lower dimension h on V_d I considered, in the previous paper, the Jacobian variety of a linear system of freedom $h+1$, when $h+1<d$. Eger, on the other hand, considers the Jacobian variety of $h+1$ linear pencils, and in the present paper we follow Eger. Our proof is inductive; we assume the existence of the system of equivalence $\{X_h\}$, for a fixed value of h, to be established on varieties of dimension less than d, and deduce the existence of the corresponding system on V_d. The "first case" of the induction is that in which $d=h+1$, and the existence of the canonical system in this case has just been established above.

It is worthy of remark that the geometrical argument which we use is exactly parallel to a familiar argument‡ which leads to the existence of the Zeuthen-Segre invariant of a surface—which is indeed merely the particular case $h=0$, $d=2$. This point seems to be worth stressing, since in most of the current accounts available of the algebro-geometric treatment of the theory of algebraic surfaces this invariant appears as an isolated phenomenon with no very clear relation to the general theory. The present paper shows, I think, that it has a very real place in the organic scheme of development of the theory of invariants.

It will be convenient for formal purposes to introduce a set of linear operators K_n, due essentially to Eger, whose field of operation consists of all proper algebraic subvarieties of V_d. These operators are defined by the condition of linearity, and by their effects on varieties lying on V_d which are as follows:

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† See, *e.g.*, O. Zariski, *Algebraic surfaces* (Berlin, 1935), 89. The proof given there for the case $d=2$ extends immediately to varieties of higher dimension.

‡ See H. F. Baker, *Principles of geometry*, 6 (Cambridge, 1933), 207.

$$\begin{aligned} K_n[V_t] &= X_n[V_t] \quad (t > n), \\ &= V_t \qquad\quad\; (t = n), \\ &= 0 \qquad\quad\;\; (t < n). \end{aligned} \tag{4}$$

3. We suppose henceforward that $h < d-1$. We consider $h+1$ irreducible linear pencils $|S_i|$ $(i = 0, 1, \ldots, h)$ on V_d, and define their Jacobian variety $J_h[S_0, S_1, \ldots, S_h]$ of h dimensions as follows. When $h = 0$ the variety consists of the set of double points of members of the pencil $|S_0|$; when $h > 0$ it consists of the points P of V_d such that the $h+1$ tangent $[d-1]$'s to the members of the pencils which pass through P have a common $[d-h]$.

Let S be a particular variety of $|S_0|$, and let (S_0^2) be the base V_{d-2} of the pencil. Denote by $|\Sigma_j|$ and $|\sigma_j|$ the pencils cut out by $|S_j|$ on S and (S_0^2) respectively, for $j = 1, 2, \ldots, h$. We prove that

$$(S \,.\, J_h[S_0, S_1, \ldots, S_h]) = J_{h-1}[\Sigma_1, \ldots, \Sigma_h] + J_{h-1}[\sigma_1, \ldots, \sigma_h]. \tag{5}$$

The sign of equality is here intentional, implying that the intersection does in fact break up into the two varieties figuring on the right of (5).

The proof of (5) is immediate. Let P be a point common to (S_0^2) and $J_h[S_0, S_1, \ldots, S_h]$. The tangent $[d-1]$'s at P to the varieties of

$$|S_1|, \ldots, |S_h|$$

which pass through P meet in a $[d-h]$. Since P lies on $J_h[S_0, S_1, \ldots, S_h]$ this $[d-h]$ lies in the tangent $[d-1]$ to some member of $|S_0|$ at P and therefore meets in a $[d-h-1]$ the $[d-2]$ common to the ∞^1 tangent $[d-1]$'s to the varieties of $|S_0|$ at P (all of which lie in the tangent $[d]$ to V_d at P). Thus the tangent $[d-3]$'s to the varieties of $|\sigma_1|, \ldots, |\sigma_h|$ which pass through P meet in a $[d-h-1]$, and the locus of P is $J_{h-1}[\sigma_1, \ldots, \sigma_h]$. On the other hand, at a point Q common to S and $J_h[S_0, S_1, \ldots, S_h]$ which does not lie on (S_0^2), the tangent $[d-1]$'s to the varieties of $|S_1|, \ldots, |S_h|$ which pass through Q meet in a $[d-h]$ lying in the tangent $[d-1]$ to S at Q, and hence the tangent $[d-2]$'s to the members of $[\Sigma_1|, \ldots, |\Sigma_h|$ which pass through Q meet in a $[d-h]$. Thus the locus of Q is $J_{h-1}[\Sigma_1, \ldots, \Sigma_h]$, and (5) follows at once.

4. We can, however, go further than this. At a point P of $J_{h-1}[\sigma_1, \ldots, \sigma_h]$ there is just one variety, S_P say, of the pencil $|S_0|$ whose tangent $[d-1]$

at P contains the $[d-h]$ common to the tangent $[d-1]$'s at P of the varieties of the pencils $|S_1|, \ldots, |S_h|$ which pass through P. We prove that this $[d-1]$ contains the tangent $[h]$ to $J_h[S_0, S_1, \ldots, S_h]$ at P.

This may be seen by the following differential argument, though a direct synthetic proof is doubtless possible. Since V_d is an analytic non-singular variety we can represent the points of V_d in a suitable neighbourhood of P by the values of d parameters $u_1, u_2, \ldots, u_d$ in such a way that P corresponds to the set of values $(0, 0, \ldots, 0)$ and that the coordinates of all points of V_d in the neighbourhood are analytic functions of the u's. Since P is supposed not to belong to the base-locus of any of the pencils $|S_1|, \ldots, |S_h|$, these parameters can be chosen so that the varieties of these pencils are given respectively, near P, by $u_1=c_1, \ldots, u_h=c_h$, where the c's are constants vanishing for the particular varieties passing through P. Since P lies on (S_0^2) the varieties of $|S_0|$ will be given near P by $\phi(u_1, \ldots, u_d)=c$ where c is a constant and ϕ is a function analytic near P which is indeterminate at the origin. Since $h<d$ the choice of u_d is certainly at our disposal, and there is therefore no loss of generality in supposing that this equation takes the form $cu_d=f(u_1, \ldots, u_d)$, where f is a power series in the u's vanishing at the origin.

The components du_i of any vector common to the tangent spaces at P to the varieties $u_1=0, \ldots, u_h=0$ satisfy

$$du_1=du_2=\ldots=du_h=0, \tag{6}$$

while the components of a vector lying in the tangent space at P to

$$cu_d=f(u_1, \ldots, u_d)$$

satisfy the single relation

$$\sum_{i=1}^{d-1}\left(\frac{\partial f}{\partial u_i}\right)_0 du_i+\left[\left(\frac{\partial f}{\partial u_d}\right)_0-c\right]du_d=0, \tag{7}$$

the suffix indicating that the derivatives are evaluated at the origin.

Now the Jacobian of the $h+1$ pencils is given by the vanishing of all $h+1$ rowed determinants of the Jacobian matrix

$$\left\| \frac{\partial(u_1, u_2, \ldots, u_h, f/u_d)}{\partial(u_1, u_2, \ldots, u_d)} \right\|,$$

a matrix which reduces to

$$\left\| \begin{matrix} I_h & 0_{d-h} \\ v_1 \quad v_2 & \ldots \quad v_d \end{matrix} \right\|.$$

where I_h is the unit matrix of h rows and columns, 0_{d-h} is a block of zeros of h rows and $d-h$ columns, and

$$v_i = \frac{\partial}{\partial u_i}\left(\frac{f}{u_d}\right) \quad (i=1, \ldots, d).$$

Thus the equations of the Jacobian are given, near P, by

$$v_{h+1} = v_{h+2} = \ldots = v_d = 0,$$

i.e. (neglecting the inessential factor u_d) by

$$\left.\begin{aligned} \frac{\partial f}{\partial u_i} &= 0 \quad (i=h+1, h+2, \ldots, d-1), \\ u_d \frac{\partial f}{\partial u_d} - f &= 0. \end{aligned}\right\} \tag{8}$$

Since P lies on this Jacobian it follows that

$$\left(\frac{\partial f}{\partial u_i}\right)_0 = 0 \quad (i=h+1, h+2, \ldots, d-1). \tag{9}$$

Hence, if we choose $c = \left(\frac{\partial f}{\partial u_d}\right)_0$, the left-hand side of (7) reduces to

$$\sum_{i=1}^{h} \left(\frac{\partial f}{\partial u_i}\right)_0 du_i = 0, \tag{10}$$

which is satisfied by any vector satisfying (6). Thus this value of c gives the variety S_P, and (10) is the equation satisfied by all vectors in the tangent $[d-1]$ to S_P at P.

On the other hand, a vector du_i lying in the tangent $[h]$ to

$$J_h[S_0, S_1, \ldots, S_h]$$

at P, whose equations are given by (8), satisfies

$$\sum_{j=1}^{d} \left(\frac{\partial^2 f}{\partial u_i \partial u_j}\right)_0 du_j = 0 \quad (i=h+1, \ldots, d-1)$$

and

$$\left(\frac{\partial f}{\partial u_d}\right)_0 du_d - \sum_{i=1}^{d} \left(\frac{\partial f}{\partial u_i}\right)_0 du_i = 0,$$

that is, when (9) is taken into account,

$$\sum_{i=1}^{h} \left(\frac{\partial f}{\partial u_i}\right)_0 du_i = 0,$$

which is just (10). Thus all such vectors lie in the tangent space to S_P at P, which is our theorem.

5. With these preliminaries we can now proceed to our main result. Let X_h denote the variety (possibly virtual) defined by

$$X_h = J_h[S_0, S_1, \ldots, S_h] - K_h \left[\prod_{i=0}^{h} (1+S_i)^2 - 1 \right], \tag{11}$$

where K_h is the operator defined in § 2, and where the expression operated on by K_h stands for its formal expansion, in which products are interpreted as intersections and meaningless symbols (intersections whose virtual dimension is negative) are replaced by zero. We shall prove that X_h defines a system of equivalence $\{X_h\}$ on V_d which is independent of the pencils $|S_i|$.

The proof is by induction on d. When $d=h+1$ (11) reduces to (3), as is easily seen by noticing that in $\left[\prod_{i=0}^{h} (1+S_i)^2 - 1 \right]$ the only varieties of dimension h are $2(S_0+\ldots+S_h)$ and all varieties of lower dimension are annihilated by the operator K_h. We therefore assume our theorem to be true for varieties of dimension less than d, and prove it for varieties of dimension d. The proof falls into two parts.

6. In the first place we must prove that (11) does actually define a system of equivalence on V_d. To prove this it is clearly sufficient to prove that, if $T_1, \ldots, T_r$ are r V_{d-1}'s on V which can vary in irreducible linear systems $|T_1|, |T_2|, \ldots, |T_r|$ (not necessarily all different), then, as they vary, the variety $X_h[T_1 T_2 \ldots T_r]$ moves in a system of equivalence.

The theorem is true trivially when $d=h+1$, since the only case which arises then is that in which $r=1$, and by definition, if T is a V_h on V_{h+1}, $X_h[T]=T$. We assume as an inductive hypothesis that the theorem is valid on varieties of dimension less than d, and prove it for varieties of dimension d.

Consider first the case in which $r=1$, and let T, T' denote two members of the linear system $|T_1|$. We consider the $h+1$ pencils $|S_i|$ on V_d and let

$$|\Sigma_i|, \quad |\Sigma_i'|$$

denote the pencils cut out by $|S_i|$ on T and T' respectively. By (11), applied to T and T' (of dimension $d-1$),

$$X_h[T] \equiv J_h[\Sigma_0, \Sigma_1, \ldots, \Sigma_h] - K_h \left[\prod_{i=0}^{h} (1+\Sigma_i)^2 - 1 \right] \quad (\text{on } T),$$

$$X_h[T'] \equiv J_h[\Sigma_0', \Sigma_1', \ldots, \Sigma_h'] - K_h \left[\prod_{i=0}^{h} (1+\Sigma_i')^2 - 1 \right] \quad (\text{on } T'),$$

where, since we are assuming our main theorem for varieties of dimension less than d, the expressions in the second term on the right of these equivalences are independent of the particular varieties of the pencils $|\Sigma_i|$ or $|\Sigma_i'|$ which we use. We can thus suppose that Σ_i' is cut out on T' by the same member of $|S_i|$ which cuts out Σ_i on T. But then, if $\Sigma_a \Sigma_b \dots \Sigma_c$ is any term appearing in the expansion,

$$X_h[\Sigma_a \Sigma_b \dots \Sigma_c] = X_h[S_a S_b \dots S_c T],$$

$$X_h[\Sigma_a' \Sigma_b' \dots \Sigma_c'] = X_h[S_a S_b \dots S_c T'],$$

and $(S_a S_b \dots S_c T)$ and $(S_a S_b \dots S_c T')$ are equivalent varieties on $(S_a S_b \dots S_c)$. Hence, applying our inductive hypothesis, since $(S_a S_b \dots S_c)$ has dimension less than or equal to $d-1$,

$$X_h[\Sigma_a \Sigma_b \dots \Sigma_c] \equiv X_h[\Sigma_a' \Sigma_b' \dots \Sigma_c']$$

on $(S_a S_b \dots S_c)$ and therefore on V_d. So

$$K_h\left[\prod_{i=0}^{h} (1+\Sigma_i)^2 - 1\right] \equiv K_h\left[\sum_{i=0}^{h} (1+\Sigma_i')^2 - 1\right] \quad \text{on} \quad V_d.$$

To prove our result when $r = 1$ it is therefore necessary only to show that

$$J_h[\Sigma_0, \Sigma_1, \dots, \Sigma_h] \equiv J_h[\Sigma_0', \Sigma_1', \dots, \Sigma_h'].$$

Now, since $T \equiv T'$, there is a linear pencil $|T^*|$ contained in $|T_1|$ to which T and T' belong. If $|\sigma_i|$ is the pencil cut out by $|S_i|$ on the base of T^*, then, by (5),

$$J_h[\Sigma_0, \Sigma_1, \dots, \Sigma_h] \equiv (T \,.\, J_{h+1}[T^*, S_0, S_1, \dots, S_h]) - J_h[\sigma_0, \sigma_1, \dots, \sigma_h].$$

As T varies in $|T^*|$ the second variety on the right remains fixed while the first describes a linear pencil on $J_{h+1}[T^*, S_0, S_1, \dots, S_h]$. Hence

$$J_h[\Sigma_0, \Sigma_1, \dots, \Sigma_h]$$

varies in a system of equivalence, which proves the result.

The extension to the case $r > 1$ is trivial. For instance, if $T_1 \equiv T_1'$ and $T_2 \equiv T_2'$, then

$$X_h[T_1 T_2] \equiv X_h[T_1 T_2'] \qquad (\text{on } T_1)$$

and

$$X_h[T_1 T_2'] \equiv X_h[T_1' T_2'] \qquad (\text{on } T_2'),$$

so that $$X_h[T_1 T_2] \equiv X_h[T_1' T_2'] \qquad (\text{on } V_d),$$

since equivalence on a subvariety implies equivalence on V_d, and the equivalence relation is transitive. The procedure when $r > 2$ is clear.

We have thus shown that (11) does define a system of equivalence. The next, and more important step, is to prove that this system is invariant.

7. The proof of the invariance of the system defined in (11) follows very closely that of the existence of the Zeuthen-Segre invariant on a surface. We introduce a further linear pencil S_{h+1}, and for convenience make the following definitions:

$$J = J_{h+1}[S_0, S_1, \ldots, S_h, S_{h+1}];$$

$$\Sigma_{ij} = (S_i S_j), \quad \sigma_{ij} = (S_i^2 S_j) \quad (i \neq j);$$

$$\Delta_i = (J S_i);$$

j_i and γ_i are the Jacobian varieties of the pencils $|\Sigma_{ij}|$, $|\sigma_{ij}|$ $(j \neq i)$ cut by $|S_j|$ on a particular variety S_i of $|S_i|$ and on (S_i^2) respectively; they are each of dimension h.

Since (11) is assumed for varieties of dimension less than d, it holds for S_i and (S_i^2). Hence

$$j_i \equiv X_h[S_i] + K_h \left[\prod_{j=0}^{h+1}{}' (1+\Sigma_{ij})^2 - 1 \right],$$

the intersections being evaluated on S_i and the dash indicating that the factor for which $j = i$ does not occur in the product. Thus

$$j_i \equiv K_h \left[S_i + S_i \left\{ \prod_{j=0}^{h+1}{}' (1+S_j)^2 - 1 \right\} \right]$$

$$\equiv K_h \left[S_i \prod_{j=0}^{h+1}{}' (1+S_j)^2 \right]. \tag{12}$$

Similarly $$\gamma_i \equiv K_h \left[S_i^2 \prod_{j=0}^{h+1}{}' (1+S_j)^2 \right]. \tag{13}$$

By (5), $$\Delta_i = (J S_i) \equiv j_i + \gamma_i. \tag{14}$$

Consider now the Jacobian locus Γ of the $h+1$ pencils Δ_i $(i = 0, 1, \ldots, h)$ on J. By (3),

$$\Gamma \equiv X_h[J] + 2 \sum_{i=0}^{h} \Delta_i. \tag{15}$$

Points of this locus arise in four different ways.

(i) Each point of γ_i (for $i=0, 1, \ldots, h$) is a fixed part of the corresponding pencil Δ_i. It appears clearly from (3) that every such fixed part separates out twice from the Jacobian locus.

(ii) At any point $J_h[S_0, S_1, \ldots, S_h]$ the sections of the tangent $[d-1]$'s to the corresponding varieties of the pencils $|S_0|, |S_1|, \ldots, |S_h|$ by the tangent $[h+1]$ to J are the tangent $[h]$'s to the varieties of the pencils

$$|\Delta_0|, \quad |\Delta_1|, \quad \ldots, \quad |\Delta_h|$$

which pass through the point, and these $[h]$'s meet in a line, the section by the tangent $[h+1]$ to J of the $[d-h]$ in which the $h+1$ $[d-1]$'s meet. Thus $J_h[S_0, S_1, \ldots, S_h]$ belongs to Γ.

(iii) The variety Γ also contains γ_{h+1}. For the reasoning of §4 shows that at any point P of γ_{h+1} the $[d-h-1]$ of intersection of the tangent spaces to the varieties of $|S_0|, |S_1|, \ldots, |S_h|$ which pass through P lies in the tangent $[d-1]$ at P to a definite member of the pencil $|S_{h+1}|$, and that the tangent $[h+1]$ to J at P lies in this $[d-1]$. It thus meets the $[d-h-1]$ in a line common to the tangent spaces of the varieties of $|\Delta_0|, |\Delta_1|, \ldots, |\Delta_h|$ passing through P, and hence P is a point of Γ. (The reader may be reminded that only $h+1$ pencils were involved in §4 as compared with $h+2$ here: this accounts for the difference in the dimensions of the intersections involved.)

(iv) At any point of Γ distinct from the foregoing there is a distinct tangent space to each of the varieties of $|S_0|, |S_1|, \ldots, |S_{h+1}|$ which pass through the point, these spaces having a common $[d-h-1]$. The tangent $[d-1]$'s to the varieties of $|S_0|, |S_1|, \ldots, |S_h|$ through the point have this $[d-h-1]$ as their complete intersection, but since the point lies on Γ the section of this $[d-h-1]$ by the tangent $[h+1]$ to J is a line. The locus Λ of such points may thus be defined as the locus of those points of J at which the $[d-h-1]$ common to the tangent spaces of the $h+2$ pencils meets the tangent $[h+1]$ of J in a line. It is a locus of contacts of the pencils of some special nature, but its only importance from our point of view is that *it is symmetrically related to the $h+2$ pencils* $|S_0|, |S_1|, \ldots, |S_{h+1}|$. We have, accordingly,

$$\Gamma = \Lambda + J_h[S_0, S_1, \ldots, S_h] + 2\sum_{i=1}^{h} \gamma_i + \gamma_{h+1}. \tag{16}$$

From (12), (13), (14), (15), and (16) it now follows that

$$
\begin{aligned}
J_h[S_0, S_1, \ldots, S_h] & \\
&\equiv \Gamma - 2 \sum_{i=0}^{h} \gamma_i - \gamma_{h+1} - \Lambda \\
&\equiv X_h[J] + 2 \sum_{i=0}^{h} j_i - \gamma_{h+1} - \Lambda \\
&\equiv X_h[J] - \Lambda + K_h \left[2 \sum_{i=0}^{h} S_i \prod_{j=0}^{h+1}{}' (1+S_j)^2 - S_{h+1}^2 \prod_{j=0}^{h} (1+S_j)^2 \right],
\end{aligned}
$$

and so

$$
\begin{aligned}
& J_h[S_0, S_1, \ldots, S_h] - K_h \left[\prod_{i=0}^{h} (1+S_i)^2 - 1 \right] \\
&\equiv X_h[J] - \Lambda + K_h \left[2 \sum_{i=0}^{h} S_i \prod_{j=0}^{h+1}{}' (1+S_j)^2 - S_{h+1}^2 \prod_{j=0}^{h} (1+S_j)^2 - \prod_{j=0}^{h} (1+S_j)^2 + 1 \right] \\
&\equiv X_h[J] - \Lambda + K_h \left[2 \sum_{i=0}^{h+1} S_i \prod_{j=0}^{h+1}{}' (1+S_j)^2 - \prod_{j=0}^{h+1} (1+S_j)^2 + 1 \right]. \qquad (17)
\end{aligned}
$$

The right-hand side of (17) is symmetrical in $S_0, S_1, \ldots, S_{h+1}$. But the left-hand side does not involve the arbitrary auxiliary pencil $|S_{h+1}|$. Hence the left-hand side of (17) is independent of the pencils $|S_0|, |S_1|, \ldots, |S_h|$, *i.e.* the system $\{X_h\}$ given by (11) is an invariant system on V_d. This completes the proof of the theorem.

We may note that if we define

$$K_h[1] = X_h[V_d], \qquad (18)$$

then (11) can be written more concisely

$$J_h[S_0, S_1, \ldots, S_d] \equiv K_h \left[\prod_{i=0}^{h} (1+S_i)^2 \right], \qquad (19)$$

which is the form obtained by Eger in the first of the notes already cited.

8. We can now show that the invariant systems thus defined possess the *general property of adjunction* expressed by the relation

$$X_{h-1}[S] \equiv (S \,.\, X_h[S] + X_h[V_d]) \quad (\text{on } S), \qquad (20)$$

where $h > 0$ and where S is an irreducible V_{d-1}. We suppose first that S belongs to an irreducible linear pencil $|S_0|$. Since (20) reduces to (2) when $d = h+1$ we assume the result to hold on varieties of dimension less than d, and prove it for varieties of dimension d.

We take h additional irreducible linear pencils $|S_i|$ on V_d, $(i = 1, 2, \dots, h)$ and denote by $|\Sigma_i|$, $|\sigma_i|$ the pencils cut out by $|S_i|$ on S and on the base (S_0^2) of $|S_0|$ respectively. Then, by the inductive hypothesis, if T is an irreducible algebraic subvariety of $k(\leqslant d-1)$ dimensions lying on V_d,

$$X_{h-1}[ST] \equiv (S \,.\, X_h[ST] + X_h[T]),$$

that is,

$$K_{h-1}[ST] \equiv (S \,.\, K_h[(1+S)T]). \tag{21}$$

Now, from (13),

$$J_{h-1}[\sigma_1, \sigma_2, \dots, \sigma_h] \equiv K_{h-1}\left[\prod_{i=1}^{h} (1+\sigma_i)^2\right] \equiv K_{h-1}\left[S^2 \prod_{i=1}^{h} (1+S_i)^2\right]. \tag{22}$$

Hence, from (11),

$$\begin{aligned}
X_{h-1}[S] &\equiv J_{h-1}[\Sigma_1, \Sigma_2, \dots, \Sigma_h] - K_{h-1}\left[\prod_{i=1}^{h} (1+\Sigma_i)^2 - 1\right] \\
&\equiv (S \,.\, J_h[S_0, S_1, \dots, S_h]) - J_{h-1}[\sigma_1, \sigma_2, \dots, \sigma_h] \\
&\qquad\qquad - K_{h-1}\left[S \prod_{i=1}^{h} (1+S_i)^2 - S\right] \quad \text{by} \quad (5) \\
&\equiv (S \,.\, J_h[S_0, S_1, \dots, S_h]) - K_{h-1}\left[S(1+S) \prod_{i=1}^{h} (1+S_i)^2 - S\right] \quad \text{by} \quad (22) \\
&\equiv (S \,.\, J_h[S_0, S_1, \dots, S_h]) \\
&\qquad - \left(S \,.\, K_h\left[(1+S) \,.\, \left\{(1+S) \prod_{i=1}^{h} (1+S_i)^2 - 1\right\}\right]\right) \quad \text{by} \quad (21) \\
&\equiv \left(S \,.\, J_h[S_0, S_1, \dots, S_h] - K_h\left[\prod_{i=0}^{h} (1+S_i)^2 - 1 - S\right]\right) \\
&\equiv (S \,.\, X_h[S] + X_h[V_d]) \quad \text{by} \quad (11).
\end{aligned}$$

This proves the theorem stated when S belongs to an irreducible linear pencil.

Now let S and T be two V_{d-1}'s on V_d belonging to irreducible linear systems $|S|$ and $|T|$ of freedom greater than zero satisfying the following conditions:

(i) The linear system $|S+T|$ is irreducible,

(ii) The linear systems cut by $|T|$ on a general S, and by $|S|$ on a general T, are irreducible and have freedom greater than zero,

(iii) The linear system cut by $|S+T|$ on the intersection (ST) of a general S and a general T is irreducible and has freedom greater than zero.

We prove that

$$X_h[S+T] \equiv X_h[S]+X_h[T]+2X_h[ST]+X_h[ST(S+T)]. \tag{23}$$

This is true trivially when $h=d-1$, since then the last two terms on the right are zero. We assume the theorem to hold for canonical systems of dimension $h+1$. Conditions (i), (ii), and (iii) are sufficient to ensure that (20) can be applied to S, T, $S+T$, on V_d, to (ST) considered as a locus lying on S or T, and to $ST(S+T)$ considered as a locus lying on (ST). Hence, using the inductive hypothesis,

$$\begin{aligned}
X_h[S+T] &\equiv (S+T\,.\,X_{h+1}[S+T]+X_{h+1}[V_d]) \\
&\equiv (S+T\,.\,X_{h+1}[S]+X_{h+1}[T]+2X_{h+1}[ST] \\
&\qquad\qquad +X_{h+1}[ST(S+T)]+X_{h+1}[V_d]) \\
&\equiv (S\,.\,X_{h+1}[S]+X_{h+1}[V_d])+(T\,.\,X_{h+1}[T]+X_{h+1}[V_d]) \\
&\qquad +(S\,.\,X_{h+1}[T]+X_{h+1}[ST])+(T\,.\,X_{h+1}[S]+X_{h+1}[ST]) \\
&\qquad +\big(S+T\,.\,X_{h+1}[ST]+X_{h+1}[ST(S+T)]\big) \\
&\equiv X_h[S]+X_h[T]+(ST\,.\,X_{h+1}[T]+X_{h+1}[ST])_T \\
&\qquad +(ST\,.\,X_{h+1}[S]+X_{h+1}[ST])_S \\
&\qquad +\big(ST(S+T)\,.\,X_{h+1}[ST]+X_{h+1}[ST(S+T)]\big)_{(ST)}
\end{aligned}$$

[the suffixes indicating that the intersections are evaluated on S, T, (ST) respectively], and the right-hand side is consequently

$$X_h[S]+X_h[T]+2X_h[ST]+X_h[ST(S+T)],$$

which proves (23).

We now use (23) to *define* $X_h[S+T]$ even when $|S+T|$ is reducible. We may also use (23) to define $X_h[S]$ on a *virtual* variety S of $d-1$ dimensions. To do this we take an effective variety U such that $T\equiv U-S$ is effective, and define

$$X_h[S]\equiv X_h[U]-X_h[T]-2X_h[ST]-X_h[STU].$$

The varieties (ST) and (STU) may be virtual, but their dimension is less than $d-1$, and they lie on the effective varieties T, (TU) respectively. If they are in fact virtual their canonical systems may be defined in the same manner. In any event, after a finite number of steps only canonical systems of effective varieties appear on the right, since when the dimension of a variety is less than h its canonical V_h does not exist†.

It follows from the proof of (23), and from Macpherson's result just cited, that the system $X_h(S)$ defined on *any* S, reducible or irreducible, effective or virtual, satisfies the relation (20). The system

$$A_h[S] \equiv X_h[S] + X_h[V_d] \tag{24}$$

we call the *adjoint* system to S, of dimension h, and we clearly have

$$X_{h-1}[S] \equiv (S \,.\, A_h[S]). \tag{25}$$

9. We have finally to identify the invariant system $\{X_h\}$ just considered with that defined in the previous paper in terms of the Jacobian of a linear ∞^{h+1} system. To do this it is sufficient to show‡ that the Jacobian $J_h[S]$ of a sufficiently general linear ∞^{h+1} system $|S|$ is given by

$$J_h[S] \equiv \sum_{i=0}^{h+2} \binom{h+2}{i} X_h[S^i]$$

that is, symbolically,

$$J_h[S] \equiv K_h[(1+S)^{h+2}]. \tag{26}$$

As Eger remarks, this may be deduced by limiting considerations from (19). It is convenient to establish the more general result§ that the Jacobian of a linear ∞^{r+1} system $|S|$ and of $h-r$ other linear pencils $|S_1|, \ldots, |S_{h-r}|$ is given by

$$J_h[S;\, S_1, \ldots, S_{h-r}] \equiv K_h\left[(1+S)^{r+2} \prod_{i=1}^{h-r} (1+S_i)^2\right]. \tag{27}$$

The proof of (27) proceeds by induction on r, since it reduces to (19) when $r=0$. When $h=d-1$, (27) is easily proved by considerations similar to those employed in the proof of (3). We suppose then that (27) has been proved for canonical systems of dimension greater than h on V_d, and prove it for the system $\{X_h\}$.

We consider a linear ∞^{r+2} system $|T|$ containing $|S|$, and a linear ∞^r system $|U|$ contained in $|S|$, and consider a general pencil $|S_0|$ lying in

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† In a note appearing in *Proc. Cambridge Phil. Soc.*, 35 (1939), 389–393, R. E. Macpherson has extended (23) to the case in which S is an isolated variety, and has shown directly that the relation of adjunction (20) is valid in this case also.

‡ *C.f.* equation (5.1) of the previous paper.

§ Eger, *loc. cit. ante.*

$|T|$. The Jacobian J_{h+1} of T, S_1, ..., S_{h-r} is of $h+1$ dimensions and contains $J_h[U;\, S_0, S_1, \ldots, S_{h-r}]$, and as $|S_0|$ varies in T this locus describes a linear system on J_{h+1}. When $|S_0|$ lies in $|S|$ it has a member S^* in common with $|U|$. As $|S_0|$ tends to the limiting position in which it lies in $|S|$ the variety $J_h[U;\, S_0,\, S_1,\, \ldots,\, S_{h-1}]$ breaks up into $J_h[S;\, S_1,\, \ldots,\, S_{h-r}]$ and a residual variety which is easily seen to be the intersection of J_{h+1} and S^*. Thus

$$J_h[U;\; S_0,\, S_1,\, \ldots,\, S_{h-r}] \equiv J_h[S;\; S_1,\, \ldots,\, S_{h-r}] + (S \,.\, J_{h+1}[\,T;\; S_1\, \ldots,\, S_{h-r}]).$$

Hence by the inductive hypotheses, on r and h,

$$\begin{aligned} J_h[S;\; S_1,\, \ldots,\, S_{h-r}] &\equiv K_h\left[(1+S)^{r+1}(1+S)^2 \prod_{i=1}^{h-r}(1+S_i)^2\right] - \left(S\,.\,K_{h+1}\left[(1+S)^{r+3}\prod_{i=1}^{h-r}(1+S_i)^2\right]\right) \\ &\equiv K_h\left[(1+S)^{r+3}\prod_{i=1}^{h-r}(1+S_i)^2 - S(1+S)^{r+2}\prod_{i=1}^{h-r}(1+S_i)^2\right] \quad \text{by} \quad (21) \\ &\equiv K_h\left[(1+S)^{r+2}\prod_{i=1}^{h-r}(1+S_i)^2\right], \end{aligned}$$

which is (27). Putting $h=r$ in (27) we obtain (26). The identity of the systems considered here with those previously obtained is therefore established.

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THE ABSOLUTE CESÀRO SUMMABILITY OF FOURIER INTEGRALS

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1. *Introduction.*

In the description of the manner in which a function tends to a limit as its argument approaches a point, the additional condition implied by the word "absolute" is that the function is of bounded variation in the neighbourhood of the point. For a summable series the sum of the moduli of the differences between successive terms of the sequence of Cesàro means must be finite; for a summable integral the partial integral must be of bounded variation.

The theory of the absolute summability of series has been treated by various authors and the results have been applied to Fourier series. Very general results have been obtained by Bosanquet*, by whom the investigation of the analogous problem for integrals was suggested to me.

In this paper results for Fourier integrals parallel to those of Bosanquet for series are obtained. They correlate the absolute summability of the Fourier integral of a function with bounded variation of the function. We first discuss absolute summability of integrals in general, then we investigate the Fourier integral problem, and finally we show that the results attained are best possible.

I wish here to express my indebtedness to Dr. Bosanquet for the indication of this problem and for the valuable help and advice he has given me in the preparation of this paper.

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* See Bosanquet, *Proc. London Math. Soc.* (2), 41 (1936), 517–528, where a full bibliography is given.

2. *The absolute summability of integrals.*

Definitions.

The integral $\int_0^\infty f(x)\,dx$ is summable (C, α), $\alpha \geqslant 0$, if

$$I_\alpha(\lambda) = \int_0^\lambda \left(1-\frac{u}{\lambda}\right)^\alpha f(u)\,du$$

exists as a Lebesgue integral for all $\lambda \geqslant 0$, and tends to a limit as $\lambda \to \infty$.

The integral $\int_0^\infty f(x)\,dx$ is absolutely summable (C, α), or summable $|C, \alpha|$, $\alpha \geqslant 0$, if $I_\alpha(\lambda)$ is of bounded variation over $(0, \infty)$.

If we differentiate formally under the sign of integration, we get, for $\alpha > 0$,

$$\frac{dI_\alpha}{d\lambda} = \alpha \int_0^\lambda \left(1-\frac{u}{\lambda}\right)^{\alpha-1} \frac{u}{\lambda^2} f(u)\,du. \tag{1}$$

This requires justification if $\alpha < 1$. We have

$$\int_0^\lambda dw \int_0^w \alpha\left(1-\frac{u}{w}\right)^{\alpha-1} \frac{u}{w^2} f(u)\,du = \alpha \int_0^\lambda uf(u)\,du \int_u^\lambda \left(1-\frac{u}{w}\right)^{\alpha-1} \frac{dw}{w^2}$$
$$= \int_0^\lambda \left(1-\frac{u}{\lambda}\right)^\alpha f(u)\,du,$$

the inversion of the integrations being justified by absolute convergence. From this it follows that (1) is true almost everywhere.

We have for $\alpha \geqslant 0$ and almost all λ,

$$\frac{dI_\alpha}{d\lambda} = \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} F_\alpha(\lambda),$$

where, for $\alpha > 0$,

$$F_\alpha(\lambda) = \frac{1}{\Gamma(\alpha)} \int_0^\lambda (\lambda-u)^{\alpha-1} uf(u)\,du,$$
$$F_0(\lambda) = \lambda f(\lambda),$$

so that $F_\alpha(\lambda)$ is the integral of order α of $uf(u)$.

It is known* that, for any fixed $v \geqslant 0$,

$$\Gamma(\alpha+1) \int_0^v |F_\alpha(u)|\, u^{-\alpha-1}\,du$$

is a non-increasing function of α.

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* Bosanquet, *Proc. Edinburgh Math. Soc.* (2), 4 (1934), 12–17.

Since $I_\alpha(\lambda)$ is absolutely continuous in any interval (δ, ξ), $0 < \delta < \xi$, we have, for any $v > 0$,

$$\int_0^v |dI_\alpha| \geqslant \int_0^v |dI_\beta|,$$

if $\beta > \alpha \geqslant 0$.

Making $v \to \infty$, we get the consistency theorem for summability $|C, \alpha|$:

If the integral $\int_0^\infty f(x)\,dx$ *is summable* $|C, \alpha|$ *for any* $\alpha \geqslant 0$, *it is summable* $|C, \beta|$ *for* $\beta > \alpha$.

3. *The absolute summability of Fourier integrals.*

We discuss the absolute summability of the integral $\int_0^\infty \psi(u)\,du$, where

$$\psi(u) = \int_{-\infty}^{\infty} f(t) \cos u(x-t)\,dt,$$

that is, of the Fourier integral for $f(t)$ at $t = x$. Write

$$\phi(t) = f(t+x) + f(t-x),$$

so that

$$\psi(u) = \int_0^\infty \phi(t) \cos ut\,dt.$$

We shall write

$$I_\alpha(\lambda) = \int_0^\lambda \left(1 - \frac{u}{\lambda}\right)^\alpha \psi(u)\,du,$$

and, for the integrals and means of fractional order of $\phi(t)$,

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \phi(u)\,du \quad (\alpha > 0),$$

$$\Phi_0(t) = \phi(t),$$

$$\phi_\alpha(t) = \Gamma(\alpha+1)\, t^{-\alpha}\, \Phi_\alpha(t) \qquad (\alpha \geqslant 0).$$

Before we proceed to the main theorems of this section, it will be convenient to define certain functions which will occur and to state known inequalities concerning them.

For $\alpha > 0$,

$$g^{\alpha}(v) = \int_0^1 (1-u)^{\alpha-1} \sin uv\, du,$$

$$\gamma^{\alpha}(v) = \int_0^1 (1-u)^{\alpha-1} \cos uv\, du.$$

The following inequalities are well-known:

$$|g^{\alpha}(v)| \begin{cases} \leqslant Av \\ \leqslant Av^{-\alpha} & (\alpha < 1), \\ \leqslant Av^{-1} & (\alpha \geqslant 1), \end{cases} \tag{a}$$

$$|\gamma^{\alpha}(v)| \begin{cases} \leqslant A \\ \leqslant Av^{-\alpha} & (\alpha \leqslant 2), \\ \leqslant Av^{-2} & (\alpha > 2). \end{cases} \tag{a'}$$

$$\left|\frac{d}{dv}\, g^{\alpha}(v)\right| \begin{cases} \leqslant A, \\ \leqslant Av^{-\alpha} & (\alpha < 2), \\ \leqslant Av^{-2} & (\alpha \geqslant 2), \end{cases} \tag{b}$$

$$\left|\frac{d}{dv}\, \gamma^{\alpha}(v)\right| \begin{cases} \leqslant A, \\ \leqslant Av^{-\alpha} & (\alpha < 3), \\ \leqslant Av^{-3} & (\alpha \geqslant 3). \end{cases} \tag{c}$$

We shall always suppose $\phi(t)$ to be integrable over any finite interval. The sense in which $\psi(u)$ exists will be specified in each case.

THEOREM I. *If $\phi(t)$ is of bounded variation over $(0, \infty)$ and tends to zero as $t \to \infty$, and if $\psi(u)$* is integrable over any finite interval, then the integral $\int_0^{\infty} \psi(u)\, du$ is summable $|C, \alpha|$ for any $\alpha > 0$.*

We can assume without loss of generality that $\phi(t)$ is non-increasing.

We have, almost everywhere,

$$\frac{dI_{\alpha}(\lambda)}{d\lambda} = \alpha \int_0^{\lambda} \frac{u}{\lambda^2} \left(1 - \frac{u}{\lambda}\right)^{\alpha-1} \psi(u)\, du.$$

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* $\psi(u)$ exists as a Cauchy limit at infinity: for this it is necessary that $\phi(t)$ tends to zero.

Now, by an easily justified inversion of order of integration,

$$\frac{dI_\alpha}{d\lambda}=\alpha\int_0^\infty \phi(t)\,dt\int_0^1(1-u)^{\alpha-1}u\cos\lambda ut\,du$$

$$=\frac{\alpha}{\lambda}\int_0^\infty \phi(t)\frac{d}{dt}g^\alpha(\lambda t)\,dt$$

$$=\frac{\alpha}{\lambda}\Big[\phi(t)\,g^\alpha(\lambda t)\Big]_0^\infty-\frac{\alpha}{\lambda}\int_0^\infty g^\alpha(\lambda t)\,d\phi(t).$$

The expression in square brackets vanishes, and so

$$\int_0^\infty |dI_\alpha(\lambda)|=\alpha\int_0^\infty\frac{d\lambda}{\lambda}\left|\int_0^\infty g^\alpha(\lambda t)\,d\phi(t)\right|$$

$$\leqslant\alpha\int_0^\infty|d\phi(t)|\int_0^\infty|g^\alpha(\lambda t)|\frac{d\lambda}{\lambda}$$

$$=\alpha\int_0^\infty|d\phi(t)|\int_0^\infty|g^\alpha(u)|\frac{du}{u}$$

$$\leqslant A\int_0^\infty|d\phi(t)|,$$

because of the inequalities (a), since $\alpha>0$. Hence, since $\phi_\alpha(t)$ is of bounded variation over $(0,\infty)$, so is $I_\alpha(\lambda)$, and this gives the theorem.

This theorem makes summability $|C,\alpha|$ depend on the variation of $\phi(t)$ over $(0,\infty)$. We shall now show that, apart from conditions on the size (and not the variation) of $\phi(t)$ at infinity, summability $|C,\alpha|$ is a local property if $\alpha>1$.

THEOREM II. *If $\phi(t)/(1+t)$ is $L(0,\infty)$, and the integral $\int_0^\infty u\phi(t)\cos ut\,dt$ is boundedly convergent in any finite interval, then the integral $\int_0^\infty\psi(u)\,du$ is summable $|C,\alpha|$ for all $\alpha>1$, provided that $\psi(u)$ is integrable over any finite interval, if either*

(i) $\displaystyle\int_0^w\frac{|\phi(t)-s|}{t}\,dt$ *exists for all w and some s,*

or (ii) *$\phi(t)$ is of bounded variation in an interval $(0,\delta)$, $\delta>0$.*

We can assume, without loss of generality, that $s=0$.

As before, we have

$$\frac{dI_\alpha}{d\lambda} = \alpha \int_0^\lambda \frac{u}{\lambda^2}\left(1-\frac{u}{\lambda}\right)^{\alpha-1} \psi(u)\,du$$

$$= \frac{\alpha}{\lambda}\int_0^\infty \phi(t)\,\frac{d}{dt}\,g^\alpha(\lambda t)\,dt,$$

on substituting the integral for $\psi(u)$ and inverting, the inversion being justified by the bounded convergence of the integral for $u\psi(u)$ and the convergence of the last integral.

$$\int_0^\infty |dI_\alpha(\lambda)| = \alpha \int_0^\infty \frac{d\lambda}{\lambda}\left|\int_0^\infty \phi(t)\,\frac{d}{dt}\,g^\alpha(\lambda t)\,dt\right|$$

$$\leqslant \alpha \int_0^\infty |\phi(t)|\,dt \int_0^\infty \left|\frac{d}{dt}\,g^\alpha(\lambda t)\right| \frac{d\lambda}{\lambda}$$

$$= \alpha \int_0^\infty \frac{|\phi(t)|}{t}\,dt \int_0^\infty \left|\frac{d}{du}\,g^\alpha(u)\right| du$$

$$\leqslant A \int_0^\infty \frac{|\phi(t)|}{t}\,dt,$$

the inner integral converging because of the inequalities (b), since $\alpha > 1$. If now the condition (i) holds, the integral is convergent and so the first part of the theorem is proved.

If condition (ii) holds, write

$$\phi(t) = p(t) + q(t),$$

where

$$p(t) = 0 \quad \text{in} \quad (0,\ \delta),$$

$$q(t) = 0 \quad \text{outside} \quad (0,\ \delta).$$

Then $p(t)$ satisfies conditions (i) and $q(t)$ satisfies the conditions of Theorem I. Hence the corresponding Fourier integrals are summable $|C, \alpha|$ for $\alpha > 1$, and therefore that of $\phi(t)$ is as well.

The next theorem is of a type converse to the last two.

THEOREM III. *If there exists a function* $\psi(u)$ *of* $L(0, \infty)$ *such that*

$$\lim_{\lambda\to\infty} \int_0^\infty \left|\psi(u) - \int_0^\lambda \left(1-\frac{t}{\lambda}\right)\phi(t)\cos ut\,dt\right| \frac{du}{1+u} = 0, \tag{1}$$

then $\phi_\alpha(t)$ *is of bounded variation over* $(0, \infty)$, *for all* $\alpha > 1$.

Write
$$f_\lambda(u) = \left(1-\frac{u}{\lambda}\right)\phi(u) \quad (0 \leqslant u < \lambda),$$
$$= 0 \quad (u > \lambda),$$
$$F_\lambda(u) = \int_0^u f_\lambda(v)\,dv,$$
and
$$P_\lambda(u) = \int_0^\lambda \left(1-\frac{t}{\lambda}\right)\phi(t)\cos ut\,dt$$
$$= \int_0^\infty f_\lambda(t)\cos ut\,dt.$$
Then, since $f_\lambda(t)$ belongs to $L(0, \infty)$, for $u \geqslant 0$ we have*
$$F_\lambda(u) = \frac{2}{\pi}\int_0^\infty P_\lambda(t)\,\frac{\sin ut}{t}\,dt.$$

If we now put
$$f_{\lambda\alpha}(t) = \alpha t^{-\alpha}\int_0^t (t-u)^{\alpha-1} f_\lambda(u)\,du,$$
then, since $\alpha > 1$,
$$f_{\lambda\alpha}(t) = \alpha(\alpha-1)\,t^{-\alpha}\int_0^t (t-u)^{\alpha-2} F_\lambda(u)\,du$$
$$= \frac{2}{\pi}\,\alpha(\alpha-1)\,t^{-\alpha}\int_0^t (t-u)^{\alpha-2}\,du\int_0^\infty P_\lambda(v)\,\frac{\sin uv}{v}\,dv$$
$$= \frac{2\alpha}{\pi}\,(\alpha-1)\,t^{-\alpha}\int_0^\infty \frac{P_\lambda(v)}{v}\,dv\int_0^t (t-u)^{\alpha-2}\sin uv\,du$$
$$= \frac{2\alpha}{\pi}\int_0^\infty P_\lambda(v)\,\gamma^\alpha(vt)\,dv.$$

Since $f_\lambda(u) \to \phi(u)$ as $\lambda \to \infty$, and $|f_\lambda(u)| < |\phi(u)|$,
$$f_{\lambda\alpha}(t) \to \phi_\alpha(t),$$
as $\lambda \to \infty$. As for the other side of the identity just proved, since $\gamma^\alpha(v)$ is $O(v^{-\alpha})$ as $v \to \infty$, it follows from (1) that
$$\int_0^\infty P_\lambda(v)\,\gamma^\alpha(vt)\,dv \to \int_0^\infty \psi(v)\,\gamma^\alpha(vt)\,dv.$$

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* See Titchmarsh, *Fourier integrals* (1937), 41, Theorem 22.

Hence, for all $t > 0$,

$$\phi_a(t) = \frac{2a}{\pi} \int_0^\infty \psi(v)\, \gamma^a(vt)\, dv.$$

On differentiating this under the integral sign, we get

$$\phi_a'(t) = \frac{2a}{\pi} \int_0^\infty \psi(v) \frac{d}{dt} \gamma^a(vt)\, dv.$$

The inequalities (c) show that the integral is uniformly convergent in any interval $t \geqslant \delta > 0$, and so the last equation holds for almost all $t > 0$.

Since $\phi_a(t)$ is absolutely continuous in any interval not containing $t = 0$, for $a > 1$, we get

$$\begin{aligned}\int_0^\infty |d\phi_a(t)| &= \frac{2a}{\pi} \int_0^\infty dt \left| \int_0^\infty \psi(v) \frac{d}{dt} \gamma^a(vt)\, dv \right| \\ &\leqslant \frac{2a}{\pi} \int_0^\infty |\psi(v)|\, dv \int_0^\infty \left| \frac{d}{dt} \gamma^a(vt) \right| dt \\ &= \frac{2a}{\pi} \int_0^\infty |\psi(v)|\, dv \int_0^\infty \left| \frac{d}{du} \gamma^a(u) \right| du \\ &\leqslant A \int_0^\infty |\psi(v)|\, dv,\end{aligned}$$

by inequalities (c), since $a > 1$.

Hence $\phi_a(t)$ is of bounded variation over $(0, \infty)$ if $a > 1$.

The definition of $\psi(u)$ needs some comment. In the first place, the definition must be such that $\phi(u)$ is given uniquely for a given $\psi(u)$: that this is the case follows immediately from the proof of the theorem. In the second place, it is a known theorem that, if $\sqrt{(\frac{1}{2}\pi)}\,\phi(t)$ is the transform of a function $\psi(u)$ of $L(0, \infty)$ then*

$$\lim_{\lambda \to \infty} \int_0^\infty \left| \psi(u) - \int_0^\lambda \left(1 - \frac{t}{\lambda}\right) \phi(t) \cos ut\, dt \right| du = 0. \tag{1a}$$

The condition (1) is therefore satisfied by the transform of any function of $L(0, \infty)$. The theorem is easily proved if we replace (1) by the stronger assumption (1a). The latter is in a sense the most natural assumption to make, for, by Wiener's result, it is equivalent to asserting that

$$\phi(u) = \frac{2}{\pi} \int_0^\infty \psi(t) \cos ut\, dt,$$

---

* See Wiener, *The Fourier integral* (1933), 82.

where $\psi(u)$ is $L(0, \infty)$. Only when it is proved that (1) involves the uniqueness of $\phi(u)$ is the assumption of (1) equivalent to this.

The theorems proved above are particular cases of more general theorems which correlate the behaviour of $\phi(u)$ and the absolute summability of the integral in further detail. Since the main points of the theory are well brought out by the theorems already proved and since the proofs of the general theorems are rather complicated while introducing nothing really new, we shall state the theorems but merely indicate the method of proof.

THEOREM I*a*. *If the integral* $u\psi(u) = \int_0^\infty u\psi(t) \cos ut\, dt$ *converges boundedly for all u in any finite interval, if* $\int_0^t \phi(v)\, dv$ *is* $O(t)$ *as* $t \to \infty$, *and if* $\phi_\alpha(t)$ *is of bounded variation over* $(0, \infty)$, *for some* $\alpha \geqslant 0$, *then the integral* $\int_0^\infty \psi(u)\, du$ *is summable* $|C, \beta|$ *for all* $\beta > \alpha$, *provided that* $\psi(u)$ *is integrable over any finite interval.*

THEOREM III*a*. *If* $\alpha \geqslant 0$, $\gamma = \min(\alpha, 1)$, *and* $\psi(t)$ *exists such that*

$$\text{(i)} \qquad \lim_{\lambda\to\infty} \int_0^\infty \left| \psi(t) - \int_0^\lambda \left(1 - \frac{u}{\lambda}\right) \phi(u) \cos ut\, du \right| \frac{dt}{(1+t)^{\gamma+1}} = 0,$$

$$\text{(ii)} \qquad \int_0^t \psi(v)\, dv = O(t), \quad \text{as} \quad t \to \infty,$$

(iii) *the integral* $\int_0^\infty \psi(u)\, du$ *is summable* $|C, \alpha|$,

then $\phi_\beta(t)$ *is of bounded variation over* $(0, \infty)$ *for all* $\beta > \alpha + 1$.

To prove Theorem I*a*, we begin by proving the expression for $\lambda(dI_\beta/d\lambda)$ obtained in Theorem I, and proceed by means of successive integrations by parts to transform this into an expression in terms of $\Phi_h(t)$, where $h = [\alpha]$, and then, by inversions of the order of integration and further integrations by parts, into an integral involving $d\phi_\alpha$. We can then prove the theorem by using the fact that $\int_0^\infty |d\phi_\alpha|$ is finite.

In the proof of Theorem III*a* we must begin by proving that

$$\phi_\beta(t) = \frac{2\beta}{\pi} \int_0^\infty \psi(v)\, \gamma^\beta(vt)\, dv$$

along the lines adopted in the proof of Theorem III. After this we proceed on lines similar to the proof[*] of Theorem Ia.

4. The theorems proved will now be shown to be best-possible as regards the values of α for which they hold.

Theorem I is false for $\alpha = 0$; for, if $\psi(u)$ is $L(0, \infty)$, $\phi(t)$ must be equivalent to a continuous function, and so any discontinuous function of bounded variation over $(0, \infty)$ is an example showing that bounded variation of $\phi(t)$ does not imply summability $|C, 0|$. It is shown by Titchmarsh, *Fourier integrals*, §6.11, that there exists a function absolutely continuous and decreasing monotonically to zero for which $\psi(u)$ is not $L(0, 1)$, and so the condition that $\psi(u)$ be integrable over any finite interval is not superfluous.

In Theorem II it is shown that summability $|C, \alpha|$ depends only on the behaviour of $\phi(t)$ near $t = 0$ as regards the variation of $\phi(t)$, if $\alpha > 1$, and we therefore ask whether summability $|C, \alpha|$ is also a local property if $\alpha \leqslant 1$. To prove that this is not the case we show that there exists a $\phi(t)$, zero save in an interval (a, b), $0 < a < b$, and $L(a, b)$, for which $\psi(u)$ is not summable $|C, 1|$.

If $\phi(t)$ is zero outside (a, b),

$$I_1(w) = \frac{2}{\pi} \int_a^b \phi(t) \frac{1 - \cos wt}{wt^2}\, dt,$$

$$\frac{dI_1}{dw} = \frac{2}{\pi} \int_a^b \phi(t) \left\{ \frac{\sin wt}{wt} - \frac{1 - \cos wt}{w^2 t^2} \right\} dt.$$

We therefore wish to prove the existence of a function $\phi(t)$ of $L(a, b)$, such that

$$\int_0^\infty dw \left| \int_a^b \phi(t) f(w, t)\, dt \right| = \infty,$$

where $$f(w, t) = \frac{\sin wt}{wt} - \frac{1 - \cos wt}{w^2 t^2}.$$

To prove our result we use the following general lemma, which will be useful in other problems[†].

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[*] Compare the proofs in the paper of Bosanquet, "The absolute summability of Fourier series", *Proc. London Math. Soc.* (2), 41 (1936), 517–528. Proofs of certain inequalities needed in the theorem can be found in Bosanquet and Hyslop, *Math. Zeitschrift*, 42 (1937), 489–512.

[†] *Cf.* Bosanquet and Kestleman, *Proc. London Math. Soc.* (2), 45 (1939), 89–97.

LEMMA. *If* $f(w, t)$ *is uniformly bounded and uniformly continuous in* t *for* t *in* (a, b) *over any finite range of values of* w, *and integrable over any finite range of* w *for all* t *in* (a, b), *then a necessary and sufficient condition that for every* $g(t)$ *of* $L(a, b)$

$$\int_0^\infty dw \left| \int_a^b g(t) f(w, t)\, dt \right| < \infty,$$

is that, for some M *and all* t *in* (a, b),

$$\int_0^\infty |f(w, t)|\, dw < M.$$

That the condition is sufficient follows from the inequality

$$\int_0^\infty dw \left| \int_0^\infty g(t) f(w, t)\, dt \right| \leqslant \int_a^b |g(t)|\, dt \int_0^\infty |f(w, t)|\, dw. \tag{1}$$

To show that it is necessary, suppose that

$$\int_0^\infty dw \left| \int_a^b g(t) f(w, t)\, dt \right| < \infty \tag{2}$$

for all functions $g(t)$ of $L(a, b)$. Then we must show that

$$J(t) = \int_0^\infty |f(w, t)|\, dw$$

is a bounded function of t.

Write

$$U_n(g) = \int_0^n dw \left| \int_a^b g(t) f(w, t)\, dt \right|.$$

$U_n(g)$ is a continuous functional in the space of functions which are $L(a, b)$, normalized by

$$\|g\| = \int_a^b |g(t)|\, dt.$$

If (2) is true, $\lim\limits_{n \to \infty} U_n(g) < \infty$ for all g in $L(a, b)$. By a theorem on continuous functionals* there exists a h of $L(a, b)$ and positive numbers δ, K, such that $U_n(g) < K$ if $\|g - h\| < \delta$, where K is independent of n.

Since, if p and q are any two functions of $L(a, b)$,

$$|U_n(p) - U_n(q)| \leqslant |U_n(p)| + |U_n(q)|,$$

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* See Banach, *Théorie des opérations linéaires* (1932), 19.

for all g, $\|g\| \leqslant \delta$,

$$\begin{aligned}|U_n(g)| &\leqslant |U_n(h)| + |U_n(g+h)| \\ &\leqslant 2K.\end{aligned}$$

Since, for any positive number k,

$$U_n(kg) = kU_n(g),$$

for all g, $\|g\| \leqslant 1$, and for all n,

$$U_n(g) \leqslant 2K/\delta = K', \text{ say.}$$

We now prove that this cannot be the case if $J(t)$ is unbounded. If it were, we could find θ in (a, b) such that

$$J(\theta) > 3K',$$

and we could find n such that

$$\int_0^n |f(w, \theta)|\,dw > 2K'.$$

Write

$$\begin{aligned} g_m(t) &= \tfrac{1}{2}m \quad (|t-\theta| \leqslant 1/m), \\ &= 0 \quad (|t-\theta| > 1/m). \end{aligned}$$

so that, for all m, $\|g_m\| = 1$.

By a Fejér argument we can easily show that, with the conditions imposed on $f(w, t)$, as $m \to \infty$,

$$\int_a^b g_m(t) f(w, t)\,dt \to f(w, \theta)$$

uniformly in every finite range of w. Hence, for the given n and any $\epsilon > 0$, we could find $m_0(\epsilon, n)$ such that for all $m > m_0$ and $w \leqslant n$,

$$\left| \int_a^b g_m(t) f(w, t)\,dt - f(w, \theta) \right| < \epsilon/n.$$

Then, taking $\epsilon < K'$, we should have

$$\begin{aligned} U_n(g_m) &= \int_0^n dw \left| \int_a^b g_m(t) f(w, t)\,dt \right| \\ &\geqslant \int_0^n |f(w, \theta)|\,dw - \epsilon \\ &> 2K' - \epsilon > K'. \end{aligned}$$

This contradicts the statement that $U_n(g) \leqslant K'$ for all g for which $\|g\| \leqslant 1$, and so $J(t)$ cannot be unbounded if (2) is true. This completes the proof of the lemma.

On returning to the problem for the solution of which the lemma was introduced, we easily see that the $f(w, t)$ there defined satisfies all the conditions of the lemma, and that the corresponding $J(t)$ is infinite for all t: the existence of a function $\phi(t)$ with the required properties is therefore secured by the lemma.

A similar problem arises from Theorem III. It is proved there that, if $\phi(u)$ is the transform of a function of $L(0, \infty)$, $\phi_a(t)$ is of bounded variation over $(0, \infty)$ if $a > 1$. The lemma enables us to show that this is not always true if $a = 1$.

There are two different ways in which $\phi_1(t)$ may fail to be of bounded variation over $(0, \infty)$. In the first place, the variation may be finite for any finite range of t, but unbounded as the range becomes infinitely great. Secondly, the breakdown may occur for small values of t, so that $\phi_1(t)$ is not of bounded variation over any range including the origin. We shall show that both cases do, in fact, occur.

Suppose, then, that

$$\phi(t) = \int_0^\infty \psi(u) \cos ut \, du,$$

where $\psi(u)$ is $L(0, \infty)$. Then

$$\phi_1(t) = \int_0^\infty \psi(u) \frac{\sin ut}{ut} \, du$$

and so, for $t > 0$,

$$\phi_1'(t) = \int_0^\infty \psi(u) \left\{ \frac{\cos ut}{t} - \frac{\sin ut}{ut^2} \right\} du,$$

the differentiation being justified by the uniform convergence of the last integral in any interval $t \geqslant \delta > 0$. Hence

$$\int_1^\infty |\phi_1'(t)| \, dt = \int_1^\infty dt \left| \int_0^\infty \psi(u) \left\{ \frac{\cos ut}{t} - \frac{\sin ut}{ut^2} \right\} du \right|.$$

We can now apply the lemma immediately, on noting that there is nothing in its proof which requires the interval (a, b) to be finite. If we put

$$f(w, t) = \frac{\cos wt}{w} - \frac{\sin wt}{w^2 t},$$

all the requirements of the lemma are satisfied throughout $(0, \infty)$ and the integral

$$\int_1^\infty |f(w, t)|\, dw$$

is infinite for all values of t. Hence the lemma shows that we can construct a function $\psi(u)$ of $L(0, \infty)$, in fact, that we can choose $\psi(u)$ to be zero outside any assigned range, such that the corresponding $\phi_1(t)$ is not of bounded variation over $(1, \infty)$.

Now consider the problem for small values of t. We must make

$$\int_0^1 |\phi_1'(t)|\, dt = \infty.$$

But

$$\int_0^1 |\phi_1'(t)|\, dt = \int_0^1 dt \left| \int_1^\infty \psi(u) \left\{ \frac{\cos ut}{t} - \frac{\sin ut}{ut^2} \right\} du \right|$$

$$= \int_1^\infty dw \left| \int_0^\infty \psi(u) \left\{ \frac{\cos (u/w)}{w} - \frac{\sin (u/w)}{u} \right\} du \right|.$$

Here, if $$f(w, u) = \frac{\cos (u/w)}{w} - \frac{\sin (u/w)}{u},$$

all the conditions of the lemma are satisfied, and a necessary and sufficient condition for the variation of $\phi_1(t)$ to be finite for all $\psi(u)$ of $L(0, \infty)$ is that $\int_1^\infty |f(w, u)|\, du$ is bounded.

$$J(u) = \int_1^\infty |f(w, u)|\, dw$$

$$= \int_1^\infty \left| \frac{\cos (u/w)}{w} - \frac{\sin (u/w)}{u} \right| dw$$

$$= \int_0^u \left| \frac{\cos t}{t} - \frac{\sin t}{t^2} \right| dt.$$

This tends to infinity as $u \to \infty$, and so there exists a function $\psi(u)$ of $L(0, \infty)$ such that the corresponding $\phi_1(t)$ is not of bounded variation over $(0, 1)$. Since $J(u)$ is bounded over any finite range of values of u, we cannot choose the function $\psi(u)$ to be zero outside a finite range; the lemma proves that for such a $\psi(u)$, $\phi_1(t)$ is of bounded variation over $(0, 1)$.

In connection with Theorem I there is a phenomenon similar to this last: this is the fact, already mentioned, that there exists a function tending monotonically to zero over $(0, \infty)$ whose Fourier cosine transform is not $L(0, 1)$. The existence of such a function is deducible from the lemma. For let

$$f(x) = \int_x^\infty g(t)\, dt,$$

where $g(t)$ is $L(0, \infty)$. Then

$$\int_0^\infty f(x) \cos xt\, dt = \left[f(x) \frac{\sin xt}{t} \right]_0^\infty + \frac{1}{t} \int_0^\infty g(x) \sin xt\, dx$$

$$= \frac{1}{t} \int_0^\infty g(x) \sin xt\, dx.$$

So
$$\int_0^1 dt \left| \int_0^\infty f(x) \cos xt\, dx \right| = \int_0^1 \frac{dt}{t} \left| \int_0^\infty g(x) \sin xt\, dx \right|.$$

By the method of the last example we can now deduce from the lemma that there exists a $g(t)$ such that the cosine transform of $f(x)$ is not $L(0, 1)$. By considering the positive and negative parts of $g(t)$ separately it is plain that we can take $f(x)$ to be non-increasing.

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ON ANALYTIC FUNCTIONS BOUNDED AT A DOUBLE SEQUENCE OF POINTS

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1. The problem of integral functions of order two, bounded at the lattice points $m+in$, $(m, n = 0, \pm 1, \pm 2, \ldots)$, has been considered by various writers. In particular, theorems have been proved by J. M. Whittaker*, G. Pólya, A. Pflüger, and V. G. Iyer. The following theorem was first proved by Iyer†, and independently a short time later by Pflüger‡.

THEOREM A. *If the integral function* $f(z)$ *satisfies the conditions*

$$\overline{\lim} \frac{\log M(r)}{r^2} < \tfrac{1}{2}\pi \quad [M(r) = \max |f(re^{i\phi})|],$$

and

$$|f(m+in)| \leqslant A \quad (m, n = 0, \pm 1, \pm 2, \ldots),$$

then $f(z)$ *is necessarily a constant.*

In this paper we consider an integral function of order two bounded at a double sequence of points $\{z_{m,n}\}$, such that

$$m-1 \leqslant \mathfrak{R}(z_{m,n}) \leqslant m,$$

$$n-1 \leqslant \mathfrak{I}(z_{m,n}) \leqslant n.$$

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* J. M. Whittaker, (**6**), (**7**), (**8**). G. Pólya, (**5**).

† V. G. Iyer, (**2**).

‡ A. Pflüger, (**3**).

We define the integral function $F(z)$ by the equation

$$F(z)=e^{-az^2}z\prod_{m=-\infty}^{\infty}\prod_{n=-\infty}^{\infty}{}'\left\{\left(1-\frac{z}{z_{m,n}}\right)\exp\left(\frac{z}{z_{m,n}}+\frac{z^2}{2z_{m,n}^2}\right)\right\},$$

where

$$2a=\lim_{R\to\infty}\sum_{|m+in|\leqslant R}{}'\frac{1}{z_{m,n}^2},$$

and the dashed notation denotes the omission of (0, 0) from the product. [We shall show later that the sum used to define a is convergent.] It is reasonable to suppose that $F(z)$ will resemble the Weierstrassian σ-function in its order and type, and Lemmas 1–6 are concerned with a comparison of the two functions.

Lemmas 5 and 6 are used to establish Lemma 8, which is an adaptation of Pólya's proof that an integral function of order two and minimum type, bounded at the lattice points, is a constant. Theorem 1 follows Iyer's proof that an integral function of order two and type less than $\frac{1}{2}\pi$, bounded at the lattice points, is of minimum type, and hence is a constant.

In Theorem 3 we use the function $F(z)$, and consider functions regular in an angle.

In Theorem 4 we consider an integral function of order two and type less than $\frac{1}{2}k\pi$, bounded at k double sequences of points.

2. *Notation.* We make use of the following notation:

$S_{m,n}$ denotes the square with upper right-hand corner $m+in$, and the curve C is said to cut $S_{m,n}$ if there is some point z of C such that

$$m-1\leqslant\mathfrak{R}(z)\leqslant m,$$

$$n-1\leqslant\mathfrak{I}(z)\leqslant n.$$

The double set of circles, $|z-z_{m,n}|=\rho>0$, are called the circles (A_ρ), and the set $|z-(m+in)|=\rho$, the circles (B_ρ). Moreover ρ is assumed to be less than $\frac{1}{2}$, so that no two members of (B_ρ) intersect.

In Lemmas 1 and 2, the variable is ζ, while z is considered to be a fixed complex number. We therefore, in these lemmas, speak of the ζ plane. $\mathcal{E}$ is the set of (m, n) such that $|m+in|\leqslant 2|z|$, and $\mathcal{G}$ is the set such that $|m+in|>2|z|$. We let

$$\zeta=\xi+i\eta,\quad z=x+iy=re^{i\theta}.$$

It is assumed throughout that $|z_{m,n}|>1$, for $m=0, +1$ and $n=0, +1$, and that $|z_{m,n}|>\frac{1}{2}|m+in|$, for all (m, n).

The notation $\sum'_{m,n}$, $\sum'_{m,n,p}$, $\prod'_{m,n}$, denotes always the omission of $(m, n) \equiv (0, 0)$ from the sum or product, and $\mathscr{E}'$ denotes $\mathscr{E}$ with $(0, 0)$ omitted.

Finally, $a_1, a_2, a_3, \ldots$ always denote numerical constants.

3. In this paragraph we investigate the order and type of $F(z)$ by comparing it with $\sigma(z)$. Lemmas 1 and 2 are concerned with a comparison of

$$\left\{\Pi_{\mathscr{E}}\left(1-\frac{z}{z_{m,n}}\right)\right\} \quad \text{and} \quad \left\{\Pi_{\mathscr{E}'}\left(1-\frac{z}{m+in}\right)\right\}.$$

In Lemma 4 we consider

$$\Pi_{\mathscr{Q}}\left\{\frac{\{1-(z/z_{m,n})\}\exp(z/z_{m,n})}{\{1-z/(m+in)\}\exp\{z/(m+in)\}}\right\}:$$

Lemmas 5 and 6 combine the inequalities of Lemmas 2 and 4, to give the results necessary in the ensuing lemmas and theorems.

3.1. LEMMA 1. *Suppose that* $f(\xi, \eta) = |1-(z/\zeta)|^2$, *and let* C_1 *and* C_2 *be the parts of*

$$\frac{\partial f}{\partial \xi} = 0 \quad \textit{and} \quad \frac{\partial f}{\partial \eta} = 0,$$

respectively, within the circle C_3, $|\zeta| = 2|z|$.

Then the number of unit squares cut by C_1, C_2, *and* C_3 *is less than* $a_1|z|$, *provided that* $|z| > 1$.

Proof. The points at which $\partial f/\partial \xi = 0$ lie on the curve

$$(3.1) \qquad x\xi^2+2y\xi\eta-x\eta^2-\xi(x^2+y^2) = 0.$$

This is a hyperbola and has at most one maximum or minimum of $\eta = \eta(\xi)$ on each branch, and at most one maximum or minimum of $\xi = \xi(\eta)$ on each branch. These points (four at most) divide C_1 into parts $C_{1,i}$ $(i = 1, 2, \ldots)$ not more than six in number, such that no line parallel to either the ξ axis or the η axis meets any $C_{1,i}$ more than once.

The length of any C_{1i} is less than or equal to the algebraic sum of its projections on $O\xi$, $O\eta$. For, if P_1, P_2 are two points on C_{1i}, the length $P_1 P_2$ is less than the algebraic sum of its projections on $O\xi$, $O\eta$, and the length of C_{1i} is the limit of a sum of chords such as $P_1 P_2$, when the length of the longest chord tends to zero. Thus the length of C_1 is less than or equal to $6 \times 8|z|$.

Now unit length of C_1 cannot cut more than six unit squares, so that C_1 cannot cut more than $6[48|z|]+6$ squares. Similarly C_2 cannot cut more than $6[48|z|]+6$ squares, and C_3 not more than $6[4\pi|z|]+6$ squares. Then, if $|z|>1$, the lemma is proved.

LEMMA 2. *If z is outside (A_ρ), (B_ρ), and $|z|>1$, we have*

$$\left(\frac{\rho}{2(1+r)}\right)^{a_2 r} < \left|\frac{\Pi_{\mathscr{E}}\{1-z/z_{m,n}\}}{z\Pi_{\mathscr{E}'}[1-\{z/(m+in)\}]}\right| < \left(\frac{2(1+r)}{\rho}\right)^{a_2 r}, \tag{3.2}$$

where $\mathscr{E}$ is the set of all (m, n) such that $|m+in|\leqslant 2|z|$, and $\mathscr{E}'$ is $\mathscr{E}$ with $(0, 0)$ omitted.

Proof. The following inequalities hold for all $z_{m,n}$ of $\mathscr{E}$ and all $(m+in)$ of $\mathscr{E}'$:

$$\left|1-\frac{z}{z_{m,n}}\right| \geqslant \frac{\rho}{2(1+r)}, \quad \left|1-\frac{z}{m+in}\right| \geqslant \frac{\rho}{2(1+r)}, \tag{3.3}$$

and

$$\left|1-\frac{z}{z_{m,n}}\right| \leqslant 1+r, \quad \left|1-\frac{z}{m+in}\right| \leqslant 1+r. \tag{3.4}$$

Let $\mathscr{E}_1$ be the sub-set of $\mathscr{E}$ corresponding to squares cut by C_1, C_2, or C_3. We see from (3.1) that $\mathscr{E}_1$ includes the squares $\{m=0, +1;\ n=0, +1\}$. The number of members of $\mathscr{E}_1$ is less than $a_1 r$. Let $\mathscr{E}_2 \equiv \mathscr{E}-\mathscr{E}_1$.

Let $z'_{m,n}$ be that vertex of $S_{m,n}$ at which $f(\xi, \eta)$ is greatest, and $z''_{m,n}$ that at which it is least. If (m, n) belongs to $\mathscr{E}_2$,

$$\left|1-\frac{z}{z''_{m,n}}\right| \leqslant \left|1-\frac{z}{z_{m,n}}\right| \leqslant \left|1-\frac{z}{z'_{m,n}}\right|, \tag{3.5}$$

where $z'_{m,n}$ and $z''_{m,n} \neq 0$. Now, if $z'_{m,n} \equiv \{s(m, n)+it(m, n)\}$, where s and t are integers, let $\mathscr{E}_3$ be the set of (s, t) corresponding to (m, n) of $\mathscr{E}_2$. Since $\partial f/\partial\xi$, $\partial f/\partial\eta$ are both of constant sign throughout blocks of adjacent squares of $\mathscr{E}_2$, and since such blocks are separated by squares of $\mathscr{E}_1$, the set $\mathscr{E}_3$ does not overlap itself; and, if (s, t) is a member of $\mathscr{E}-\mathscr{E}_3$, then $s+it$ is a vertex of some square of $\mathscr{E}_1$. The number of members of $\mathscr{E}-\mathscr{E}_3$ is therefore at most $4a_1|z|$.

Now we have

$$\left|\Pi_{\mathscr{E}_2}\left(1-\frac{z}{z_{m,n}}\right)\right| < \left|\Pi_{\mathscr{E}_3}\left(1-\frac{z}{m+in}\right)\right|,$$

where $\mathcal{E}_3$ does not include (0, 0). Hence

$$(3.6)\qquad \left|\frac{\Pi_{\mathcal{E}}\{1-(z/z_{m,n})\}}{z\Pi_{\mathcal{E}'}[1-\{z/(m+in)\}]}\right| < \left|\frac{\Pi_{\mathcal{E}_1}\{1-(z/z_{m,n})\}}{z\Pi_{\mathcal{E}'-\mathcal{E}_3}[1-\{z/(m+in)\}]}\right|$$

$$< \frac{(1+r)^{a_1 r}}{r}\left(\frac{2(1+r)}{\rho}\right)^{4a_1 r}:$$

for $\mathcal{E}_1$ has at most $a_1 r$ members, and $\mathcal{E}'-\mathcal{E}_3$ has at most $4a_1 r$ members.

A similar proof shows that, if $z''_{m,n} \equiv \{u(m,n)+i\,v(m,n)\}$, and $\mathcal{E}_4$ is the set of (u, v) corresponding to the (m, n) of $\mathcal{E}_2$,

$$(3.7)\qquad \left|\frac{\Pi_{\mathcal{E}}\{1-(z/z_{m,n})\}}{z\Pi_{\mathcal{E}'}[1-\{z/(m+in)\}]}\right| > \left|\frac{\Pi_{\mathcal{E}_1}\{1-(z/z_{m,n})\}}{z\Pi_{\mathcal{E}'-\mathcal{E}_4}[1-\{z/(m+in)\}]}\right|$$

$$> \frac{1}{r}\left(\frac{\rho}{2(1+r)}\right)^{a_1 r}\left(\frac{1}{1+r}\right)^{4a_1 r}.$$

Hence, if $a_2 = 5a_1+1$, the lemma is proved.

3.2. It is useful to have the following information set out as a lemma.

LEMMA 3.

(*a*)
$$\sum_{n=-\infty}^{\infty}\ \sum_{m=-\infty}^{\infty}{}' \frac{1}{|m+in|^3} = O(1).$$

(*b*)
$$\sum_{|m+in|>R} \frac{1}{|m+in|^3} = O\left(\frac{1}{R}\right).$$

(*c*)
$$\sum_{|m+in|\leqslant R}{}' \frac{1}{|m+in|^2} = O(\log R).$$

Proof. (*a*) The proof of (*a*) is given by Whittaker and Watson, *Modern analysis* (1927), 3.4.

(*b*) The sum over $|m+in| > R$ is less than the corresponding sum over all $m+in$ outside the square with vertices $(\pm\frac{1}{2}R\sqrt{2}, \pm\frac{1}{2}R\sqrt{2})$. Thus we have

$$\tfrac{1}{4}\sum_{|m+in|>R}\frac{1}{|m+in|^3} < \sum_{m=[\frac{1}{2}R\sqrt{2}]+1}^{\infty}\left(\frac{1}{2\sqrt{2}\,.\,m^3}\right)+2\sum_{m=[\frac{1}{2}R\sqrt{2}]+1}^{\infty}\ \sum_{n=0}^{m-1}\frac{1}{(m^2+n^2)^{\frac{3}{2}}}$$

$$< \frac{1}{2\sqrt{2}}\int_{[\frac{1}{2}R\sqrt{2}]}^{\infty}\frac{1}{t^3}\,dt + 2\int_{[\frac{1}{2}R\sqrt{2}]}^{\infty}\frac{1}{t^2}\,dt.$$

Therefore

$$\sum_{|m+in|>R} \frac{1}{|m+in|^3} = O\left(\frac{1}{R}\right).$$

(*c*) The sum over $|m+in| \leqslant R$ is less than the sum over those $m+in$ lying within or on the square $(\pm R, \pm R)$. Thus

$$\begin{aligned}\sum_{|m+in|\leqslant R}' \frac{1}{|m+in|^2} &< 4 \sum_{m=1}^{[R]} \frac{1}{2m^2} + 8 \sum_{m=1}^{[R]} \cdot \sum_{n=0}^{(m-1)} \left(\frac{1}{m^2+n^2}\right) \\ &< 2\left(1+\int_1^R \frac{1}{t^2}\,dt\right) + 8 \sum_{m=1}^{[R]} \frac{1}{m} \\ &< 4+8\left(1+\int_1^R \frac{1}{t}\,dt\right) \\ &< 12+8\log R.\end{aligned}$$

Therefore $$\sum_{|m+in|\leqslant R}' \frac{1}{|m+in|^2} = O(\log R).$$

3.3. LEMMA 4. *If z is greater than some constant r_1, there exists a constant a_3 such that*

$$e^{-a_3|z|} < \left| \Pi_{\mathcal{Q}} \left\{ \frac{\{1-(z/z_{m,n})\}\, e^{z/z_{m,n}}}{[1-\{z/(m+in)\}]\, e^{z/(m+in)}} \right\} \right| < e^{a_3|z|}, \tag{3.8}$$

where $\mathcal{Q}$ is the set of all (m, n) such that $|m+in| > 2|z|$.

Proof. We have

$$\begin{aligned}&\left| \log \left\{ \frac{\{(1-(z/z_{m,n})\} \exp(z/z_{m,n})}{[1-\{z/(m+in)\} \exp\{z/(m+in)\}]} \right\} \right| \\ &\qquad = \left| \frac{z^2}{2} \left(\frac{1}{z^2_{m,n}} - \frac{1}{(m+in)^2}\right) + \frac{z^3}{3} \left(\frac{1}{z^3_{m,n}} - \frac{1}{(m+in)^3}\right) + \dots \right| \\ &\qquad \leqslant \frac{r^2 \sqrt{2} \max(|z_{m,n}|, |m+in|)}{|z_{m,n}|^2 |m+in|^2} + \frac{r^3 \sqrt{2} \max(|z_{m,n}|^2, |m+in|^2)}{|z_{m,n}|^3 |m+in|^3} + \dots \\ &\qquad < \frac{12 r^2 \sqrt{2}}{|m+in|^3}.\end{aligned}$$

For, as we have already remarked, $|z_{m,n}| > \frac{1}{2}|m+in|$, and

$$|z_{m,n}-(m+in)| \leqslant \sqrt{2},$$

and, if r is large, it follows that $1-r/|z_{m,n}| > \frac{1}{3}$, since

$$1-r/|m+in| > \tfrac{1}{2}.$$

It now follows from Lemma 3(b) that there are constants a_3 and r_1 such that, if $r > r_1$, then

$$\left|\log\left\{\Pi_{\mathcal{G}} \frac{\{1-(z/z_{m,n})\} \exp(z/z_{m,n})}{[1-\{z/(m+in)\} \exp\{z/(m+in)\}]}\right\}\right| < a_3|z|.$$

This proves the lemma.

3.4. LEMMA 5. *If* z *is outside all* (A_ρ), *and* ϵ *is any small positive number, then*

$$\left|\frac{\log|F(z)|}{|z|^2} - \tfrac{1}{2}\pi\right| < \epsilon,$$

provided that $|z| > R_1(\epsilon)$.

Proof. We must first show that a exists. Since

$$\sum_{r_1 < |m+in| \leqslant r_2} \frac{1}{(m+in)^2} = 0,$$

we have

$$\left|\left(\sum_{|m+in| \leqslant r_2}{}' - \sum_{|m+in| \leqslant r_1}{}'\right)\left(\frac{1}{z_{m,n}^2}\right)\right| = \left|\sum_{r_1 < |m+in| \leqslant r_2} \left(\frac{1}{z_{m,n}^2} - \frac{1}{(m+in)^2}\right)\right|$$

$$< 8\sqrt{2} \sum_{|m+in| > r_1} \frac{1}{(m+in)^3},$$

as shown in the above lemma. From Lemma 3(b) we see that

$$\sum_{|m+in| \leqslant R}{}' \left(\frac{1}{z_{m,n}^2}\right)$$

tends to a limit as $R \to \infty$. We call this limit a.

Now we know that

$$\left|z\left(\frac{1}{z_{m,n}} - \frac{1}{(m+in)}\right)\right| < \frac{r\sqrt{2}}{|z_{m,n}||m+in|} < \frac{2\sqrt{2}r}{|m+in|^2}.$$

Thus, by Lemma 3(c), there exist constants a_4 and r_2 such that, if $r > r_2$,

$$e^{-a_4 r \log r} < \left|\Pi_{\mathcal{E}'} \left\{\frac{\exp(z/z_{m,n})}{\exp\{z/(m+in)\}}\right\}\right| < e^{a_4 r \log r}. \tag{3.9}$$

Also

$$(3.10)\qquad \left|\prod_{m,n}{}' \left\{\frac{\exp(z^2/z^2_{m,n})}{\exp\{z^2/(m+in)^2\}}\right\}\right| = e^{2\alpha|z|^2}.$$

From Lemmas 2 and 4, and equations (3.9), (3.10), it follows that, for any z outside (A_ρ) and (B_ρ),

$$\left|\log\left|\frac{F(z)}{\sigma(z)}\right|\right| < a_2 r \log\left(\frac{2(1+r)}{\rho}\right) + a_3 r + a_4 r \log r + \left\{\log r + \left|\log\left|1 - \frac{r}{|z_{0,0}|}\right|\right|\right\},$$

provided that $r > r_3$ (say). Hence, for any z outside (A_ρ) and (B_ρ),

$$(3.11)\qquad e^{(\frac{1}{2}\pi-\epsilon)|z|^2} < |F(z)| < e^{(\frac{1}{2}\pi+\epsilon)|z|^2},$$

if $|z| > R_0(\epsilon, \rho)$.

The equation (3.11) holds for any z outside $(A_{\frac{1}{4}\rho})$, and $(B_{\frac{1}{4}\rho})$, provided that $|z| > R_0(\epsilon, \frac{1}{4}\rho)$. We now observe that, if b is any circle of the set (B_ρ), it meets no other circle of (B_ρ), and at most four circles of (A_ρ). Let b' be the circle concentric with b, and of radius $\frac{1}{4}\rho$, and let R_b be the closed region consisting of all points of b not interior to any (A_ρ). Then R_b and circles of $(A_{\frac{1}{4}\rho})$ do not meet. If R_b and b' do not overlap, the equation (3.11) holds in R_b for $|z| > R_0(\epsilon, \frac{1}{4}\rho)$. If, on the other hand, R_b and b' have points in common, such points are distant at least $(\frac{3}{4}\rho)$ from any circle of $(A_{\frac{1}{4}\rho})$. Thus the closed circle b' does not meet any circle of $(A_{\frac{1}{4}\rho})$, and we can apply the maximum and minimum modulus principles to $F(z)$ in the region formed by R_b together with b'. If $|z| > \{R_0(\frac{1}{2}\epsilon, \frac{1}{4}\rho) + 2\rho\}$ we have, for any z in R_b,

$$e^{(\frac{1}{2}\pi-\frac{1}{2}\epsilon)(r-2\rho)^2} < |F(z)| < e^{(\frac{1}{2}\pi+\frac{1}{2}\epsilon)(r+2\rho)^2}.$$

Hence if $|z| > R_1(\epsilon, \rho)$, say, (3.11) holds for z in any region such as R_b, and the lemma is thus proved.

LEMMA 6. *Suppose that any two members of the double sequence* $\{z_{m,n}\}$ *are at a distance greater than* $\rho_1 > 0$ *apart. Then, if* ϵ *is any small positive number,*

$$(3.12)\qquad \left|\frac{\log|F'(z_{m,n})|}{|z_{m,n}|^2} - \tfrac{1}{2}\pi\right| < \epsilon,$$

provided that $|z_{m,n}| > R_2(\epsilon)$, *where* $R_2(\epsilon)$ *does not depend on the particular set* $\{z_{m,n}\}$.

Proof. Let $F_0(z)$ be the integral function defined by the equation

$$F_0(z) = \frac{F(z)}{z-z_{M,N}}.$$

Then $$F_0(z_{M,N}) = F'(z_{M,N}).$$

and

$$\left| \frac{\log|F_0(z)|}{|z|^2} - \tfrac{1}{2}\pi \right| < \left| \frac{\log|F(z)|}{|z|^2} - \tfrac{1}{2}\pi \right| + \left| \frac{\log|z_{M,N}-z|}{z^2} \right|.$$

Now suppose that $\rho < \frac{1}{2}\rho_1$, so that no two circles of (A_ρ) intersect. If z is any point on $|z-z_{M,N}| = \rho$, and if $|z_{M,N}| > R_1(\frac{1}{2}\epsilon)+\rho$, it follows from Lemma 5 that

$$\left| \frac{\log|F_0(z)|}{|z|^2} - \tfrac{1}{2}\pi \right| < \tfrac{1}{2}\epsilon + \frac{|\log\rho|}{|z|^2}.$$

Since $F_0(z)$ is regular and has no zeros within $|z-z_{M,N}| = \rho$, we have

$$\left| \frac{\log|F_0(z_{M,N})|}{|z_{M,N}|^2} - \tfrac{1}{2}\pi \right| < \epsilon,$$

if $|z_{M,N}| > R_2(\epsilon)$.

4. We are now in a position to prove that an integral function of order two and type less than $\frac{1}{2}\pi$, bounded at a double sequence of points $\{z_{m,n}\}$, is a constant. What follows is an adaptation of Iyer's method of proof.

4.1. LEMMA 7. *Suppose that an integral function is of order two and type less than $\frac{1}{2}\pi$. Then, if any two $z_{m,n}$ are distant at least ρ_1 apart,*

$$f(z) = F(z) \left\{ \sum_{m,n}{}' \frac{f(z_{m,n})}{(z-z_{m,n})\,F'(z_{m,n})} + \frac{f(0)}{z} \right\},$$

where the series on the right-hand side is absolutely convergent

Proof. Let $C_{R'}$ be a contour consisting of the parts of $|z| = R'$ outside all circles of $(A_{2\rho})$, and joined together by arcs of these circles. We assume $\rho < \frac{1}{4}\rho_1$. Then by Cauchy's theorem,

$$(4.1) \qquad \frac{1}{2\pi i} \int_{C_{R'}} \frac{f(\zeta)\,d\zeta}{(z-\zeta)\,F(\zeta)} = \frac{f(0)}{zF'(0)} + \sum_{m,n}{}' \frac{f(z_{m,n})}{(z-z_{m,n})\,F'(z_{m,n})} - \frac{f(z)}{F(z)},$$

where the summation on the right-hand side is over all $z_{m,n}$ within $C_{R'}$.

It follows from Lemma 5 and the hypothesis concerning $f(z)$ that the left-hand side of (4 . 1) tends to 0 as R' tends to infinity. Hence

$$f(z)=F(z)\left\{\frac{f(0)}{z}+\sum_{m,n}{}'\frac{f(z_{m,n})}{(z-z_{m,n})F'(z_{m,n})}\right\},$$

where the sum on the right-hand side is suitably bracketed. As it follows from Lemma 6 that the sum is absolutely convergent, the brackets may be removed.

LEMMA* 8. *Suppose that $f(z)$ is an integral function of order two and minimum type, bounded at a double sequence of points $\{z_{m,n}\}$. Then $f(z)$ is a constant.*

Proof. If k is any positive integer, let $f_k(z)\equiv f(zk)$, and

$$z_{k,m,n}=z_{km,kn}/k.$$

Then $z_{k,m,n}$ lies in, or on the boundary of, $S_{m,n}$, and

$$|(m+in)-z_{k,m,n}|<\frac{\sqrt{2}}{k}.$$

If k is large the assumptions in § 2 about $\{z_{m,n}\}$ are justified.

It follows from Lemma 7 that

$$f_k(z)=F_k(z)\left\{\sum_{m,n}{}'\left\{\frac{f_k(z_{k,m,n})}{(z-z_{k,m,n})F_k'(z_{k,m,n})}\right\}+\frac{f_k(0)}{z}\right\}, \tag{4.2}$$

where

$$F_k(z)=e^{-a_k z^2}\left\{z\prod_{m,n}{}'\left(1-\frac{z}{z_{k,m,n}}\right)\exp\left(\frac{z}{z_{k,m,n}}+\frac{z^2}{2z_{k,m,n}^2}\right)\right\}, \tag{4.3}$$

and

$$a_k=\tfrac{1}{2}\sum_{m,n}{}'\left(\frac{1}{z_{k,m,n}^2}-\frac{1}{(m+in)^2}\right). \tag{4.4}$$

By the method of Lemma 4 we can see at once that the convergence of the product on the right-hand side of (4 . 3) is uniform in k, and in z for

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* It should be noted that this lemma is an immediate consequence of Theorem 8 of J. M. Whittaker's *Interpolatory function theory*, (8). The above proof avoids introducing the "flat regions" of an integral function into the present paper.

$|z| < R$ (say). Also

$$|a_k| < \tfrac{1}{2} \sum_{m,n}' \left| \frac{1}{z^2_{k,m,n}} - \frac{1}{(m+in)^2} \right| < \frac{a}{k},$$

where a is a numerical constant. Combining these results we have

$$\lim_{k\to\infty} F_k'(z_{k,m,n}) = \sigma'(m+in), \tag{4.5}$$

and

$$\lim_{k\to\infty} \left(\frac{k}{z}\right) F_k\left(\frac{z}{k}\right) = 1. \tag{4.6}$$

Now write (4.2) in the form

$$f(z) = kF_k\left(\frac{z}{k}\right) \left\{ \sum_{m,n}' \frac{f(z_{mk,nk})}{(z-z_{mk,nk})\, F_k'(z_{mk,nk}/k)} + \frac{f(0)}{z} \right\}. \tag{4.7}$$

For $k > k_0$ (say) we have $|z - z_{mk,nk}| > 1$ for all (m, n), and since $f(z_{mk,nk})$ is bounded, it follows from Lemma 6 that the sum on the right-hand side converges uniformly for $k > k_0$. Now let $k \to \infty$, so that, using (4.5),

$$\sum_{m,n}' \left\{ \frac{f(z_{mk,nk})}{(z-z_{mk,nk})\, F_k'(z_{k,m,n})} \right\} \to 0.$$

Then
$$f(z) = \lim_{k\to\infty} kF_k\left(\frac{z}{k}\right) \left\{ \frac{f(0)}{z} \right\} = f(0),$$

and $f(z)$ is a constant.

4.2. We shall prove that an integral function of order two and type less than $\frac{1}{2}\pi$, bounded at a double sequence of points, must be of minimum type, provided that the points are distant at least ρ_1 apart.

It is first necessary to prove the following lemma.

LEMMA 9. *Suppose that the double sequence* $\{Z_{m,n}\}$ *is bounded for all* (m, n), *and that the integral function* $H(z)$ *is defined by the equation*

$$H(z) = F(z) \sum_{m,n}' \left\{ \frac{Z_{m,n}}{(z-z_{m,n})\, F'(z_{m,n})} \right\}, \tag{4.8}$$

where any two members of $\{z_{m,n}\}$ *are distant at least* ρ_1 *apart.*

Then $H(z)$ *is of order two and type less than or equal to* $\frac{1}{2}\pi$.

Proof. Let $C_{m,n}$ be the circle with centre $z_{m,n}$ and radius $|m+in|^{-3}$. From Lemma 3(a) it follows that for any sufficiently large R, and $a > 1$, there is an r_ν such that

$$R \leqslant r_\nu < aR,$$

and that $|z| = r_\nu$ does not intersect any $C_{m,n}$. From Lemma 6 we have

$$\left| \sum_{m,n}' \frac{Z_{m,n}}{(z-z_{m,n})\,F'(z_{m,n})} \right| < (\text{upper bound } |Z_{m,n}|) \sum' \frac{|m+in|^3}{|F'(z_{m,n})|}$$

$$< a_5,$$

provided that z is outside all $C_{m,n}$. Hence

$$M(R,H) \leqslant M(r_\nu, H) < a_5 M(r_\nu, F) \leqslant a_5 M(aR, F),$$

where
$$M(r,H) = \max_{0 \leqslant \theta < 2\pi} |H(re^{i\theta})|,$$

and
$$F(r,H) = \max_{0 \leqslant \theta < 2\pi} |F(re^{i\theta})|.$$

The order of $H(z)$ is therefore at most two, and the type is less than $\tfrac{1}{2}a^2(\pi+\epsilon)$, for any $a > 1$ and $\epsilon > 0$. That is to say, the type of $H(z)$ is less than or equal to $\tfrac{1}{2}\pi$.

THEOREM 1. *Suppose that an integral function $f(z)$ is of order two and type less than $\tfrac{1}{2}\pi$, and that it is bounded at a double sequence of points $\{z_{m,n}\}$, which satisfy the additional condition that any two members of the sequence are at distance greater than ρ_1 apart. Then $f(z)$ is a constant.*

Proof. We may assume without loss of generality that $f(0) = 0$, and that $|z_{m,n}| > 1$, $|z_{m,n}| > \tfrac{1}{2}|m+in|$ for all (m, n).

Since, for any finite $h \geqslant 0$, the integral function $\{f(z)z^h\}$ is of order two and type less than $\tfrac{1}{2}\pi$, from it follows Lemma 7 that

$$\frac{f(z)\,z^h}{F(z)} = \sum_{m,n}' \frac{f(z_{m,n})\,z_{m,n}^h}{F'(z_{m,n})(z-z_{m,n})}. \tag{4.9}$$

Now let $\psi(z)$ be an integral function of order two and type less than $\tfrac{1}{2}\pi$, and suppose that

$$\psi(z) \equiv a_0 + a_1 z + a_2 z^2 + \dots .$$

Then
$$\sum_{p=0}^{\infty} |a_p|\,|z_{m,n}|^p \leqslant \exp(\eta\,|z_{m,n}|^2),$$

for some $\eta < \frac{1}{2}\pi$, and large $|z_{m,n}|$. The triple series

$$\sum_{m,n,p} \left\{ \frac{f(z_{m,n})\, a_p\, z_{m,n}^p}{F'(z_{m,n})(z-z_{m,n})} \right\}$$

converges absolutely except for z equal to any $z_{m,n}$. Summing it in two different ways and using (4.9), we have

$$\frac{f(z)\,\psi(z)}{F(z)} = \sum_{m,n}{}' \frac{f(z_{m,n})\,\psi(z_{m,n})}{F'(z_{m,n})(z-z_{m,n})}. \tag{4.10}$$

Since $f(z)$ is of type less than $\frac{1}{2}\pi$, it follows by putting $\psi(z) \equiv \{f(z)\, z^h\}$, that

$$\frac{z^h [f(z)]^2}{F(z)} = \sum_{m,n}{}' \frac{[f(z_{m,n})]^2\, z_{m,n}^h}{F'(z_{m,n})(z-z_{m,n})}, \tag{4.11}$$

for $h = 0, 1, 2, \ldots$.

Then starting from (4.9), and repeating the argument that leads up to (4.11) from (4.9), we have

$$\frac{z^h [f(z)]^3}{F(z)} = \sum_{m,n}{}' \frac{[f(z_{m,n})]^3\, z_{m,n}^h}{F'(z_{m,n})(z-z_{m,n})},$$

for $h = 0, 1, 2, \ldots$. Ultimately, for any finite p,

$$[f(z)]^p = F(z) \left\{ \sum_{m,n}{}' \frac{[f(z_{m,n})]^p}{F'(z_{m,n})(z-z_{m,n})} \right\}. \tag{4.12}$$

Now suppose that the type of $f(z)$ is $k > 0$, and let p be the smallest positive integer such that $kp > \frac{1}{2}\pi$. Then the left-hand side of (4.12) is of type greater than $\frac{1}{2}\pi$, while from Lemma 9 we know that the right-hand side is of type less than or equal to $\frac{1}{2}\pi$. The assumption that $k > 0$ is therefore false. But, if $k = 0$, we know from Lemma 8 that $f(z)$ is a constant.

4.3. If the double sequence $\{z_{m,n}\}$ does not necessarily satisfy the condition that there exists some positive ρ_1, such that any two members of the sequence are distant at least ρ_1 apart, we have the following theorem.

THEOREM 2. *Suppose that $f(z)$ is an integral function, of order two and type less than $\frac{1}{8}\pi$, bounded at a double sequence of points $\{z_{m,n}\}$. Then $f(z)$ is a constant.*

Proof. The function $f(2z)$ is of order two and type less than $\frac{1}{2}\pi$, and is bounded at the double sequence of points $\{\frac{1}{2}z_{2m,2n}\}$. Any two points of this sequence are distant at least $\frac{1}{2}$ apart, so that the result follows.

Since members of the double sequence $\{z_{m,n}\}$ may actually coincide, the function $\sigma(\frac{1}{2}z)$, which is of order two and type $\frac{1}{8}\pi$, and vanishes at all points $2m+2ni$, suffices to show that Theorem 2, as regards the type of $f(z)$, is "best possible."

5. Theorems concerning functions regular in an angle, and bounded at lattice points, have been proved by J. M. Whittaker, Pflüger, Cartwright*, and others. Pflüger† has moreover proved the following theorem.

THEOREM B. *Suppose that*

(i) *$f(z)$ is regular and of exponential type in the angle*

$$|\arg z| \leqslant \beta < \tfrac{1}{2}\pi,$$

$$\text{(ii)} \qquad \varlimsup_{n \to \infty} \frac{1}{r} \log |f(re^{\pm i\beta})| \leqslant \pi B \sin\beta \quad (B > 0),$$

$$\text{(iii)} \qquad \varlimsup_{n \to \infty} \frac{1}{\rho_n} \log |f(a_n)| \leqslant 0,$$

for an unbounded increasing sequence $a_n = \rho_n e^{i\alpha_n}$, such that

$$\text{(iv)} \qquad \lim_{n \to \infty} \frac{n}{\rho_n} = D \geqslant \frac{B \sin\beta}{\sin(\beta - \alpha)},$$

$$\rho_{n+1} - \rho_n \geqslant g > 0, \quad |\alpha_n| \leqslant \alpha < \beta.$$

Then it follows that

$$\text{(v)} \qquad \varlimsup_{r \to \infty} \frac{1}{r} \log |f(r)| \leqslant \frac{\pi B \sin\beta \sin\alpha}{\sin(\beta - \alpha)}.$$

A consideration of Pflüger's theorem suggested Theorem 3, given below. Moreover the method of proof is analogous to Pflüger's proof of Theorem B.

THEOREM 3. *Suppose that the function $f(z)$ is regular in the angle $\delta \leqslant \arg z \leqslant \frac{1}{2}\pi - \delta$. Also suppose that*

$$\text{(i)} \qquad \varlimsup_{r \to \infty} \frac{\log\log M(r)}{\log r} = 2,$$

where $M(r) = \max |f(re^{i\theta})|$ for $\delta \leqslant \theta \leqslant \frac{1}{2}\pi - \delta$,

$$\text{(ii)} \qquad \varlimsup_{r \to \infty} \frac{1}{r^2} \log |f(re^{i\delta})| < \tfrac{1}{2}\pi,$$

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* J. M. Whittaker, (6), (7), (8). A. Pflüger, (3). M. L. Cartwright, (1).

† A. Pflüger, (4).

and
$$\overline{\lim_{r\to\infty}}\,\frac{1}{r^2}\log|f(re^{i(\frac{1}{2}\pi-\delta)})|<\tfrac{1}{2}\pi,$$

(iii)
$$\overline{\lim_{z_{m,n}\to\infty}}\,\frac{1}{|z_{m,n}|^2}\log|f(z_{m,n})|<\tfrac{1}{2}\pi,$$

where there is one $z_{m,n}$ *corresponding to every* $S_{m,n}$ *whose area within the angle is greater than* $\frac{1}{2}$, *and the distance of any* $z_{m,n}$ *from the boundary of the angle, or from any other* $z_{m,n}$, *is greater than some positive* ρ_1.

Then, for all θ *such that* $\delta\leqslant\theta\leqslant\frac{1}{2}\pi-\delta$,

$$\overline{\lim_{r\to\infty}}\,\frac{\log|f(re^{i\theta})|}{r^2}\leqslant\tfrac{1}{2}\pi.$$

The Phragmén-Lindelöf result is sufficient to show, from (i) and (ii), only that

$$\overline{\lim_{r\to\infty}}\,\frac{\log|f(re^{i\theta})|}{r^2}\leqslant\frac{\pi\sin 2\theta}{2\sin 2\delta}.$$

5.1. In the proof of Theorem 3 we use the integral function $D(z)$, defined by the equation

$$D(z)=z\prod_{m=1}^{\infty}\prod_{n=1}^{\infty}\left(1-\frac{z^4}{z_{m,n}^4}\right). \tag{5.1}$$

Then $D(z)$ is a particular function $F(z)$, except that $1+z/z_{1,1}$ occurs in the product, and this corresponds to $S_{0,0}$.

Proof of Theorem 3. We suppose that

$$\overline{\lim_{r_{m,n}\to\infty}}\,\frac{1}{r_{m,n}^2}\log|f(z_{m,n})|=\tfrac{1}{2}\pi-2\epsilon, \tag{5.2}$$

where $\epsilon>0$ ($r_{m,n}=|z_{m,n}|$); and we define $z'_{m,n}$ as $z_{m,n}$, when this exists, and otherwise as the centre of the area of $S_{m,n}$ outside, and distant at least ρ_1 from the boundary of, the angle. Then, if

$$D(z)=z\prod_{m=1}^{\infty}\prod_{n=1}^{\infty}\left(1-\frac{z^4}{z'^4_{m,n}}\right),$$

any two of the $z'_{m,n}$ are distant at least ρ_1 apart, where ρ_1 is supposed small. We choose some $\rho<\frac{1}{2}\rho_1$.

It follows from Lemmas 5 and 6 that, if z tends to infinity outside (A_ρ),

$$(5.3) \qquad \lim_{|z|\to\infty} \frac{\log|D(z)|}{|z|^2} = \tfrac{1}{2}\pi,$$

and also that

$$(5.4) \qquad \lim_{z_{m,n}\to\infty} \frac{\log|D'(z_{m,n})|}{|z_{m,n}|^2} = \tfrac{1}{2}\pi.$$

The function $\phi(z)$ is defined by the equation

$$(5.5) \qquad \phi(z) = D(z)\,\Sigma\left\{\frac{f(z_{m,n})}{(z-z_{m,n})\,D'(z_{n,m})}\right\},$$

where the summation is over all $z_{m,n}$ (but not all $z'_{m,n}$). From (5.2) and (5.4) it follows that the sum converges uniformly in z, for z outside (A_ρ). Therefore it represents a function regular, and bounded by a constant $k(\rho)$, outside (A_ρ).

The above is true however small ρ is, so that the only possible pole of $\phi(z)$ within a circle of (A_ρ) is at a $z_{m,n}$. But, at any $z_{m,n}$, the zero of $D(z)$ cancels with the $z-z_{m,n}$ of the denominator, so that $\phi(z)$ is regular in the whole angle $(\delta, \frac{1}{2}\pi-\delta)$. Moreover, outside all (A_ρ),

$$(5.6) \qquad |\phi(z)| < K(\rho)\,|D(z)| + e^{o(|z|)^2},$$

where $K(\rho)$ is a constant, and $o(|z|^2)/|z|^2 \to 0$ as $|z|\to\infty$. By the maximum modulus principle this holds throughout the whole angle.

We now define $\Phi(z)$ by the equation

$$(5.7) \qquad \Phi(z) = \frac{f(z)-\phi(z)}{D(z)}.$$

Then $\Phi(z)$ is regular in the angle $(\delta, \frac{1}{2}\pi-\delta)$: for numerator and denominator are regular, and the simple zeros of $D(z)$ at the points $z_{m,n}$ cancel the simple zeros of the numerator. Since $|D(z)|$ is bounded below outside (A_ρ), and since the maximum modulus principle can be applied to $\Phi(z)$ in all circles (A_ρ), we have

$$(5.8) \qquad \overline{\lim}\, \frac{\log\log M(r, \Phi)}{\log r} \leqslant 2,$$

where $\quad M(r, \Phi) = \max|\Phi(re^{i\theta})| \quad$ and $\quad \delta \leqslant \theta \leqslant \frac{1}{2}\pi-\delta.$

Hence

(a) $\Phi(z)$ is regular and satisfies (5.8) in the angle.

(b) $\Phi(z) < k_2$ (say) on the arms of the angle: for $\left|\bigl(f(z)/D(z)\bigr)\right|$ is bounded, by the hypothesis (ii) and (5.3), and $|\phi(z)/D(z)|$ is bounded by (5.6).

From the Phragmén-Lindelöf theorem it follows that $|\Phi(z)| < k_2$ for any z such that $\delta \leqslant \arg z \leqslant \frac{1}{2}(\pi-\delta)$. Then

$$|f(z)| < k_2 |D(z)| + |\phi(z)| < (k_1+k_2)|D(z)| \quad \text{(say)},$$

and hence, if $\delta \leqslant \theta \leqslant \frac{1}{2}\pi - \delta$,

$$\overline{\lim} \frac{\log|f(re^{i\theta})|}{r^2} \leqslant \overline{\lim} \frac{\log|D(re^{i\theta})|}{r^2} = \tfrac{1}{2}\pi.$$

6. We return to integral functions, and obtain the following generalization of Theorem 1.

THEOREM 4. *Suppose that $f(z)$ is an integral function of order two and type less than $\frac{1}{2}k\pi$, where k is any positive integer; and suppose that $f(z)$ is bounded at k double sequences of points [that is the points $\{z^{(t)}_{m,n}\}$, $t = 1, 2, \ldots, k$] such that any two points are at a distance greater than some $\rho_1 > 0$ apart. Then $f(z)$ is a constant.*

The proof is sufficiently indicated by the statement that it depends on the function $F(z)$, defined by the equation

$$F(z) = z^{1-k} \prod_{t=1}^{k} F_{(t)}(z),$$

where $F_{(t)}(z)$ is the function defined as in § 1, corresponding to the double sequence $\{z^{(t)}_{m,n}\}$.

The following results are almost immediate deductions from Theorem 4.

COROLLARY 1. *Suppose that an integral function $f(z)$, of order two and type less than $\pi/(2pq)$, where p, q are any integers, is bounded at the points $mp+inq$, for all integral values of m, n. Then $f(z)$ is a constant.*

COROLLARY 2. *Suppose that an integral function of order two and type less than $\pi/(2k_1 k_2 \sin\omega)$ is bounded at the points $(k_1 m + k_2 n\cos\omega,\ k_2 n \sin\omega)$, where k_1, k_2 are any real numbers, and ω is an angle such that $0 < \omega \leqslant \frac{1}{2}\pi$, for all integral values of m, n. Then the function is a constant.*

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PARSEVAL'S THEOREM FOR HANKEL TRANSFORMS

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The Hankel repeated integral

$$f(x) = \int_0^\infty J_\nu(xy)\, y\, dy \int_0^\infty J_\nu(yt)\, t f(t)\, dt \tag{1.1}$$

gives rise to the reciprocity formulae

$$f(x) = \int_0^\infty \sqrt{(xy)}\, J_\nu(xy)\, F(y)\, dy, \tag{1.11}$$

$$F(x) = \int_0^\infty \sqrt{(xy)}\, J_\nu(xy) f(y)\, dy \tag{1.12}$$

connecting two functions f and F, and each of the two functions so connected is said to be the *Hankel transform* of the other and we write $f = F^*$, or $F = f^*$.

Parseval's theorem for Hankel transforms can be stated in the following form: If F and f are two functions satisfying the relations (1.11) and (1.12), and G and g are similarly related, so that $F = f^*$ and $G = g^*$, then

$$(1.2) \qquad J = \int_0^\infty F(x)\, G(x)\, dx = \int_0^\infty f(x)\, g(x)\, dx = j. \qquad \text{(P)}$$

The object of this paper is to attempt a systematic discussion of the formula (P) on the lines set out by Hardy and Titchmarsh[†] in their treatment of the Fourier cosine case.

In Theorems I–V, I give the extensions corresponding to Theorems 1, 2, 4, 5, 6 of their paper. Theorems I and II follow from a theorem proved by Titchmarsh, while Theorem III is immediate. Theorem IV is a consequence of theorems which I have proved elsewhere, while Theorem V requires independent, and somewhat lengthy, treatment. It is assumed throughout that $\nu \geqslant -\frac{1}{2}$.

[†] See Hardy and Titchmarsh, **1**.

2. THEOREM I. *If f and g are L^2, then F and G exist as M_2 integrals for almost all x, and* (P) *is true, both integrals being L integrals.*

3. THEOREM II. *If f and G are L^2, then F and g exist as M_2 integrals for almost all x, and* (P) *is true, both integrals being L integrals.*

Both these theorems are a consequence of a theorem proved by Titchmarsh†.

4. THEOREM III. *If f and G are L, then* (P) *is true, both integrals being L integrals.*

In this case, F and g, defined as the transforms of f and G, exist as L integrals for all x and they are continuous, so that we get

$$\begin{aligned} (4.1) \qquad \int_0^\infty F(x)\,G(x)\,dx &= \int_0^\infty G(x)\,dx \int_0^\infty f(t)\,\sqrt{(xt)}\,J_\nu(xt)\,dt \\ &= \int_0^\infty f(t)\,dt \int_0^\infty \sqrt{(xt)}\,J_\nu(xt)\,G(x)\,dx \\ &= \int_0^\infty f(t)\,g(t)\,dt, \end{aligned}$$

the inversion of the repeated integral being justified on account of its absolute convergence.

5. THEOREM IV. (i) *If f is L and g is L in $(0, c)$ and g/x^2 is L in (c, ∞), where $c > 0$, and if G exists, for all x, as a Cauchy integral at ∞, and is L, then* (P) *is true, the integrals being L integrals.*

(ii) *If $g(x) \to 0$, as $x \to \infty$, the conditions on G may be broken for a finite number of values of x.*

(iii) *In particular* (P) *is true whenever f, g, and G are all L.*

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† See Titchmarsh, 4, 473.

For, by Theorem III, we have only to prove that $g = G^*$ and this follows from theorems which I have proved previously[†].

(i) follows from Theorem II, (ii) from Theorems II and IV, and (iii) from Theorem I. In case (i) we notice that the conditions of Theorems II of the paper referred to are satisfied since the condition G is L is enough to ensure that $x^{\nu+\frac{1}{2}} G(x)$ is L in every finite interval.

6. THEOREM V. *If f is L, g is of bounded variation in* $(0, \infty)$, *and* $g(u) \to 0$ *when* $u \to \infty$, *then* (P) *is true, J being a Cauchy integral at* 0 *and* ∞, *and j an L integral.*

If we substitute for F and G their expressions as Hankel transforms of f and g, we obtain formally

$$\int_\delta^X F(x)G(x)\,dx = \int_\delta^X G(x)\,dx \int_0^\infty f(t)\sqrt{(xt)}\,J_\nu(xt)\,dt$$

$$= \int_0^\infty \sqrt{t}f(t)\,dt \int_\delta^X G(x)\sqrt{x}\,J_\nu(xt)\,dx$$

$$= \int_0^\infty \sqrt{t}f(t)\,dt \int_\delta^X \sqrt{x}\,J_\nu(xt)\,dx \int_0^\infty g(u)\sqrt{(xu)}\,J_\nu(xu)\,du \tag{6.1}$$

$$= \int_0^\infty \sqrt{t}f(t)\,dt \int_0^\infty \sqrt{u}\,g(u)\,du \int_\delta^X xJ_\nu(xt)\,J_\nu(xu)\,dx$$

$$= \int_0^\infty \sqrt{t}f(t)\,dt \int_0^\infty \frac{\sqrt{u}\,g(u)\,du}{t^2-u^2} \times \Big[xt\,J_{\nu+1}(xt)\,J_\nu(xu) - xu\,J_\nu(xt)\,J_{\nu+1}(xu)\Big]_\delta^X \tag{6.2}$$

$$= \int_0^\infty f(t)\{\phi(t, X) - \phi(t, \delta)\}\,dt$$

$$\to \int_0^\infty f(t)\,g(t)\,dt, \tag{6.3}$$

as $X \to \infty$ and $\delta \to 0$, where

$$\phi(t, x) = \sqrt{t}\int_0^\infty \frac{\sqrt{u}\,g(u)\,du}{t^2-u^2}\{xt\,J_{\nu+1}(xt)\,J_\nu(xu) - xu\,J_\nu(xt)\,J_{\nu+1}(xu)\}; \tag{6.4}$$

[†] See Macaulay-Owen, **3**, Theorems I, II, IV.

and in order to prove the theorem, we have to justify the two inversions (6.1) and (6.2) and the passage to the limit (6.3).

(6.1) is justified by the absolute convergence of (1.12) for all x, $\sqrt{(xt)}\, J_\nu(xt)$ being bounded for all x and t.

To justify (6.2) we have to prove that

$$\int_0^\infty g(u) \sqrt{(xu)}\, J_\nu(xu)\, du$$

is uniformly convergent in the range

$$\delta \leqslant x \leqslant X.$$

Now, g being of bounded variation, we may write

$$g = \psi_1 - \psi_2, \tag{6.51}$$

where ψ_1 and ψ_2 are positive, decreasing, monotonic functions. If, as $u \to \infty$, $\psi_1(u) \to A$, then also will $\psi_2(u) \to A$, since $g(u) \to 0$, so that, if we write

$$\psi_1 - A = \chi_1, \tag{6.52}$$

$$\psi_2 - A = \chi_2, \tag{6.53}$$

we get

$$\int_U^{U'} g(u) \sqrt{(xu)}\, J_\nu(xu)\, du \tag{6.6}$$
$$= \int_U^{U'} \chi_1(u) \sqrt{(xu)}\, J_\nu(xu)\, du - \int_U^{U'} \chi_2(u) \sqrt{(xu)}\, J_\nu(xu)\, du,$$

where χ_1 and χ_2 are positive functions, decreasing monotonically to zero.

By the second mean value theorem, we can write

$$\int_U^{U'} \chi_1(u) \sqrt{(xu)}\, J_\nu(xu)\, du = \chi_1(U) \int_U^{U''} \sqrt{(xu)}\, J_\nu(xu)\, du, \tag{6.61}$$

where

$$U \leqslant U'' \leqslant U',$$

provided that $\int_U^{U'} \sqrt{(xu)}\, J_\nu(xu)\, du$ exists.

Now

$$\int_U^{U'} \sqrt{(xu)}\, J_\nu(xu)\, du \tag{6.62}$$
$$= \sqrt{\left(\frac{2}{\pi}\right)} \int_U^{U'} \left\{\cos(xu-\omega) - \frac{4\nu^2-1}{8}\, \frac{\sin(xu-\omega)}{xu} + O\left(\frac{1}{u^2}\right)\right\} du,$$

if we take $U > U_0$, where U_0 is independent of x, and

$$(6.63) \qquad \omega = \tfrac{1}{2}\nu\pi + \tfrac{1}{4}\pi,$$

so that the given integral exists and is uniformly bounded for $U' > U > U_0$. Similarly we can write

$$(6.64) \qquad \int_U^{U'} \chi_2(u)\sqrt{(xu)}\,J_\nu(xu)\,du = \chi_2(U)\int_U^{U'''} \sqrt{(xu)}\,J_\nu(xu)\,du,$$

where $$U \leqslant U''' \leqslant U'.$$

We thus obtain, when $U' > U > U_0$,

$$(6.65) \qquad \left|\int_U^{U'} g(u)\sqrt{(xu)}\,J_\nu(xu)\,du\right| \leqslant H\{\chi_1(U)+\chi_2(U)\},$$

H being an absolute constant.

Also, since χ_1 and χ_2 both tend to zero, we can make

$$\chi_1 < \frac{\epsilon}{2H}, \qquad \chi_2 < \frac{\epsilon}{2H}$$

by taking $U > U_1(\epsilon)$. We thus get

$$(6.66) \qquad \left|\int_U^{U'} g(u)\sqrt{(xu)}\,J_\nu(xu)\,du\right| < \epsilon,$$

where $U' > U > U_2$, U_2 being independent of x; this establishes the uniform convergence, and consequently (6.2).

To justify (6.3) we appeal to Lebesgue's convergence theorem†. Here the first condition of the theorem is satisfied since f is L. To show that the second and third conditions are also satisfied, we have to prove that $\phi(t, X)$ and $\phi(t, \delta)$ are bounded for all t, X, and δ; and that they tend to $g(t)$ and 0 respectively, when $X \to \infty$ and $\delta \to 0$, for almost all values of t. In order to do this we shall require the following lemma:

7. LEMMA. *The integral*

$$\int_U^{U'} \frac{\sqrt{(tu)}\,du}{t^2-u^2}\{atJ_{\nu+1}(at)\,J_\nu(au) - au\,J_\nu(at)\,J_{\nu+1}(au)\}$$

is uniformly bounded for all a, t, U, $U' > 0$.

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† See, *e.g.*, Hobson, 2, 289–290.

The substitutions $at = T$, $au = U$ reduce the lemma to its special case $a = 1$.

Suppose first that $t < 1$ and write

$$(7.1)\quad \int_U^{U'} \frac{\sqrt{(tu)}\,du}{t^2-u^2}\{tJ_{\nu+1}(t)\,J_\nu(u)-uJ_\nu(t)\,J_{\nu+1}(u)\}$$

$$= \int_U^{\frac{1}{2}t}+\int_{\frac{1}{2}t}^{2t}+\int_{2t}^{U'} = K_1+K_2+K_3.$$

Now

$$(7.11)\qquad K_1 = \int_U^{\frac{1}{2}t} \frac{\sqrt{(tu)}\,du}{t^2-u^2}\{tJ_{\nu+1}(t)\,J_\nu(u)-uJ_\nu(t)\,J_{\nu+1}(u)\}$$

$$= J_1-J_2,$$

and, since $\sqrt{t}\,J_{\nu+1}(t)$ is bounded for all t and $u/(t^2-u^2)$ is $O(1/t)$ in the range $U \leqslant u \leqslant \frac{1}{2}t$, we may write

$$(7.12)\qquad J_1 = O(t)\int_U^{\frac{1}{2}t} \frac{\sqrt{u}\,J_\nu(u)}{t^2-u^2}\,du$$

$$= O(t)\,O\int_U^{\frac{1}{2}t} \frac{u^{\nu-\frac{1}{2}}}{t}\,du = O(1).$$

In the same way, since $\sqrt{t}\,J_\nu(t)$ is bounded for all t and $u^2/(t^2-u^2)$ is bounded and increases monotonically in the range $U \leqslant u \leqslant \frac{1}{2}t$, we may write

$$(7.13)\qquad J_2 = O(1)\int_U^{\frac{1}{2}t} \frac{u^{\frac{3}{2}}\,J_{\nu+1}(u)}{t^2-u^2}\,du$$

$$= O(1)\int_{U''}^{\frac{1}{2}t} \frac{J_{\nu+1}(u)}{\sqrt{u}}\,du,$$

where $U \leqslant U'' \leqslant \frac{1}{2}t$, by the second mean value theorem.

This gives

$$(7.14)\qquad J_2 = O(1),$$

since

$$\int_0^\infty \frac{J_{\nu+1}(u)}{\sqrt{u}}\,du$$

is bounded.

We thus get

$$(7.15)\qquad K_1 = O(1).$$

To deal with K_2, we write it in the form

$$(7.2)\quad K_2 = \int_{\frac{1}{2}t}^{2t} \frac{\sqrt{(tu)}\,du}{t^2-u^2} \{tJ_{\nu+1}(t)\,J_\nu(u) - uJ_\nu(t)\,J_{\nu+1}(u)\}$$

$$= \int_{\frac{1}{2}t}^{2t} \frac{\sqrt{(tu)}\,du}{t+u} \{J_\nu(t)\,J_{\nu+1}(\tau) + \tau J_\nu(t)\,J'_{\nu+1}(\tau) - tJ_{\nu+1}(t)\,J'(\tau)\},$$

where $$\tfrac{1}{2}t \leqslant \tau \leqslant 2t.$$

This is plainly

$$(7.21)\qquad O(t)\,O\{t^{\nu+\nu+1} + t^{1+\nu+\nu} + t^{\nu+2+\nu-1}\} = O(t^{2\nu+2}) = O(1).$$

We are left with K_3, which we write

$$(7.3)\quad K_3 = \int_{2t}^{U'} \frac{\sqrt{(tu)}\,du}{u^2-t^2} \{uJ_\nu(t)\,J_{\nu+1}(u) - tJ_{\nu+1}(t)\,J_\nu(u)\} = J_3 - J_4.$$

Now

$$(7.31)\qquad J_3 = \sqrt{t}\,J_\nu(t) \int_{2t}^{U'} \frac{u^{\frac{3}{2}}\,J_{\nu+1}(u)}{u^2-t^2}\,du$$

$$= O(1) \int_{2t}^{U''} \frac{J_{\nu+1}(u)}{\sqrt{u}}\,du = O(1),$$

where $2t \leqslant U'' \leqslant U'$, since $u^2/(u^2-t^2)$ is bounded and decreases monotonically in the range $2t \leqslant u \leqslant U'$.

Also

$$(7.32)\qquad J_4 = t^{\frac{3}{2}}\,J_{\nu+1}(t) \int_{2t}^{U'} \frac{\sqrt{u}\,J_\nu(u)}{u^2-t^2}\,du$$

$$= O(t)\,\frac{1}{3t} \int_{2t}^{U''} \frac{\sqrt{u}\,J_\nu(u)}{u-t}\,du,$$

where $2t \leqslant U'' \leqslant U'$, since $1/(u+t)$ decreases monotonically.

Also $u/(u-t)$ is bounded and decreases monotonically in the range $2t \leqslant u \leqslant U'$, so that we get

$$(7.33)\qquad J_4 = O(1) \int_{2t}^{U'''} \frac{J_\nu(u)}{\sqrt{u}}\,du = O(1),$$

where $2t \leqslant U''' \leqslant U''$.

This gives

$$(7.34)\qquad K_3 = O(1),$$

and the lemma is proved for the case $t < 1$.

Consider now $t \geqslant 1$. In this case we write

$$\int_U^{U'} = \int_U^{t-\frac{1}{2}} + \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} + \int_{t+\frac{1}{2}}^{U'} = K_4 + K_5 + K_6. \tag{7.4}$$

We write

$$K_4 = \int_U^{t-\frac{1}{2}} \frac{\sqrt{(tu)}\,du}{t^2-u^2} \{tJ_{\nu+1}(t)\,J_\nu(u) - uJ_\nu(t)\,J_{\nu+1}(u)\} \tag{7.41}$$

$$= J_5 - J_6.$$

Now

$$J_5 = t^{\frac{3}{2}} J_{\nu+1}(t) \int_U^{t-\frac{1}{2}} \frac{\sqrt{u}\,J_\nu(u)}{t^2-u^2}\,du \tag{7.42}$$

$$= O(t) \int_U^{t-\frac{1}{2}} \frac{\sqrt{u}\,J_\nu(u)}{t^2-u^2}\,du = O(1) \int_U^{U''} \frac{\sqrt{u}\,J_\nu(u)}{t-u}\,du,$$

where $U \leqslant U'' \leqslant t-\frac{1}{2}$, since $1/(t+u)$ decreases monotonically and is $O(1/t)$ in the range $U \leqslant u \leqslant t-\frac{1}{2}$.

Also, since $1/(t-u)$ is bounded and increases monotonically in the range $U \leqslant u \leqslant U''$ and $\int_0^\infty \sqrt{u}\,J_\nu(u)\,du$ is bounded, we obtain

$$J_5 = O(1) \int_{U'''}^{U''} \sqrt{u}\,J_\nu(u)\,du = O(1), \tag{7.43}$$

where $U \leqslant U''' \leqslant U''$.

$$J_6 = \sqrt{t}\,J_\nu(t) \int_U^{t-\frac{1}{2}} \frac{u^{\frac{3}{2}}\,J_{\nu+1}(u)}{t^2-u^2}\,du \tag{7.44}$$

$$= O(1) \int_{U''}^{t-\frac{1}{2}} \frac{\sqrt{u}\,J_{\nu+1}(u)}{t-u}\,du,$$

where $U \leqslant U'' \leqslant t-\frac{1}{2}$, since $u/(t+u)$ is bounded and increases monotonically in the range $U \leqslant u \leqslant t-\frac{1}{2}$.

We thus get, as in the case of J_5,

$$J_6 = O(1) \tag{7.45}$$

and consequently

$$K_4 = O(1). \tag{7.46}$$

We write

$$(7.5)\quad K_5=\int_{t-\frac12}^{t+\frac12}\frac{\sqrt{(tu)}\,du}{t^2-u^2}\,\{tJ_{\nu+1}(t)\,J_\nu(u)-uJ_\nu(t)\,J_{\nu+1}(u)\}$$

$$=\int_{t-\frac12}^{t+\frac12}\frac{\sqrt{(tu)}\,du}{t+u}\,\{J_\nu(t)\,J_{\nu+1}(\tau)+\tau J_\nu(t)\,J'_{\nu+1}(\tau)-tJ_{\nu+1}(t)\,J_\nu{}'(\tau)\},$$

where $t-\frac12\leqslant\tau\leqslant t+\frac12$, and this is

$$(7.51)\qquad O(1)\left\{O\left(\frac{1}{t}\right)+O(1)+O(1)\right\}=O(1),$$

since $t\geqslant 1$ and $\tau\geqslant t-\frac12\geqslant\frac12$.

There remains K_6.

Now

$$(7.6)\quad K_6=\int_{t+\frac12}^{U'}\frac{\sqrt{(tu)}\,du}{u^2-t^2}\,\{uJ_\nu(t)\,J_{\nu+1}(u)-tJ_{\nu+1}(t)\,J_\nu(u)\}=J_7-J_8.$$

$$(7.61)\qquad J_7=\sqrt{t}\,J_\nu(t)\int_{t+\frac12}^{U'}\frac{u^{\frac32}\,J_{\nu+1}(u)}{u^2-t^2}\,du$$

$$=O(1)\int_{U''}^{U'}\frac{\sqrt{u}\,J_{\nu+1}(u)}{u-t}\,du,$$

where $t+\frac12\leqslant U''\leqslant U'$, since $u/(u+t)$ is bounded and increases monotonically in the range $t+\frac12\leqslant u\leqslant U'$.

Also, since $1/(u-t)$ is bounded and decreases monotonically in the range $U''\leqslant u\leqslant U'$, we obtain

$$(7.62)\qquad J_7=O(1)\int_{U''}^{U'''}\sqrt{u}\,J_{\nu+1}(u)\,du=O(1).$$

$$(7.63)\qquad J_8=t^{\frac32}\,J_{\nu+1}(t)\int_{t+\frac12}^{U'}\frac{\sqrt{u}\,J_\nu(u)}{u^2-t^2}\,du$$

$$=O(t)\int_{t+\frac12}^{U'}\frac{\sqrt{u}\,J_\nu(u)}{u^2-t^2}\,du$$

$$=O(1)\int_{t+\frac12}^{U''}\frac{\sqrt{u}\,J_\nu(u)}{u-t}\,du,$$

where $t+\frac12\leqslant U''\leqslant U'$, since $1/(u+t)$ decreases monotonically and is $O(1/t)$ in the range $t+\frac12\leqslant U\leqslant U'$.

We thus get, as in the case of J_7,

$$J_8 = O(1), \tag{7.64}$$

and consequently

$$K_6 = O(1). \tag{7.65}$$

Thus the lemma is completely established.

8. To prove that $\phi(t, X)$ and $\phi(t, \delta)$ are bounded for all t, X, and δ, we write

$$g = \psi_1 - \psi_2, \tag{8.1}$$

where ψ_1 and ψ_2 are positive, increasing, monotonic functions.

Then

$$\int_0^U \frac{\psi_1(u)\sqrt{(tu)}\,du}{t^2-u^2}\{atJ_{\nu+1}(at)\,J_\nu(au) - auJ_\nu(at)\,J_{\nu+1}(au)\} \tag{8.2}$$

$$= \psi_1(U)\int_{U'}^U \frac{\sqrt{(tu)}\,du}{t^2-u^2}\{atJ_{\nu+1}(at)\,J_\nu(au) - auJ_\nu(at)\,J_{\nu+1}(au)\},$$

where $0 < U' < U$, and where a is written in turn for X and δ.

The integral involving ψ_2 takes a similar form and we have, by Lemma (7)

$$|\phi(t, a)| < H\{\psi_1(U) + \psi_2(U)\}, \tag{8.21}$$

where H is an absolute constant, and, since ψ_1 and ψ_2 are bounded, we see that $\phi(t, a)$, and consequently $\phi(t, X)$ and $\phi(t, \delta)$, are bounded for all t, X, and δ.

9. To prove that $\phi(t, X) \to g(t)$ for almost all t, we write

$$\phi(t, X) = \sqrt{t}\int_0^\infty \frac{\sqrt{u}\,g(u)\,du}{t^2-u^2}\{XtJ_{\nu+1}(Xt)\,J_\nu(Xu) - Xu\,J_\nu(Xt)\,J_{\nu+1}(Xu)\} \tag{9.1}$$

$$= \int_0^{t-\eta} + \int_{t-\eta}^{t+\eta} + \int_{t+\eta}^\infty = K_1 + K_2 + K_3.$$

We prove that, as X tends to infinity, K_2 tends to $g(t)$ for almost all values of t, while K_1 and K_3 tend to zero. K_2 is dominated by the asymptotic behaviour of the Bessel functions, and we write

$$
\begin{aligned}
(9.2)\quad K_2 &= \int_{t-\eta}^{t+\eta} \frac{\sqrt{(tu)}\,g(u)\,du}{t^2-u^2}\,\{XtJ_{\nu+1}(Xt)\,J_\nu(Xu)-XuJ_\nu(Xt)\,J_{\nu+1}(Xu)\} \\
&= \int_{t-\eta}^{t+\eta} \frac{\sqrt{(tu)}\,g(u)\,du}{t^2-u^2}\left\{XJ_\nu(Xt)\frac{d}{dX}J_\nu(Xu)-XJ_\nu(Xu)\frac{d}{dX}J_\nu(Xt)\right\} \\
&= \int_{t-\eta}^{t+\eta} \frac{\sqrt{(tu)}\,g(u)\,du}{t^2-u^2}\Big\{\sqrt{X}J_\nu(Xt)\frac{d}{dX}\sqrt{X}J_\nu(Xu) \\
&\qquad\qquad -\sqrt{X}J_\nu(Xu)\frac{d}{dX}\sqrt{X}J_\nu(Xt)\Big\} \\
&= \int_{t-\eta}^{t+\eta} \frac{g(u)\,du}{t^2-u^2}\,\{u\sqrt{(Xt)}\,J_\nu(Xt)\,[\sqrt{(Xu)}\,J_\nu(Xu)]' \\
&\qquad\qquad -t\sqrt{(Xu)}\,J_\nu(Xu)\,[\sqrt{(Xt)}\,J_\nu(Xt)]'\} \\
&= \int_{t-\eta}^{t+\eta} \frac{g(u)\,du}{t^2-u^2}\,\frac{2}{\pi}\,\{t\cos(Xu-\omega)\sin(Xt-\omega) \\
&\qquad\qquad -u\cos(Xt-\omega)\sin(Xu-\omega)\} \\
&\quad +\int_{t-\eta}^{t+\eta} \frac{g(u)\,du}{t^2-u^2}\Big\{u\sqrt{(Xt)}\,J_\nu(Xt)\,[\sqrt{(Xu)}\,J_\nu(Xu)]' \\
&\qquad\qquad -t\sqrt{(Xu)}\,J_\nu(Xu)\,[\sqrt{(Xt)}\,J_\nu(Xt)]' \\
&\qquad\qquad -\frac{2}{\pi}\,t\cos(Xu-\omega)\sin(Xt-\omega) \\
&\qquad\qquad +\frac{2}{\pi}\,u\cos(Xt-\omega)\sin(Xu-\omega)\Big\} \\
&= K_4+K_5,
\end{aligned}
$$

where ω satisfies (6.63).

Now

$$
\begin{aligned}
(9.21)\qquad K_4 &= \frac{1}{\pi}\int_{t-\eta}^{t+\eta} g(u)\,du\left\{\frac{\sin(t-u)X}{t-u}+\frac{\sin(\overline{t+u}\,X-2\omega)}{t+u}\right\} \\
&= K_6+K_7.
\end{aligned}
$$

Suppose now that $t > 0$. Then

$$(9.22) \qquad |K_7| \leqslant \frac{1}{\pi t} \int_{t-\eta}^{t+\eta} |g(u)|\, du = o(1) \quad \text{as} \quad \eta \to 0.$$

K_6 is similar to an integral which occurs in the theory of summation of Fourier series, and, since g is of bounded variation in $(0, \infty)$, we know that

$$(9.23) \qquad K_6 = g(t) + o(1),$$

for almost all values of t, the constant in $o(1)$ depending on η.

By the first mean value theorem, we have

$$\begin{aligned}(9.24) \quad K_5 = \int_{t-\eta}^{t+\eta} \frac{g(u)\, du}{t+u} \Big\{ & Xt[\sqrt{(VX)}\, J_\nu(VX)]' \, [\sqrt{(tX)}\, J_\nu(tX)]' \\ & -VX[\sqrt{(tX)}\, J_\nu(tX)][\sqrt{(VX)}\, J_\nu(VX)]'' \\ & -\sqrt{(tx)}\, J_\nu(tX)[\sqrt{(VX)}\, J_\nu(XV)]' \\ & -\frac{2}{\pi} Xt \sin(VX-\omega) \sin(tX-\omega) \\ & -\frac{2}{\pi} \cos(tX-\omega) \sin(VX-\omega) \\ & -\frac{2}{\pi} VX \cos(Xt-\omega) \cos(VX-\omega) \Big\},\end{aligned}$$

where

$$(9.241) \qquad V = t + \theta(u-t),$$

$$\theta = \theta(t, u, X),$$

and $$0 < \theta < 1.$$

Also, since $$t-\eta \leqslant V \leqslant t+\eta,$$

we see that $$\alpha \leqslant V \leqslant \beta,$$

where $\alpha > 0$ and β is finite, so that VX is greater than a constant, and tX is greater than a constant. If we substitute asymptotic expansions, we see that the part of the integrand in K_5 which is inside the brackets is equal to

$$\frac{2}{\pi}Xt\left\{-\sin(VX-\omega)-\frac{4\nu^2-1}{8VX}\cos(VX-\omega)+O\left(\frac{1}{X^2}\right)\right\}$$
$$\times\left\{-\sin(Xt-\omega)-\frac{4\nu^2-1}{8Xt}\cos(Xt-\omega)+O\left(\frac{1}{X^2}\right)\right\}$$
$$-\frac{2}{\pi}VX\left\{\cos(Xt-\omega)-\frac{4\nu^2-1}{8Xt}\sin(Xt-\omega)+O\left(\frac{1}{X^2}\right)\right\}$$
$$\times\left\{-\cos(VX-\omega)+\frac{4\nu^2-1}{8VX}\sin(VX-\omega)+O\left(\frac{1}{X^2}\right)\right\}$$
$$-\left\{\cos(Xt-\omega)-\frac{4\nu^2-1}{8Xt}\sin(Xt-\omega)+O\left(\frac{1}{X^2}\right)\right\}$$
$$\times\left\{-\sin(VX-\omega)-\frac{4\nu^2-1}{8VX}\cos(VX-\omega)+O\left(\frac{1}{X^2}\right)\right\}$$
$$-\frac{2}{\pi}Xt\sin(VX-\omega)\sin(Xt-\omega)$$
$$-\frac{2}{\pi}VX\cos(Xt-\omega)\cos(VX-\omega)$$
$$-\frac{2}{\pi}\cos(Xt-\omega)\sin(VX-\omega)$$
$$=\frac{4\nu^2-1}{8}\,\frac{2}{\pi}\left\{\frac{t}{V}\cos(VX-\omega)\sin(Xt-\omega)\right.$$
$$\left.-\frac{V}{t}\sin(Xt-\omega)\cos(VX-\omega)\right\}+O\left(\frac{1}{X}\right)$$
$$=\frac{2}{\pi}(4\nu^2-1)\frac{t^2-V^2}{8tV}\cos(VX-\omega)\sin(tX-\omega)+O\left(\frac{1}{X}\right)$$
$$=O(1).$$

Thus

(9.25) $$|K_5|<A\int_{t-\eta}^{t+\eta}|g(u)|\,du \quad (A \text{ being a constant})$$
$$=o(1) \quad \text{as } \eta\to 0.$$

Summing up, we get

(9.26) $$K_2=g(t)+o(1)+o(1),$$

for almost all values of t, the last term tending to zero with η, and the constant in the second term depending on η.

Having thus fixed η, we see that

$$K_2 = g(t) + o(1), \tag{9.27}$$

the constant in $o(1)$ being now independent of η.

To prove that K_1 tends to zero as X tends to infinity, we write

$$\begin{aligned} K_1 &= \int_0^{t-\eta} \frac{\sqrt{(tu)}\, g(u)\, du}{t^2-u^2} \{Xt\, J_{\nu+1}(Xt)\, J_\nu(Xu) - Xu\, J_\nu(Xt)\, J_{\nu+1}(Xu)\} \\ &= K_8 - K_9. \end{aligned} \tag{9.3}$$

Now

$$\begin{aligned} K_8 &= t^{\frac{3}{2}} X J_{\nu+1}(Xt) \int_0^{t-\eta} \frac{\sqrt{u}\, g(u)}{t^2-u^2} J_\nu(Xu)\, du \\ &= O(\sqrt{X}) \int_0^{t-\eta} \frac{\sqrt{u}\, g(u)}{t^2-u^2} J_\nu(Xu)\, du, \end{aligned} \tag{9.31}$$

and, since

$$\begin{aligned} \int_0^{t-\eta} \frac{|g(u)|\, du}{t^2-u^2} &< \frac{1}{\eta(2t-\eta)} \int_0^{t-\eta} |g(u)|\, du \\ &< A \quad \text{(a constant)}, \end{aligned}$$

we have†, since η is now fixed,

$$K_8 = O(\sqrt{X})\, o(1/\sqrt{X}) = o(1). \tag{9.32}$$

Also

$$\begin{aligned} K_9 &= \sqrt{t}\, X J_\nu(Xt) \int_0^{t-\eta} \frac{u^{\frac{3}{2}} g(u)}{t^2-u^2} J_{\nu+1}(Xu)\, du \\ &= O(\sqrt{X}) \int_0^{t-\eta} \frac{u^{\frac{3}{2}} g(u)}{t^2-u^2} J_{\nu+1}(Xu)\, du, \end{aligned} \tag{9.33}$$

and, since

$$\begin{aligned} \int_0^{t-\eta} \frac{u\,|g(u)|\, du}{t^2-u^2} &= \int_0^{t-\eta} \frac{|g(u)|\, du}{(t-u)(1+t/u)} \\ &< \frac{1}{\eta} \int_0^{t-\eta} |g(u)|\, du < A \quad \text{(a constant)}, \end{aligned}$$

we have, as before,

$$K_9 = O(\sqrt{X})\, o(1/\sqrt{X}) = o(1). \tag{9.34}$$

---

† See Macaulay-Owen, 3, Theorem A, 302.

Thus

(9.35) $$K_1 = o(1) \quad \text{as} \quad X \to \infty.$$

To deal with K_3, we write

(9.4) $$K_3 = \int_{t+\eta}^{\infty} \frac{\sqrt{(tu)}\, g(u)\, du}{t^2-u^2} \{Xt\, J_{\nu+1}(Xt)\, J_\nu(Xu) - Xu\, J_\nu(Xt)\, J_{\nu+1}(Xu)$$

$$= \int_{t+\eta}^{U} + \int_{U}^{\infty} = K_{10} + K_{11}.$$

Since g is of bounded variation in $(0, \infty)$, we write

(9.41) $$g = \psi_1 - \psi_2,$$

where ψ_1 and ψ_2 are positive, decreasing, monotonic functions. If, as $u \to \infty$, $\psi_1(u) \to A$, then also will $\psi_2(u) \to A$, since $g(u) \to 0$, so that, if we write

(9.42) $$\psi_1 - A = \chi_1,$$

(9.43) $$\psi_2 - A = \chi_2,$$

we get

(9.44) $$\int_U^\infty = \int_U^\infty \chi_1(u) \ldots - \int_U^\infty \chi_2(u) \ldots,$$

where χ_1 and χ_2 are positive functions, decreasing monotonically to zero.

By the second mean value theorem, we can write

(9.45) $$\int_U^\infty \chi_1(u) \ldots = \chi_1(U) \int_U^{U'} \frac{\sqrt{(tu)}\, du}{t^2-u^2} \{Xt J_{\nu+1}(Xt)\, J_\nu(Xu) - Xu J_\nu(Xt)\, J_{\nu+1}(Xu)\},$$

where $$U \leqslant U' < \infty,$$

with a similar expression for the integral involving χ_2.

We thus get, by Lemma (7),

(9.44) $$|K_{11}| < H\{\chi_1(U) + \chi_2(U)\} < \epsilon,$$

if $U > U_1(\epsilon)$.

Having thus fixed U, we prove as in the case of K_1 that

(9.45) $$K_{10} = o(1)$$

and consequently

(9.46) $$K_3 = o(1) \quad \text{as} \quad X \to \infty.$$

10. To prove that $\phi(t, \delta) \to 0$ when $\delta \to 0$, for almost all values of t, write

$$(10.1) \quad \phi(t, \delta) = \int_0^\infty \frac{\sqrt{(tu)}\, g(u)\, du}{t^2-u^2} \{\delta t\, J_{\nu+1}(\delta t)\, J_\nu(\delta u) - \delta u\, J_\nu(\delta t)\, J_{\nu+1}(\delta u)\}$$

$$= \int_0^{t-\eta} + \int_{t-\eta}^U + \int_U^\infty = I_1 + I_2 + I_3.$$

I_3 is substantially the same as K_{11}, and we prove in the same way that

$$(10.2) \qquad I_3 = o(1), \quad \text{when} \quad U > U_1.$$

Having thus fixed U, we write

$$(10.3) \quad I_2 = \int_{t-\eta}^U \frac{g(u)\, du}{t+u} \{\delta t [\sqrt{(\delta V)}\, J_\nu(\delta V)]' \, [\sqrt{(\delta t)}\, J_\nu(\delta t)]' \, . \\ - \delta V [\sqrt{(\delta t)}\, J_\nu(\delta t)][\sqrt{(\delta V)}\, J_\nu(\delta V)]'' \\ - \sqrt{(\delta t)}\, J_\nu(\delta t)\, [\sqrt{(\delta V)}\, J_\nu(\delta V)]'\},$$

where $t-\eta \leqslant V \leqslant U$.

We see that $$\alpha \leqslant V \leqslant \beta,$$

where $\alpha > 0$ and β is finite, so that δt is less than a constant and δV is less than a constant, and we can write the part of the integrand inside the brackets in the form

$$\left[\frac{1}{2^\nu \Gamma(\nu+1)}\right]^2 [(\nu+\tfrac{1}{2})(\delta V)^{\nu-\frac{1}{2}} + O(\delta^{\nu+\frac{3}{2}})][(\nu+\tfrac{1}{2})(\delta t)^{\nu+\frac{1}{2}} + O(\delta^{\nu+\frac{5}{2}})] \\ - \left[\frac{1}{2^\nu \Gamma(\nu+1)}\right]^2 [(\nu^2-\tfrac{1}{4})(\delta V)^{\nu-\frac{1}{2}} + O(\delta^{\nu+\frac{3}{2}})][(\delta t)^{\nu+\frac{1}{2}} + O(\delta^{\nu+\frac{5}{2}})] \\ - \left[\frac{1}{2^\nu \Gamma(\nu+1)}\right]^2 [(\nu+\tfrac{1}{2})(\delta V)^{\nu-\frac{1}{2}} + O(\delta^{\nu+\frac{3}{2}})][(\delta t)^{\nu+\frac{1}{2}} + O(\delta^{\nu+\frac{5}{2}})]$$

$$= \left[\frac{1}{2^\nu \Gamma(\nu+1)}\right]^2 (\delta V)^{\nu-\frac{1}{2}} (\delta t)^{\nu+\frac{1}{2}} [(\nu+\tfrac{1}{2})^2 - (\nu^2-\tfrac{1}{4}) - (\nu+\tfrac{1}{2})] + O(\delta^{2\nu+2})$$

$$= O(\delta^{2\nu+2}).$$

Thus

$$(10.31) \qquad I_2 = O(\delta^{2\nu+2}) \int_{t-\eta}^U |g(u)|\, du$$

$$= o(1), \quad \text{as} \quad \delta \to 0.$$

Also

$$(10.4) \quad I_1 = O \int_0^{t-\eta} |g(u)|\, du \{O(\delta^{\nu+2})\, O(\delta^{\nu} u^{\nu+\frac{1}{2}}) + O(\delta^{\nu+1})\, O(\delta^{\nu+1} u^{\nu+\frac{5}{2}})\}$$

$$= O(\delta^{2\nu+2}) \int_0^{t-\eta} |g(u)|\, du$$

$$= o(1), \quad \text{as} \quad \delta \to 0.$$

We have thus proved that $\phi(t, \delta) \to 0$, as $\delta \to 0$, for almost all values of t.

The theorem is thus proved.

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CORRIGENDA

ON THE PRODUCT OF THREE HOMOGENEOUS LINEAR FORMS (II)

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[*Proc. London Math. Soc.* (2), 44 (1938), 412–431.]

P. 413. On line 16, *for* "$\phi = 0 - {\cdot}44504$" *read* "$\phi = -0{\cdot}44504$."

P. 415. On line 5 from below, *for* "Adding and (7) (8)" *read* "Adding (7) and (8)".

P. 420. On line 11 insert a comma after "*where m is an integer*."

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